$$W_o = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k \tag{3}$$

The grammians of equations (2) and (3) are the solutions of the Lyapunov equations of (4) and (5) respectively:

$$AW_c A^T - W_c = -BB^T \tag{4}$$

$$A^T W_o A - W_o = -C^T C \tag{5}$$

It has been shown that similarty transformation can be found such that the system (1) is internally balanced, that is, the matrices W_c and W_o are equal and diagonal:

$$W_c = W_o = \Sigma = diag\,(\sigma_1, \sigma_2, ..., \sigma_n) \tag{6}$$

where $\sigma_i \geq \sigma_{i+1}$, *i=1,2,...,n-1* are the grammians singular values and are invariant under similarty transformation.
Based on the order of magnitude of singular values, this balanced system and grammian can be partitioned as below:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$
$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{7}$$

And it can be shown that if $\sigma_i >> \sigma_{i+1}$, the subsystem (A_{11}, B_1, C_1) is a good reduced order approximation of the main full order system (A, B, C). technique is called Balance Truncation (BT).

III. New Controller Reduction Approach

In this section, the new controllability and observability frequency based grammians from the input/output energy distribution viewpoint will be proposed. These grammianns will be used in next step for controller reduction. Consider the closed loop system of figure 1.
Where in this figure, the transfer function of the plant, G(z) has a state space realization as equation (1). K(z) is a high order stable controllable and observable controller with state space realization as (8).

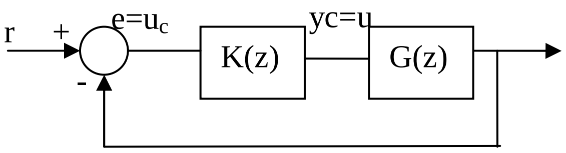

Fig. 1 The closed loop system

$$\begin{cases} x_c(k+1) = A_c x_c(k) + B_c u_c(k) \\ \quad\; y_c(k) = C_c x_c(k) \end{cases} \tag{8}$$

Where $x_c \in R^{n_c}$ is a state vector, $u_c \in R^q$ represents the input vector of the controller, and $y_c \in R^p$ is the output of the controller with matrices A_c, B_c, and C_c in the appropriate dimensions. The order of the plant is n and the order of controller is n_c

For the closed loop system of fig. 1, consider the following equations:

$$X_c(e^{j\omega}) = (e^{j\omega} I - A_c)^{-1} B_c U_c(e^{j\omega}) \tag{9}$$

$$U_c = R - Y = R - GC_c X_c \tag{10}$$

This can be simplified to result the relation between controller states and system input as:

$$\begin{aligned} X_c &= (e^{j\omega} I - A_c + B_c G C_c)^{-1} B_c R \\ &= (e^{j\omega} I - A_c)^{-1} B_c (I + GK)^{-1} R \end{aligned} \tag{11}$$

where the dependence on $e^{j\omega}$ has been dropped for simplicity.

By forming the following energy related quantity due to input *R* as in [5] and using Parsaval theorem we have:

$$\begin{aligned} E_i &= \sum_{k=-\infty}^{\infty} x_c(k) x_c^*(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_c(e^{j\omega}) X_c^*(e^{j\omega}) d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{j\omega} I - A_c)^{-1} B_c (I + GK)^{-1} RR^* (I + GK)^{-*} B_c^* (e^{j\omega} I - A_c)^{-*} d\omega \end{aligned} \tag{12}$$

By considering the input as white noise, the frequency domain closed loop controllability gammian of controller W$_{cc}$ is defined as:

$$W_{cc} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{j\omega} I - A_c)^{-1} B_c (I + GK)^{-1} RR^* (I + GK)^{-*} B_c^* (e^{j\omega} I - A_c)^{-*} d\omega \tag{13}$$

A similar interpretation for the output of the closed loop system can hold. This time the output energy of the system due to controller state will be considered. Because $Y = GY_c = GC_c X_c$ then using relation (10) to (12) and Parsaval theorem, we have:

$$\begin{aligned} E_o &= \sum_{k=-\infty}^{\infty} y^*(k) y(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y^*(e^{j\omega}) Y(e^{j\omega}) d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{-j\omega} I - A_c^*)^{-1} C_c^* (I + GK)^{-*} G^* R^* R G_c (I + GK)^{-1} C_c (e^{j\omega} I - A_c)^{-1} d\omega \end{aligned} \tag{14}$$

Relation (14) was obtained by considering the states as white noise and eliminating the input of the closed loop system. The frequency domain closed loop observability gammian of controller W_{oc} is defined as (15).

$$W_{oc} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{-j\omega} I - A_c^*)^{-1} C_c^* (I + GK)^{-*} G^* R^* R G_c (I + GK)^{-1} C_c (e^{j\omega} I - A_c)^{-1} d\omega \tag{15}$$

Because most systems work in a certain frequency domain it is desirable to tight the frequency domain. In this case, consider the input signal $r(e^{j\omega})$ which its energy density spectrum is unity in frequency range $[\omega_0, \omega_1]$ and zero elsewhere, i. e:

$$\left|R(e^{j\omega})\right| = \begin{cases} 1, & \omega \in [\omega_0, \omega_1] \\ 0, & Otherwise \end{cases} \tag{16}$$

So the grammians in (13) and (15) for finite frequency range will be:

$$W_{cc} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{j\omega}I - A_c)^{-1}B_c(I+GK)^{-1}RR^*(I+GK)^{-*}B_c^*(e^{j\omega}I - A_c)^{-*}d\omega \tag{17}$$

$$W_{oc} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{-j\omega}I - A_c^*)^{-1}C_c^*(I+GK)^{-*}G^*R^*RG_c(I+GK)^{-1}C_c(e^{j\omega}I - A_c)^{-1}d\omega \tag{18}$$

The following lemma shows that by using a transformation on these grammians the eingenvalues of the grammians will be invariant so these closed loop grammians can be used for controller reduction:

Lemma 3.1. For a discrete linear time-invariant controller (A_c, B_c, C_c) the following properties hold:

A. If a coordinate transformation such as $x(k) = T\hat{x}(k)$ is considered, the transformed closed loop controllability and observability grammians of the controller can be calculated as: $\tilde{W}_{cc} = T^{-1}W_{cc}T^{-*}$, $\tilde{W}_{co} = T^*W_{co}T$

B. Under the selected transformation *T*, the singular values of product W_{cc}, W_{co} hat is, $\sigma_i = \sqrt{\lambda_i(W_{cc}W_{oo})}$ are invariant.

C. There exists a special transformation $\bar{T}$ such that the closed loop controllability and observability grammians of controller can be diagnolized and equal

Proof. The result follows using direct substitutions. □

Based on lemma 3.1, the singular values of the closed loop system are invariant so one can find a similarty transformation which can balance the controllability and observability grammians of the closed loop system (17),(18). This transformation can be calculated by the procedure presented in [1]. After using this transformation, these grammians which are diagnolized and equal can be represented in the new coordinate can be as $\sum_f$, where

$$\sum_f = diag(\sigma_1, \sigma_2, ..., \sigma_{n_c}), \quad \sigma_i \ge \sigma_{i+1}, \quad i = 1,2,.., n_c - 1$$

Partitioning the balanced controller and Grammian gives that

$$A_c = \begin{bmatrix} A_{c11} & A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix}, \quad B_c = \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix}, \quad C_c = \begin{bmatrix} C_{c1} & C_{c2} \end{bmatrix}$$
$$\Sigma_f = \begin{bmatrix} \Sigma_{f1} & 0 \\ 0 & \Sigma_{f2} \end{bmatrix}, \quad x_c(k) = \begin{bmatrix} x_{c1}(k) \\ x_{c2}(k) \end{bmatrix} \tag{19}$$

where A_{c11} and $\sum_{f1}$ are $r \times r$ ($r<n_c$) matrices. The system $(A_{c11}, B_{c1}, C_{c1})$ is the reduced-order controller.

The following steps described an algorithm for the proposed controller reduction method:

A. Calculate closed loop controllability and observability grammians W_{cc} and W_{co} in the given frequency range. They can be obtained by (17), (18).

B. Find a similarity transformation T that makes the controller balanced, that is, $W_{cc} = W_{co} = \Sigma_f$.

C. Partition the transformed controller as (19) based on the grammians singular values. The subsystem $(A_{c11}, B_{c1}, C_{c1})$ is the reduced controller

Now we show the stability of the reduced order controller. Consider equation (17) where the finite frequency domain can be considered with a window $W(e^{i\omega})$ where

$$W(e^{j\omega}) = \begin{cases} 1, & \omega_0 < \omega < \omega_1 \\ 0, & Elsewhere \end{cases} \tag{20}$$

So we have

$$W_{cc} = \frac{1}{2\pi}\int_{-\pi}^{\pi}W(e^{j\omega}I - A_c)^{-1}B_cW_iW_i^*B_c^*(e^{j\omega}I - A_c^*)^{-1}W^*d\omega \tag{21}$$

where $W = (I + GK)^{-1}$

Using Parsaval theorem, the equation (21) in discrete time domain can be written as:

$$W_{cc} = \sum_{k=0}^{\infty} A_c^k B_k * W_2W_2^* * B_k^*(A_c^*)^k = \sum_{k=0}^{\infty} A_c^k z_k z_k^*(A_c^*)^k \tag{22}$$

where * denotes convolution and $W_2 = W * W_i$ and $z = B_k * W_2$. By forming the Lyapunov equation for controller we have:

$$A_cW_{cc}A_c^* - W_{cc} = A_c^\infty z_k(\infty)z_k^*(\infty)(A_c^*)^\infty - z_k(0)z_k^*(0) \tag{23}$$

If the original controller is stable, in infinite time the states of the controller converge to zero, so we have:

$$A_cW_{cc}A_c^* - W_{cc} = -z_k(0)z_k^*(0) \le 0 \quad \Rightarrow$$
$$A_cW_{cc}A_c^* - W_{cc} = -NN^* \tag{24}$$

By the same procedure for the observability grammian, we have:

$$A_c^* W_{co} A_c - W_{co} = -LL^* \tag{25}$$

So the controllability and observability grammians are negative semidefinite and based on [6] we can say that the reduced order controller is stable.

IV. EXAMPLE

In [2] a plant is considered as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -50 & -79 & -33 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 50 & 15 & 1 & 0 \end{bmatrix}$$

We design a controller for this plant as:

$$C(z) = \frac{1.228 - 1.075z^{-1} + .3323z^{-2}}{1 - 2.207z^{-1} + 1.777z^{-2} - 0.5122z^{-3}}$$

We reduce this controller by the new method in frequency range of $\left[0, \frac{\pi}{4}\right]$ and Moore method. The result of the error between the reduced order controller and the full order controller $\|K(z) - K_r(z)\|_\infty$ are summarized in Table 1. Moore method does not consider the loop. So the results show that this method cannot be so much accurate.

Method	**Reduced Order Controller of Degree 2**	**Reduced Order Controller of Degree 1**
BT	**0.3177**	**0.3760**
New Method	**0.1857**	**0.1937**

Table 1 The error of the closed loop system

Figure 2 shows the magnitude of the reduced order controller of degree 2 and 1 with the new method and original controller. It is obvious that the reduced order controller has the performance similar to original controller in the defined frequency range.

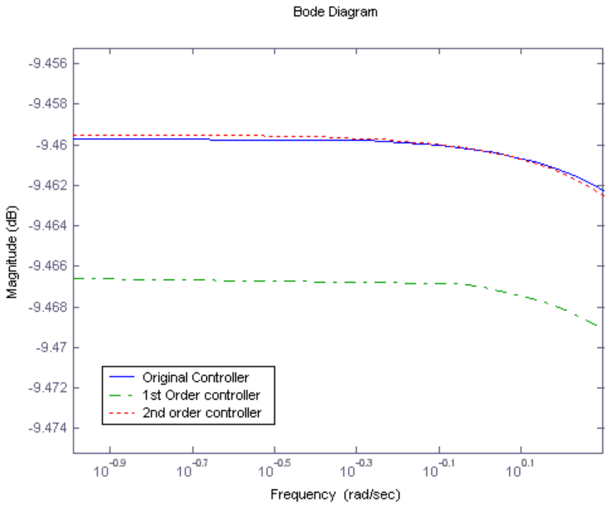

Fig. 2 Original controller and controller of degree 2 (--) and 1 (-.)

V. CONCLUSION

In this paper a new frequency based controller reduction method for discrete linear time invariant system was proposed. The method is based on new frequency-domain controllability and observability grammians. The stability of the reduced order controller was discussed and the simulation results show the effectiveness of this method.

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