$$u(0,t) = a(t), \qquad 0 < t < 1, \tag{7}$$

$$u(\pi,t) = b(t), \qquad 0 < t < 1, \tag{8}$$

We say that $u \in C^{2,1}(D)$ if u has a continuous second order partial derivative with respect to x and a continuous first order partial derivative with respect to t. Let $X = C(\overline{D})$ be the Banach space of real-valued continuous functions on $\overline{D}$, equipped with the norm $\|u\|_\infty = \max\{|u(x,t)|;\ (x,t) \in D\}$ for $u \in X$. A strong solution of the above problem is a function $u \in C^{2,1}(D) \cap C(\overline{D})$. The following result can be found in [13] and [21].

Assume that the functions $f,\ u_0$, are Hölder continuous, and the functions a and b are continuous. Then, Problem (5), (6), (7), (8) has a unique strong solution given by for each $(x,t) \in D = (0,\pi) \times (0,1)$,

$$\begin{aligned} u(x,t) &= \textstyle\int_0^t \int_0^\pi G(x,t;y,s)\, f(y,s)\, dyds \\ &+ \textstyle\int_0^\pi G(x,t;y,0)\, u_0(y)\, dy \\ &+ \textstyle\int_0^t \frac{\partial G}{\partial y}(x,t;0,s)\ a(s)ds \\ &- \textstyle\int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s)\ b(s)ds, \end{aligned} \tag{9}$$

where $G(x,t;y,s)$ is the Green's function corresponding to the linear homogeneous problem. This function satisfies the following

(i) $G_t - G_{xx} = \delta(t-s)\,\delta(x-y) \quad s < t,$
$0 < x, y < \pi$

(ii) $G(x,t;y,s) = 0 \quad s > t, 0 < x, y < \pi$

(iii) $G(0,t;y,s) = G(\pi,t;y,s) = 0 \quad s < t$

(iv) $G(x,t;y,s) > 0$ for $(x,t) \in D$

(v) $G,\ G_t,\ G_x,\ G_{xx}$ are continuous functions of $(x,t), (y,s) \in D,\ t - s > 0$.

(vi) there exist $d_0 > 0$ and $\mu \in (0,1)$ such that

$$|G(x,t;y,s)| \leq \frac{d_0\,|x-y|^{1-\mu}}{(t-s)^\mu},$$

and $\displaystyle\left|\frac{\partial G}{\partial y}(x,t;y,s)\right| \leq \frac{d_0\,|x-y|^{\kappa-2+2\mu}}{(t-s)^\mu},$
with $1 - \frac{\kappa}{2} < \mu < 1$.

Lemma 1 *Let* y_0 *be a fixed number in* $[0,\pi]$.

Then there exists a constant $\delta_0 > 0$ such that $\max_{(x,t)\in D} \int_0^t \left|\frac{\partial G}{\partial y}(x,t;y_0,s)\right| ds \leq \delta_0$.

Proof. This follows from the estimate on $\left|\frac{\partial G}{\partial y}(x,t;y,s)\right|$.

We write (9) in the following convenient form, for each $(x,t) \in D$,

$$u(x,t) = G\,(f\ + u_0)\,(x,t) + \gamma\,(a,b)\,(x,t) \tag{10}$$

where

$$\begin{aligned} &G\,(f\ + u_0)\,(x,t) = \\ &\textstyle\int_0^t \int_0^\pi G(x,t;y,s)\, f(y,s)\, dyds \\ &+ \textstyle\int_0^\pi G(x,t;y,0)\, u_0(y)\, dy, \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\gamma\,(a,b)\,(x,t) = \\ &\textstyle\int_0^t \frac{\partial G}{\partial y}(x,t;0,s)\ a(s)ds \\ &- \textstyle\int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s)\ b(s)ds. \end{aligned} \tag{12}$$

The operators $G,\ \gamma$ map $C(\overline{D})$ into $C^{2,1}(D)$. Moreover, $v = G\,(f\ + u_0)$ solves the problem
$v_t - v_{xx} = f,\ v(x,0) = u_0(x)$ and $w = \gamma\,(a,b)$ solves the problem $w_t - w_{xx} = 0,\ w(x,0) = 0,\ w(0,t) = a(t),\ w(\pi,t) = b(t)$.

III. MULTIVALUED FUNCTIONS

We, now, introduce some useful definitions and properties from set-valued analysis. For complete details on multivalued maps we refer the interested reader to the books [1], [2] and [9].

Let $(Y, |\cdot|)$ be a normed space. We shall denote the set of all subsets of Y having property ℓ by $P_\ell(Y)$. For instance, $U \in P_{cl}(Y)$ means U closed in Y; when $\ell = b$ we have the bounded subsets of Y, $\ell = cv$ for convex subsets, $\ell = cp$ for compact subsets and $\ell = cp, cv$ for compact and convex subsets. A multivalued map $R : Y \to 2^Y$ is convex (closed) valued if $R(z)$ is convex (closed) for each $z \in Y$. R is bounded on bounded sets if $R(B) = \cup_{z\in B} R(z)$ is bounded in Y for all $B \in P_b(Y)$ (i.e. $\sup_{z\in B}\{\sup\{|y|\,; y \in R(z)\}\} < \infty$). The multivalued map R is called upper semicontinuous (usc) on Y if for each $z \in Y$ the set $R(z) \in P_{cl}(Y)$ and is nonempty, and for each open subset Λ of Y containing $R(z)$, there exists an open neighborhood Π of z such that $R(\Pi) \subset \Lambda$. The set-valued map R is called completely continuous if

$R(B)$ is relatively compact for every $B \in P_b(Y)$. If R is completely continuous with nonempty compact values, then R is usc if and only if R has a closed graph (i.e. $z_n \to z,\ w_n \to w,\ w_n \in R(z_n) \Rightarrow w \in R(z)$). R has a fixed point if there exists $z \in Y$ such $z \in R(z)$. A multivalued map $R : \overline{D} \to P_{cl}(\mathbb{R})$ is called measurable if for every $\theta \in \mathbb{R}$, the function $v \longmapsto dist(\theta, R(v)) = \inf\{|\theta - z|\,; z \in R(v)\}$ is measurable.

Definition 2 $\digamma : D \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \emptyset$ *is called*

an L^2-Carathéodory multifunction if
(i) $\digamma(.,.,u) : D \to 2^{\mathbb{R}}$ is measurable for all $u \in \mathbb{R}$,
(ii) $\digamma(x,t,.) \to \mathbb{R} \to 2^{\mathbb{R}}$ is usc for almost all $(x,t) \in D$
(iii) for each $\varrho > 0$ there exists $\omega_\varrho \in L^2(D)$ such that $|u| \le \varrho$ implies
$|\digamma(x,t,u)| := \{|w|\,; w \in F(x,t,u)\} \le \omega(x,t)$ for a.e.$(x,t) \in D$.

Definition 3 *Let $u \in X$. Then $S_{\digamma,u}$ denotes the set of L^2-selections of the set-valued map $\digamma : \mathbb{R} \to 2^{\mathbb{R}}$, and is the set*

$\{w \in L^2(D);\ w(x,t) \in \digamma(x,t,u(x,t)),\ \forall (x,t) \in D\}$.
The fact that this set is not empty follows from Lemma 3 in [18].

Definition 4 *Let $\digamma : D \times \mathbb{R} \to 2^{\mathbb{R}}$ have nonempty compact values. The Nemitsky operator $\mathcal{F}$ of $\digamma$ is the set-valued operator defined by*

$\mathcal{F} : C(D) \to L^2(D),\ \mathcal{F}(u)$ is the set of all

$w : D \to \mathbb{R}$ measurable such that
$w(x,t) \in \digamma(x,t,u(x,t)),\ \forall (x,t) \in D$.
It can be shown (see [14, page 40], [23]) that if $\digamma$ is usc with convex bounded values then the operator $\mathcal{F}$ is well defined, usc, bounded on bounded sets in $C(D)$, and has convex values.

Definition 5 *u is a strong solution of (1), (2), (3), (4) if there exists a Lipschitz selection $f \in S_{F,u}$ and u has the integral representation (9).*

Remark. If F is a Lipschitz multifunction then it admits a Lipschitz selection. See [17].

Definition 6 *Let (Z,d) be a metric space and let A, B be two nonempty subsets of Z. The Hausdorf distance between A and B is defined by*

$$d_H(A,B) = \max\{\sup_{a \in A} d(a,B),\ \sup_{b \in B} d(A,b)\}.$$

Here $d(a,B) = \inf\{d(a,b); b \in B\}$.Then $(P_{cl,b}(Z),\ d_H)$ is a metric space.

Definition 7 *A multivalued operator $\mathcal{L}$: $Z \to P_{cl}(Z)$ is called*

(i) $\delta-$Lipschitz if and only if there exists $\delta > 0$ such that $d_H(\mathcal{L}(u), \mathcal{L}(v)) \le \delta\, d(u,v)$ for all $u, v \in Z$

(ii) a contraction if and only if it is $\delta-$Lipschitz with $\delta < 1$.

The following theorems play an important role in our existence results.

Theorem 8 *[19] Let E be a Banach space and $\mathcal{L} : E \to P_{cp,cv}(E)$ a condensing map. If the set $S := \{z \in E;\ \lambda z \in \mathcal{L}(z)$ for some $\lambda > 1\}$ is bounded, then $\mathcal{L}$ has a fixed point.*

We remark that a compact map is the simplest example of condensing maps.

Theorem 9 *[10]Let $B_r(0)$ and $\overline{B_r(0)}$ denote respectively the open and closed balls in a Banach space $(E, \|\cdot\|)$ centered at 0 and having radius r. Let $\mathcal{L}_1 : \overline{B_r(0)} \to P_{cl,cv,b}(E)$ and $\mathcal{L}_2 : \overline{B_r(0)} \to P_{cp,cv}(E)$ be two multivalued operators satisfying*

(i) $\mathcal{L}_1$ is a contraction,
(ii) $\mathcal{L}_2$ is compact and usc.
Then either
(j) the operator inclusion $u \in \mathcal{L}_1 u + \mathcal{L}_2 u$ has a solution in $\overline{B_r(0)}$, or
(jj) there exists $u \in E$ with $\|u\| = r$ such that $\lambda u \in \mathcal{L}_1 u + \mathcal{L}_2 u$ for some $\lambda > 1$.

Theorem 10 *[14, page 11]Let E be a normed linear space, C convex subset in E and U open in C with $0 \in U$. Let $\Lambda : U \to 2^C$ be an usc, compact multivalued operator with closed and convex values. Then either*

(a) Λ has a fixed point, or
(b) there exists $z \in \partial U$ such that $z \in \lambda \Lambda z$ for some $\lambda \in (0,1)$.

IV. MAIN RESULTS

In this section, we shall state and prove our main results.We shall assume throughout the remainder of the paper that the following conditions hold.

(H0) The multifunction $F : D \times \mathbb{R} \to 2^{\mathbb{R}}$ is nonempty, usc and has compact and convex values. Moreover, there exists a Lipschitz selection $f \in S_{F,u}$ for each $u \in X$.

(H1) $u_0 \in C([0, \pi])$.

Theorem 11 *Suppose that, in addition to (H0) and (H1) the following assumptions are satisfied,*

(H2) $g, \ h : C(D) \to \mathbb{R}$ are continuous and bounded,

(H3) F maps bounded sets into relatively compact sets, and there are positive constants $c_1, \ c_2$ such that $|F(x, t, u)| \leq c_1 + c_2 |u|$.

Then Problem (1), (2), (3), (4) has at least one strong solution.

Proof. It follows from (9), (10), and (11) that u is a solution of (1), (2), (3), (4) if and only if u is a fixed pont of the multivalued operator $\mathcal{L}$, defined by

$$\mathcal{L}u = G\mathcal{F}(u) + \gamma(u), \tag{13}$$

where $\mathcal{F}$ is the Nemitski operator of F.
In fact, we have

$$\mathcal{L}u(x,t) = \int_0^t \int_0^\pi G(x,t;y,s)F(y,s,u(y,s))dyds$$
$$+ \int_0^\pi G(x,t;y,0)\, u_0(y)\, dy$$
$$+ \int_0^t \frac{\partial G}{\partial y}(x,t;0,s) \ \int_0^\pi g(u(y,s))dyds$$
$$- \int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s) \ \int_0^\pi h(u(y,s))dyds,$$

where

$$\int_0^t \int_0^\pi G(x,t;y,s)\,(\mathcal{F}u)(y,s)\,dyds$$

is the Aumann integral of $\mathcal{F}$. We see that $\mathcal{L}$ is the sum of a multivalued operator $G\mathcal{F}$ and a single valued operator $\gamma(\cdot)$. We apply Theorem 6 to the operator $\mathcal{L}$.
Let $u \in X$. We show that $\mathcal{L}u \in P_{cp,cv}(X)$.
(a) $\mathcal{L}u$ is a convex subset of X for each $u \in X$. Let $v_1, v_2 \in \mathcal{L}u$. Then there exists $w_1, w_2 \in S_{F,u}$ such that for each $(x,t) \in D$ we have for $i = 1, 2$

$$\begin{aligned} v_i(x,t) &= \int_0^t \int_0^\pi G(x,t;y,s)w_i(y,s)\,dyds \\ &\quad + \int_0^\pi G(x,t;y,0)\,u_0(y)\,dy \\ &\quad + \int_0^t \frac{\partial G}{\partial y}(x,t;0,s) \ \int_0^\pi g(u(y,s))dyds \\ &\quad - \int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s) \ \int_0^\pi h(u(y,s))dyds. \end{aligned}$$

Since $S_{F,u}$ is convex, it is clear from the above relation that any convex combination of v_1, v_2 is an element of $\mathcal{L}u$.
(b) $\mathcal{L}u$ is a compact subset of X for each $u \in X$. Let $(w_n)_{n\in\mathbb{N}}$ be a bounded sequence in $S_{F,u}$. By (H0) and (H3) the Nemitski operator $\mathcal{F}$ of F is well defined, usc and maps bounded sets into relatively compact sets. The sequence $(v_n)_{n\in\mathbb{N}}$ given by, for each $n \in \mathbb{N}$

$$\begin{aligned} v_n(x,t) &= \int_0^t \int_0^\pi G(x,t;y,s)w_n(y,s)\,dyds \\ &\quad + \int_0^\pi G(x,t;y,0)\,u_0(y)\,dy \\ &\quad + \int_0^t \frac{\partial G}{\partial y}(x,t;0,s) \ \int_0^\pi g(u(y,s))dyds \\ &\quad - \int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s) \ \int_0^\pi h(u(y,s))dyds. \end{aligned}$$

is relatively compact in $\mathcal{L}u$. This implies that $\mathcal{L}u$ is a compact subset of X.
(c) We show that $\mathcal{L} = G\mathcal{F} + \gamma(\cdot)$ is a compact operator. To achieve this, we show that $\mathcal{L}$ is uniformly bounded and maps bounded sets into equicontinuous sets.
Let B be a bounded subset of X, and let $u \in B$. Then there is $M > 0$ such that $\|u\|_\infty \leq M$.
Now, for each $v \in \mathcal{L}u$ there exists $w \in S_{F,u}$ such that

$$\begin{aligned} v(x,t) &= \int_0^t \int_0^\pi G(x,t;y,s)w(y,s)\,dyds \\ &\quad + \int_0^\pi G(x,t;y,0)\,u_0(y)\,dy \\ &\quad + \int_0^t \frac{\partial G}{\partial y}(x,t;0,s) \ \int_0^\pi g(u(y,s))dyds \\ &\quad - \int_0^t \frac{\partial G}{\partial y}(x,t;\pi,s) \ \int_0^\pi h(u(y,s))dyds. \end{aligned}$$

Hence, if m_g and m_h denote the bounds on g and h respectively,

$$|v(x,t)| \leq$$
$$\int_0^t \int_0^\pi G(x,t;y,s)[c_1 + c_2 |u(y,s)|]dyds$$
$$+ \int_0^\pi G(x,t;y,0)\,|\,u_0(y)|\,dy$$
$$+\pi m_g \max_D \int_0^1 \left| \frac{\partial G}{\partial y}(x,t;0,s) \right| ds$$
$$+\pi m_h \max_D \int_0^1 \left| \frac{\partial G}{\partial y}(x,t;\pi,s) \right| ds.$$

It follows from Lemma 1
$|v(x,t)| \leq \pi\,(c_1 + \|u_0\|_\infty)\,\|G\|_\infty$

$$+(\pi m_g+\pi m_h)\delta_0$$
$$+c_2\int_0^1\int_0^\pi G(x,t;y,s)\left|u\left(y,s\right)\right|dyds$$
$$\leq M_0+c_2\int_0^1\int_0^\pi G(x,t;y,s)\left|u\left(y,s\right)\right|dyds$$

where
$M_0=\pi\left(c_1+\|u_0\|_\infty\right)\|G\|_\infty+(\pi m_g+\pi m_h)\delta_0$.
Then $|v(x,t)|$
$\leq M_0+\pi c_2\|G\|_\infty\|u\|_\infty.\leq M_0+\pi c_2\|G\|_\infty M$.
This shows that $\mathcal{L}u$ is uniformly bounded.

Next, let $(x,t)\,,\ (\xi,\tau)\in D$. Then

$$|v\left(x,t\right)-v\left(\xi,\tau\right)|\leq$$
$$(c_1+c_2M)\int_0^1\int_0^\pi|G(x,t;y,s)-G(\xi,\tau;y,s)|\,dyds$$

$$+\|u_0\|_\infty\int_0^\pi G(x,t;y,0)-\ G(\xi,\tau;y,0)dy$$

$$+\pi m_g\int_0^1\left|\frac{\partial G}{\partial y}\left(x,t;0,s\right)\ -\frac{\partial G}{\partial y}\left(\xi,\tau;0,s\right)\right|ds$$

$$+\pi m_h\int_0^1\left|\frac{\partial G}{\partial y}\left(x,t;\pi,s\right)\ -\frac{\partial G}{\partial y}\left(\xi,\tau;\pi,s\right)\right|ds$$

It follows from the properties of the Green's function that, as $|x-\xi|+|t-\tau|\to 0$, the right hand of the last inequality tends to zero. This shows that $\mathcal{L}u$ is equicontinuous.

(d) Now, consider the set $S=\{u\in X;\lambda u\in\mathcal{L}u$, for some $\lambda>1\}$. We show that this set is bounded.
We proceed as before to obtain

$$|\lambda u(x,t)|\leq$$
$$M_0+c_2\int_0^1\int_0^\pi G(x,t;y,s)\left|u\left(y,s\right)\right|dyds$$

Since $\lambda>1$ it follows from the above inequalities that,

$$|u(x,t)|\leq$$
$$M_0+c_2\int_0^1\int_0^\pi G(x,t;y,s)\left|u\left(y,s\right)\right|dyds.$$

Gronwall's inequality implies

$$\|u\|_\infty\leq M_0\exp\left(\pi c_2\|G\|_\infty\right).$$

Therefore the set S is bounded and consequently, $\mathcal{L}$ has a fixed point in X. This fixed point is the solution to our original problem.□

For our second result, we shall assume, in addition to (H0) and (H1), that the following conditions are satisfied.

(H4) g and h are Lipschitz continuous, with Lipschitz constants k_g and k_h respectively, with

$$\Delta:=(k_g\ +k_h)\delta_0<1$$

and further $g(0)=h(0)=0$.

(H5) F has compact, convex values and there exists $\Psi:[0,\infty)\to(0,\infty)$ continuous and nondecreasing such that $|F(x,t,u)|\leq\Psi\left(|u|\right)$

(H6) $\displaystyle\sup_{\rho\in(0,\infty)}\frac{\rho\left(1-\pi\Delta\right)}{\pi\|G\|_\infty\left(\|u_0\|_\infty+\pi\Psi\left(\rho\right)\right)}>1$

Theorem 12 *If the conditions (H0), (H1), (H4), (H5), and (H6) are satisfied. Then Problem (1), (2), (3), (4) has at least one solution.*

Proof. Condition (H6) implies that there exists $r>0$ such that

$$\frac{r\left(1-\pi\Delta\right)}{\pi\|G\|_\infty\left(\|u_0\|_\infty+\pi\Psi\left(r\right)\right)}>1. \tag{14}$$

Consider the closed ball $\overline{B_r(0)}$ in the Banach space X. Let $u\in\overline{B_r(0)}$. Write $\mathcal{L}u$ as $\mathcal{L}_1u+\mathcal{L}_2u$,
with

$$\mathcal{L}_1u(x,t)=$$

$$\int_0^t\frac{\partial G}{\partial y}\left(x,t;0,s\right)\ \int_0^\pi g(u(y,s))dyds \tag{15}$$

$$-\int_0^t\frac{\partial G}{\partial y}\left(x,t;\pi,s\right)\ \int_0^\pi h(u(y,s))dyds,$$

and

$$\mathcal{L}_2u(x,t)=$$

$$\int_0^t\int_0^\pi G(x,t;y,s)F(y,s,u\left(y,s\right))dyds \tag{16}$$

$$+\int_0^\pi G(x,t;y,0)\ u_0\left(y\right)dy.$$

Claim 1. $\mathcal{L}_1:\overline{B_r(0)}\to P_{cl,cv,b}(X)$ is a contraction.
Notice that $\mathcal{L}_1$ is a single valued operator. The continuity of the functions g and h implies that $\mathcal{L}_1u\in P_{cl,cv,b}(X)$.
Now, let $u,v\in\overline{B_r(0)}$. Then

$$|\mathcal{L}_1u(x,t)-\mathcal{L}_1v(x,t)|\leq$$

$$\int_0^t\left|\frac{\partial G}{\partial y}\left(x,t;0,s\right)\right|\ \int_0^\pi|g(u(y,s))-g(v(y,s))|\,dyds$$

$$+\int_0^t\left|\frac{\partial G}{\partial y}\left(x,t;\pi,s\right)\right|\ \int_0^\pi|h(u(y,s))-h(v(y,s))|\,dyds$$

Hence

$$\|\mathcal{L}_1 u - \mathcal{L}_1 v\|_\infty \le$$

$$k_g \int_0^t \left|\frac{\partial G}{\partial y}(x,t;0,s)\right| \int_0^\pi |u(y,s)-v(y,s)|\,dyds$$

$$+k_h \int_0^t \left|\frac{\partial G}{\partial y}(x,t;\pi,s)\right| \int_0^\pi |u(y,s)-v(y,s)|\,dyds.$$

Condition (H4) implies that

$$d_H(\mathcal{L}_1 u, \mathcal{L}_1 v) = \|\mathcal{L}_1 u - \mathcal{L}_1 v\|_\infty \le \Delta \|u-v\|_\infty .$$

Since $\Delta < 1$ it follows (see Definition 5) that $\mathcal{L}_1$ is a contraction.□

Claim 2. $\mathcal{L}_2 : \overline{B_r(0)} \to P_{cp,cv}(X)$ is compact and usc.

·Let $u \in \overline{B_r(0)}$. We proceed as in the proof of the previous theorem to show that $\mathcal{L}_2 u$ is a compact and convex subset of X.

·We show that $\mathcal{L}_2$ is a compact operator on $\overline{B_r(0)}$. For each $v \in \mathcal{L}_2 u$, there exists $w \in S_{F,u}$ such that for each $(x,t) \in D$ we have

$$v(x,t) = \int_0^t \int_0^\pi G(x,t;y,s) w(y,s)\,dyds$$
$$+ \int_0^\pi G(x,t;y,0)\, u_0(y)\,dy.$$

Condition (H5) implies that

$$|v(x,t)| \le \int_0^1 \int_0^\pi G(x,t;y,s)\,|w(y,s)|\,dyds$$
$$+ \int_0^\pi G(x,t;y,0)\ |u_0(y)|\,dy$$
$$\le \int_0^1 \int_0^\pi G(x,t;y,s)\Psi(|u(y,s)|)dyds$$
$$+ \int_0^\pi G(x,t;y,0)\ |u_0(y)|\,dy$$
$$\le \int_0^1 \int_0^\pi G(x,t;y,s)\Psi(\|u\|_\infty)dyds$$
$$+\pi \|G\|_\infty \|u_0\|_\infty$$

Thus,

$$\|v\|_\infty \le \pi \|G\|_\infty [\Psi(r) + \|u_0\|_\infty].$$

·Next, we show that $\mathcal{L}_2$ maps bounded sets into equicontinuous subsets of X.

Let (x,t) and $(\xi,\tau) \in D$. For each $v \in \mathcal{L}_2 u$ there is $w \in S_{F,u}$ such that

$$|w(y,s)| \le \Psi(|u(y,s)|).$$

Thus,

$$|v(x,t) - v(\xi,\tau)| \le$$
$$\int_0^1 \int_0^\pi |G(x,t;y,s) - G(\xi,\tau;y,s)|\,|w(y,s)|\,dyds$$
$$\le \Psi(r) \int_0^1 \int_0^\pi |G(x,t;y,s) - G(\xi,\tau;y,s)|\,dyds.$$

The continuity of the Green's function implies that the right hand side of the above inequality tends to zero as $|x-\xi| + |t-\tau|$ tends to zero. By the Ascoli-Arzela theorem, we conclude that the operator $\mathcal{L}_2$ is compact.

·$\mathcal{L}_2$ has a closed graph. Let $(u_n, v_n) \in Gr(\mathcal{L}_2)$ converge to (u,v). We must show that $v \in \mathcal{L}_2 u$. We have $v_n \in \mathcal{L}_2 u_n$, and there exists $w_n \in S_{F,u_n}$ such that for each $(x,t) \in D$

$$v_n(x,t) = \int_0^t \int_0^\pi G(x,t;y,s) w_n(y,s)\,dyds$$
$$+ \int_0^\pi G(x,t;y,0)\, u_0(y)\,dy.$$

Obviously,

$$\|v_n - v\|_\infty \to 0 \text{ as } n \to \infty.$$

Consider the continuous operator $\Gamma : L^2(D) \to X$, defined by

$$(\Gamma w)(x,t) = \int_0^t \int_0^\pi G(x,t;y,s) w(y,s)\,dyds$$
$$+ \int_0^\pi G(x,t;y,0)\, u_0(y)\,dy.$$

Then $\Gamma \circ S_F$ has a closed graph (see [16, Theorem 2]). Also,

$$v_n \in \Gamma \circ S_{F,u_n}.$$

Since $u_n \to u$, uniformly, it follows that

$$v \in \Gamma \circ S_{F,u}.$$

Hence, there exists $w \in S_{F,u}$ such that

$$v(x,t) = \int_0^t \int_0^\pi G(x,t;y,s) w(y,s)\,dyds$$
$$+ \int_0^\pi G(x,t;y,0)\, u_0(y).$$

This shows that $v \in \mathcal{L}_2 u$, and hence $\mathcal{L}_2$ has a closed graph.

·Since $\mathcal{L}_2$ has compact values, it follows that $\mathcal{L}_2$ is usc.□

Claim 3. The second alternative in Theorem 7 does not hold.

Suppose, on the contrary, that there exists $u \in X$ with $\|u\|_\infty = r$ and $\lambda > 1$ such that $\lambda u \in \mathcal{L}_1 u + \mathcal{L}_2 u$. There exists $z \in S_{F,u}$ such that

$$\lambda u(x,t) = \int_0^t \int_0^\pi G(x,t;y,s)z\,(y,s)\,dyds$$

$$+\int_0^\pi G(x,t;y,0)\,u_0\,(y)\,dy$$

$$+\int_0^t \tfrac{\partial G}{\partial y}\,(x,t;0,s)\ \int_0^\pi g(u(y,s))dyds$$

$$-\int_0^t \tfrac{\partial G}{\partial y}\,(x,t;\pi,s)\ \int_0^\pi h(u(y,s))dyds.$$

Then

$$\begin{aligned}
&|u(x,t)| \\
\leq\ & \int_0^1 \int_0^\pi G(x,t;y,s)\,|z\,(y,s)|\,dyds \\
&+\int_0^\pi G(x,t;y,0)\ |u_0\,(y)|\,dy \\
&+\int_0^t \left|\frac{\partial G}{\partial y}\,(x,t;0,s)\right| \int_0^\pi |g(u(y,s))|\,dyds \\
&+\int_0^t \left|\frac{\partial G}{\partial y}\,(x,t;\pi,s)\right| \int_0^\pi |h(u(y,s))|\,dyds.
\end{aligned}$$

Hence by (H5)

$$\begin{aligned}
&|u(x,t)| \\
\leq\ & \int_0^1 \int_0^\pi G(x,t;y,s)\,|\Psi(u\,(y,s))|\,dyds \\
&+\pi\,\|G\|_\infty\,\|u_0\|_\infty \\
&+k_g\pi \int_0^1 \left|\frac{\partial G}{\partial y}\,(x,t;0,s)\right| ds\,\|u\|_\infty \\
&+\pi k_h \int_0^1 \left|\frac{\partial G}{\partial y}\,(x,t;\pi,s)\right| ds\,\|u\|_\infty\,.
\end{aligned}$$

This last inequality implies that

$$\begin{aligned}
&|u(x,t)| \\
\leq\ & \pi\Delta\,\|u\|_\infty + \pi\,\|G\|_\infty\,\|u_0\|_\infty \\
&+\int_0^1 \int_0^\pi G(x,t;y,s)\,|\Psi(\|u\|_\infty)|\,dyds \\
\leq\ & \pi\Delta r + \pi\,\|G\|_\infty\,\|u_0\|_\infty + \pi\,\|G\|_\infty\,\Psi\,(r)\,.
\end{aligned}$$

This last inequality infer that

$$r \leq \pi\Delta r + \pi\,\|G\|_\infty\,\|u_0\|_\infty + \pi\,\|G\|_\infty\,\Psi(r),$$

which, in turn, implies that

$$r(1-\pi\Delta) \leq \pi(\|G\|_\infty\,\|u_0\|_\infty + \|G\|_\infty\,\Psi(r)).$$

This contradicts the definition of r (see(14)).
Therefore the first alternative holds, which means that $u \in \mathcal{L}_1 u + \mathcal{L}_2 u$ has a solution in $\overline{B_r(0)}$. This proves that our problem has at least one solution.□

Theorem 13 *Suppose that (H1) holds and F is an L^2-Carathéodory multifunction satisfying (H0) and*

(H7) $|F(x,t,u| \leq H(x,t,|u|)$ for a.e $(x,t) \in D$, all $u \in \mathbb{R}$, where $H : D \times [0,\infty) \to [0,\infty)$ is an L^2-Carathéodory function, nondecreasing with respect to its third argument and such that

$$\limsup_{\varrho\to\infty} \frac{\|G\|_\infty}{\varrho} \int_0^T \int_\Omega H(x,t,\varrho)dxdt < 1.$$

(H8) g, h are continuous, nondecreasing and

$$\limsup_{u\to\infty} \frac{g(u)}{u} = 0 = \limsup_{u\to\infty} \frac{h(u)}{u}.$$

Then problem (1), (2), (3), (4) has at least one solution.

Proof. For $\lambda \in [0,1]$, consider the following one-parameter family of problems

$$u_t - u_{xx} \in \lambda F(x,t,u),\ 0<x<\pi,\ 0<t<1,$$

$$\begin{aligned}
&u(x,0) = \lambda u_0(x) \qquad 0 \leq x \leq \pi, \\
&u(0,t) = \lambda \int_0^\pi g(u(x,t))dx,\ 0<t<1,
\end{aligned}$$

$$u(\pi,t) = \lambda \int_0^\pi h(u(x,t))dx,\ 0<t<1.$$

Notice that this problem has only the trivial solution for $\lambda = 0$, while its solutions of are fixed points of the multivalued operator $\mathcal{L}_\lambda := \lambda G\mathcal{F} + \gamma$, where $\mathcal{L}_1 = \mathcal{L}$ is given by (13).
We have

$$u(x,t) \in \lambda \int_0^t \int_0^\pi G(x,t;y,s)F(y,s,u\,(y,s))dyds$$

$$+\lambda \int_0^\pi G(x,t;y,0)\,u_0\,(y)\,dy$$

$$+\lambda \int_0^t \tfrac{\partial G}{\partial y}\,(x,t;0,s)\ \int_0^\pi g(u(y,s))dyds$$

$$-\lambda \int_0^t \tfrac{\partial G}{\partial y}\,(x,t;\pi,s)\ \int_0^\pi h(u(y,s))dyds,$$

so that $|u(x,t)|$

$$\begin{aligned}
&\leq \int_0^t \int_0^\pi G(x,t;y,s)H(y,s,|u\,(y,s)|)dyds \\
&+\int_0^\pi G(x,t;y,0)\,|\,u_0\,(y)|\,dy \\
&+\int_0^t \left|\tfrac{\partial G}{\partial y}\,(x,t;0,s)\right|\ \int_0^\pi |g(u(y,s))|\,dyds \\
&+\int_0^t \left|\tfrac{\partial G}{\partial y}\,(x,t;\pi,s)\right|\ \int_0^\pi |h(u(y,s))|\,dyds.
\end{aligned}$$

Then

$$\begin{aligned}
&R_0 := \|u\|_\infty \leq \\
&\|G\|_\infty \int_0^T \int_0^\pi H(y,s,R_0)dyds + \pi\,\|G\|_\infty\,\|u_0\|_\infty \\
&\qquad +\delta_0\pi\,(g(R_0)+h(R_0)).
\end{aligned}$$

Thus

$$\left\{\begin{array}{l} 1 \leq \frac{\|G\|_\infty}{R_0}\int_0^T\int_0^\pi H(y,s,R_0)dyds \\ \\ +\frac{\pi\|G\|_\infty\|u_0\|_\infty}{R_0}+\frac{\delta_0\pi\left(g(R_0)+h(R_0)\right)}{R_0}\end{array}\right.$$

On the other hand, it follows from the conditions on the functions H, g, h that there exists $R^* > 0$ such that for all $\varrho > R^*$ we have

$$\frac{\|G\|_\infty}{\varrho}\int_0^T\int_0^\pi H(y,s,\varrho)dyds+\frac{\pi\|G\|_\infty\|u_0\|_\infty}{\varrho}$$

$$+\frac{\delta_0\pi\left(g(\varrho)+h(\varrho)\right)}{\varrho} < 1.$$

Comparing the last two inequalities we see that $R_0 \leq R^*$.

Hence, all possible solutions of $(17.\lambda)$ are a priori bounded, independently of λ.

Let $U := \{u \in X; \|u\|_\infty < R^* + 1\}$. Then U is open in X with $0 \in U$.

Assume that there exists $z \in \partial U$ such that $z \in \mathcal{L}_\lambda z$ for some $\lambda \in (0,1)$. This implies that z is a solution of $(17.\lambda)$ with $\|z\|_\infty = R^* + 1$, which is not possible. This implies that the first alternative in Theorem 8 holds. Consequently, $\mathcal{L}_1 = \mathcal{L}$ has a fixed point z_0, which is a solution of the above family of problems for $\lambda = 1$, which is exactly our original problem. This completes the proof.

V. LOWER AND UPPER SOLUTIONS

In this section we study a general case of problem (1), (2), (3), (4) by the method of lower and upper solutions. More specifically, we shall consider the case where the multifunction $F(x,t,u)$ has the form $[\varphi(x,t,u), \psi(x,t,u)]$, where $\varphi, \psi : D \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions

(j) $\varphi(.,.,u), \psi(.,.,u) : D \to \mathbb{R}$ are measurable,
(jj) $\varphi(x,t,.) : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous,
(jjj) $\psi(x,t,.) : \mathbb{R} \to \mathbb{R}$ is upper semicontinuous,
(jv) $\varphi(x,t,u) \leq \psi(x,t,u)$

Then $F = [\varphi, \psi]$ is the general upper semicontinuous multifunction with compact, convex values (see [9, page 5]).

We will refer to the original problem, in this case, by problem (P).

Definition 14 *A solution of our problem is a function* $u \in C(D)$ *such that the exists* $f \in L^2(D)$, $\varphi(x,t,u) \leq f(x,t) \leq \psi(x,t,u)$ *and* u *satisfies* $u_t - u_{xx} = f$ *for a.e.* (x,t), *(2), (3), (4).*

Definition 15 $\theta \in C(D)$ *is a lower solution of* (P) *if it satisfies*

(l.1) $\theta_t - \theta_{xx} \leq \varphi(x,t,\theta)$
(l.2) $\theta(x,0) \leq u_0(x)$
(l.3) $\theta(0,t) \leq \int_0^\pi g(\theta(x,t))\,dx$
(l.4) $\theta(\pi,t) \leq \int_0^\pi h(\theta(x,t))\,dx$

Definition 16 $\Theta \in C(D)$ *is an upper solution of* (P) *if the above inequalities are reversed when we substitute* Θ *for* θ.

Definition 17 *Let* $\theta \leq \Theta$ *be as above. Then* $[\theta, \Theta]$ *denotes the set of all* $u \in C(D)$ *such that* $\theta(x,t) \leq u(x,t) \leq \Theta(x,t)$ *for all* $(x,t) \in D$.

Theorem 18 *Assume that*

(1) there exists $\beta \in C(D; \mathbb{R}_+)$ such that $\max(|\varphi(x,t,u)|, |\psi(x,t,u)|) \leq \beta(x,t)$

(2) (P) has a lower solution θ and an upper solution Θ such that $\theta \leq \Theta$,

(3) the functions g and h are continuous and bounded.

Then (P) has at least one solution $u \in [\theta, \Theta]$.

Proof. Define a truncation operator $T : C(D) \to [\theta, \Theta]$ by $T(u) = \max\{\theta, \min(u, \Theta)\}$. Then, it can be shown that T is continuous and bounded. Consider the modified problem

$u_t - u_{xx} \in F(x,t,T(u)),\ (x,t) \in D$
$u(x,0) = u_0(x)$
$u(0,t) = \int_0^\pi g\left(T(u(x,t))\right)dx$
$u(\pi,t) = \int_0^\pi h\left(T(u(x,t))\right)dx.$

Notice that that the multifunction $F_1 : D \times \mathbb{R} \to 2^{\mathbb{R}}$, given by $F_1(x,t,u) = F(x,t,T(u))$ is nonempty L^2-Carathéodory multifunction with compact and convex values, and bounded. So, we can apply Theorem 8 to obtain a solution u of (P).

We show that $u \geq \theta$.

Suppose on the contrary that the set $\omega := \{(x,t) \in D;\ u(x,t) < \theta(x,t)\}$ has positive measure. Then for all $(x,t) \in \omega$ we have $T(u(x,t)) = \theta(x,t)$.

Hence $F_1(x,t,u(x,t)) = [\varphi(x,t,\theta(x,t)), \psi(x,t,\theta(x,t))]$.

Let $w(x,t) = u(x,t)$-$\theta(x,t)$. Then (recall that θ is a lower solution),

$\cdot$ for all $(x,t) \in \omega$ we have $w(x,t) < 0$

$\cdot w_t - w_{xx} = (u_t - u_{xx}) - (\theta_t - \theta_{xx}) \geq$

$\varphi(x,t,\theta(x,t)) - \varphi(x,t,\theta(x,t)) = 0$

$\cdot w(x,0) \geq u_0(x) - u_0(x) = 0$

$\cdot w(0,t) \geq \int_0^\pi g\left(T(u(x,t))\right)dx - \int_0^\pi g(\theta(x,t))dx =$

$\int_0^\pi g(\theta(x,t))dx - \int_0^\pi g(\theta(x,t))dx = 0$

$$\cdot w(\pi,t) \geq \int_0^\pi h\left(T(u(x,t))\right) dx - \int_0^\pi h\left(\theta(x,t)\right) dx =$$

$$\int_0^\pi h\left(\theta(x,t)\right) dx - \int_0^\pi h\left(\theta(x,t)\right) dx = 0.$$

The maximum principle (see [13], [21]) implies that $w\left(x,t\right) \geq 0$ for all $(x,t) \in \omega$.

This is a contradiction. Hence the set ω has measure zero, and so $u\left(x,t\right) \geq \theta\left(x,t\right)$ for all $(x,t) \in D$.

Similarly, we can show that $u\left(x,t\right) \leq \Theta\left(x,t\right)$ for all $(x,t) \in D$.

Thus $T(u(x,t)) = u(x,t)$. We infer that $F_1(x,t,u\left(x,t\right)) = F(x,t,u\left(x,t\right))$.

Therefore problem (P) has a solution u in the order interval $[\theta,\Theta]$.□

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