A Lie Symmetry analysis of the Prieto-Langraica model for bacteria activity on a surface of a medical implant

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***Abstract*—** The traditional Lie symmetry analysis to equations usually does not deliver. One common reason is the rarity of symmetries. Where they do exist, the results are often unintegrable. In this paper, we propose an infinitesimal parameter ω to Lie's definition of point transformations, allowing for symmetries to exist where none could. The analysis leads to expressions that can be evaluated through basic limit and continuity principles. This we apply to the model developed by Alicia Prieto-Langraica et. al., that describes the interaction of blood cells and bacteria on a surface of a medical implant.

***Keywords*—** Medical implants, Partial differential equations, Symmetry analysis, Variation of parameters.

I. Introduction

THIS contribution hinges on the work of Alicia Prieto-Langarica, Hristo V. Kojouharov, Liping Tang, and Benito M. Chen-Charpentier: [1] and [2]. Theirs were on models and interpretations on the five known hepatitis viruses. that is, A, B, C, D, and E, or HAV, HBV, HCV, HDV and HEV. The HDV can only propagate in the presence of HBV, hence the need to study both. We seek solutions to the modeling in particular to the system

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} - \frac{\partial}{\partial x}(V_0 U) + (kV + a)U, \qquad (1)$$

$$\frac{\partial V}{\partial t} = D_\omega \frac{\partial^2 V}{\partial x^2} + (r - k_e U)V, \qquad (2)$$

where U is the number of neutrophil cells, V is the number of epidermidis cells. They depend on time t and position x. The quantities V_0 and D are the advection and diffusion coefficients, respectively, in the neutrophils equation, k is the rate at which neutrophil cells call on other neutrophil cells depending on the presence of epdidermidis while a is the rate at which neutrophil cells call on other neutrophil cells independently of the presence of epidermidis, D_ω is the diffusion coefficient in the epidermidis equation, r is the growth rate of epidermidis, and k_e is the rate at which neutrophil cells kill epidermidis cells.

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The method we use is based on Sophus Lie's symmetry group theoretical methods first introduced through his now famous article [3]. The pure Lie approach can run into unforeseeable obstacles, such as the necessary symmetries necessary for finding the solutions not possible to find, or the analysis becoming too complicated to find them. Alternatively, the symmetries may exist, but leading to expression not integrable, forcing the analyst to incorporate numerical techniques, which does not seem right. We introduce here what we call modified symmetries which allows to avoid all the mentioned difficulties.

We intend solving the model using Lie's symmetry group theoretical methods, a technique first introduced by Marius Sophus Lie (1842 – 1899). That is, a slightly modified version thereof. The pure Lie approach tend to run into difficulties. In most studies, the symmetry groups never materials, thus rendering the whole exercise futile. Where they exist other difficulties are encountered. For example, the analyses lead to integrals that cannot be evaluated. Some practitioners tend to avoid these situations by modifying the models parameters. Unfortunately such acts have adverse effects on applications. Literature is abound with evidence, and this extends to different fields of applications. For a basic idea on how the theory proceeds, and some of the current developments, one is referred to Kallianpur and Karandikar [4], Kwok [5], Hui [6], Longstaff [7], Platen [8], Naicker, Andriopoulos and Leach [9], Pooe, Mahomed and Soh [10], Sinkala, Leach and O' Hara [11], Gazizov and Ibragimov [12]. We believe we may have found a remedy. This we discuss in the next section.

Section II is on the foundation of Lie' symmetry group theoretical approach to differential equations, including our suggestions on modified symmetries as an improvement. The first subsection, Subsection II-A, is on the contemporary form of the theory. Our suggestions follow in the next subsection, Subsection II-B. We then complete the section by providing a formula that will make it easy the proposed symmetries. This we do in Subsection II-C.

In Section III we apply Lie's approach to the model. It is basically to show that Lie's theory when applied in its purest form, frequently runs into difficulties.

In Section IV we discuss the application of the ideas of this papers, the modified symmetries with variation of parameters, to the model equations. Basically an analysis. This is extended to the solutions that arise from the analysis.

Section V is displays intrinsic details and manipulation of our version of variation of parameters through the model. A special case of a theorem introduced in Section IV is applied. The plots are eventually realised.

II. The Theoretical Basis

Smart symmetries, or modified one-parameter local point symmetries in this case, or simply modified symmetries for short, is a new concept that we are introducing, and want others to try. It is for this reason that we see a need for more depth and details. We first present the traditional approach.

A. Traditional symmetries

By Traditional symmetries here we are referring to local one-parameter point transformations, and not all symmetries in general. A broader discussion would take a lot of space. In here, we dwell on symmetries that apply to second order ordinary differential equations.

To begin, we first define a group.

Definition 1: A group G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:

(i) *Closure*. For $\{G_1, G_2\} \subset G$, we have $\phi(G_1, G_2) \in G$.
(ii) *Associativity*. For $\{G_1, G_2, G_2\} \subset G$, we have $\phi(G_1, \phi(G_2, G_3)) = \phi(\phi(G_1, G_2), G_3) \in G$.
(iii) *Identity*. There exists $G_0 \in G$, such that $\phi(G_0, G_i) = \phi(G_i, G_0) = G_i$, for every element G_i in G. The element G_0 is called the identity element of G.
(iv) *Inverse*. There exists $G_i^{-1} \in G$ for every $G_i \in G$, such that $\phi(G_i^{-1}, G_i) = \phi(G_i, G_i^{-1}) = G_0 \in G$. The element G_i^{-1} is called the inverse of G_i.

That done, we next turn to group of transformations.

Definition 2: Let

$$\bar{\mathbf{x}} = \psi(\mathbf{x}; \epsilon) \tag{3}$$

be a family of invertible transformations, of points $\mathbf{x} = (x^1, \cdots, x^N) \in D \subset \mathbb{R}^N$ into points $\bar{\mathbf{x}} = (\bar{x}^1, \cdots, \bar{x}^N) \in R \subset \mathbb{R}^{\mathbf{N}}$, with the parameter $\epsilon \in S \subset \mathbb{R}$. These are called one-parameter group of point transformations if the following hold.

(i) For each $\epsilon \in S$, we have the transformations being one-to-one and onto D, meaning D is not different from R, as x^N is not different from $\bar{x}^N$.
(ii) The set S is a group, say G, with $\phi(\epsilon, \delta)$ defining the composition law.
(iii) The case $\bar{\mathbf{x}} = \mathbf{x}$ corresponds to $\epsilon = \epsilon_0$: The identity element of G. That is,

$$\bar{\mathbf{x}}|_{\epsilon=\epsilon_0} = \mathbf{x}. \tag{4}$$

or,

$$\psi(\mathbf{x}; \epsilon)\Big|_{\epsilon=\epsilon_0} = \mathbf{x}. \tag{5}$$

(iv) If $\bar{\mathbf{x}} = \psi(\mathbf{x}; \epsilon)$ and $\bar{\bar{\mathbf{x}}} = \psi(\bar{\mathbf{x}}; \delta)$, then

$$\bar{\bar{\mathbf{x}}} = \psi(\mathbf{x}; \phi(\epsilon, \delta)).$$

Theorem 1: **Lie's First Fundamental Theorem:** There exists a parametrization $\tau(\epsilon)$ such that the Lie group of transformations is equivalent to the solution of an initial value problem for a system of first-order ODEs given by

$$\frac{d\bar{x}}{d\tau} = \xi(\bar{x}), \tag{6}$$

with

$$\frac{d\bar{x}}{d\tau}\Big|_{\tau=0} = x. \tag{7}$$

1) Local one-parameter point transformation groups : The transformation can be expanded using the Taylor-Maclaurin series expansion with respect to the parameter. That is,

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon\left(\frac{\partial \mathbf{G}}{\partial \epsilon}\Big|_{\epsilon=0}\right) + \frac{\epsilon^2}{2}\left(\frac{\partial^2 \mathbf{G}}{\partial \epsilon^2}\Big|_{\epsilon=0}\right) + \cdots = \mathbf{x} + \epsilon\left(\frac{\partial \mathbf{G}}{\partial \epsilon}\Big|_{\epsilon=0}\right) + O(\epsilon^2). \tag{8}$$

Letting

$$\xi(\mathbf{x}) = \frac{\partial \mathbf{G}}{\partial \epsilon}\Big|_{\epsilon=0}, \tag{9}$$

reduces the expansion to

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon\xi(\mathbf{x}) + O(\epsilon^2). \tag{10}$$

Definition 3: The expression

$$\bar{\mathbf{x}} = \mathbf{x} + \epsilon\xi(\mathbf{x}), \tag{11}$$

is called a local one-parameter point transformation.

The set G is a group since the following properties hold under binary operation +:

1) **Closure**. If $\bar{\mathbf{x}}_{\epsilon_1}, \bar{\mathbf{x}}_{\epsilon_2} \in G$ and $\epsilon_1, \epsilon_2 \in \mathbb{R}$, then

$$\bar{\mathbf{x}}_{\epsilon_1} + \bar{\mathbf{x}}_{\epsilon_2} = (\epsilon_1 + \epsilon_2)\xi(\mathbf{x}) = \bar{\mathbf{x}}_{\epsilon_3} \in G, \tag{12}$$

and

$$\epsilon_3 = \epsilon_1 + \epsilon_2 \in \mathbb{R}. \tag{13}$$

2) **Identity**. If $\bar{\mathbf{x}}_0 \equiv I \in G$ such that for any $\epsilon \in \mathbb{R}$

$$\bar{\mathbf{x}}_0 + \bar{\mathbf{x}}_\epsilon = \bar{\mathbf{x}}_\epsilon = \bar{\mathbf{x}}_\epsilon + \bar{\mathbf{x}}_0, \tag{14}$$

then $\bar{\mathbf{x}}_0$ is an identity in G.

3) **Inverses**. For $\bar{\mathbf{x}}_\epsilon \in G$, $\epsilon \in \mathbb{R}$, there exists $\bar{\mathbf{x}}_\epsilon^{-1} \in G$, such that

$$\bar{\mathbf{x}}_\epsilon^{-1} + \bar{\mathbf{x}}_\epsilon = \bar{\mathbf{x}}_\epsilon + \bar{\mathbf{x}}_\epsilon^{-1}, \quad \bar{\mathbf{x}}_\epsilon^{-1} = \bar{\mathbf{x}}_{\epsilon^{-1}}, \tag{15}$$

and $\epsilon^{-1} = -\epsilon \in D$, where + is a binary composition of transformations and it is understood that $\bar{\mathbf{x}}_\epsilon = \bar{\mathbf{x}}_\epsilon - \mathbf{x}$. Associativity follows from the closure property.

2) The Lie operator : For the multivariate function $\psi = \psi(x^1, \cdots, x^N; \epsilon)$, the expression (11) can be rewritten in the form

$$\bar{\mathbf{x}} = x + \epsilon\xi \frac{\partial x}{\partial x}, \tag{16}$$

or

$$\bar{x}^i = \left(1 + \epsilon\xi^i \frac{\partial}{\partial x^i}\right) x^i. \tag{17}$$

That is,

$$\bar{x}^i = (1 + \epsilon\xi \cdot \nabla)\, x^i, \tag{18}$$

where $\xi = (\xi^1, \cdots, \xi^N)$. That is,

$$\bar{x}^i = (1 + \epsilon X)\, x^i, \tag{19}$$

with

$$X = \sum_{i=1}^{N} \xi^i(x^1, \cdots, x^N) \frac{\partial}{\partial x^i}. \tag{20}$$

This operator is the symmetry generator.

3) Prolongations formulas : The operator X is not adequate generating symmetries for differential equations, where it applies. This, however, can be remedied through prolongations.

The case $N = 2$ has $x^1 = x$ and $x^2 = y$ so that

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{21}$$

In determining the prolongations, it is convenient to use the operator of total differentiation

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots, \tag{22}$$

where

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}, \quad \cdots. \tag{23}$$

The derivatives of the transformed point is then

$$\bar{y}' = \frac{d\bar{y}}{d\bar{x}}. \tag{24}$$

Since

$$\bar{x} = x + \epsilon\xi \quad \text{and} \quad \bar{y} = y + \epsilon\eta, \tag{25}$$

then

$$\bar{y}' = \frac{dy + \epsilon d\eta}{dx + \epsilon d\xi}. \tag{26}$$

That is,

$$\bar{y}' = \frac{dy/dx + \epsilon d\eta/dx}{dx/dx + \epsilon d\xi/dx}. \tag{27}$$

Now introducing the operator D:

$$\bar{y}' = \frac{y' + \epsilon D(\eta)}{1 + \epsilon D(\xi)} = \frac{(y' + \epsilon D(\eta))(1 - \epsilon D(\xi))}{1 - \epsilon^2 (D(\xi))^2}. \tag{28}$$

Hence

$$\bar{y}' = \frac{y' + \epsilon(D(\eta) - y'D(\xi)) - \epsilon^2 D(\xi)D(\eta)}{1 - \epsilon^2(D(\xi))^2}. \tag{29}$$

That is,

$$\bar{y}' = y' + \epsilon(D(\eta) - y'D(\xi)), \tag{30}$$

or

$$\bar{y}' = y' + \epsilon\zeta^1, \tag{31}$$

with

$$\zeta^1 = D(\eta) - y'D(\xi). \tag{32}$$

It expands into

$$\zeta^1 = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)\eta - y'\left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)\xi, \tag{33}$$

so that

$$\zeta^1 = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y. \tag{34}$$

The first prolongation of X is then

$$X^{[1]} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \zeta^1\frac{\partial}{\partial y'}. \tag{35}$$

For the second prolongation, we have

$$\bar{y}'' = \frac{y'' + \epsilon D(\zeta^1)}{1 + \epsilon D(\xi)} \approx y'' + \epsilon\zeta^2, \tag{36}$$

with

$$\zeta^2 = D(\zeta^1) - y''D(\xi). \tag{37}$$

This expands into

$$\begin{aligned}\zeta^2 \;&=\; \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 \\ &-\; y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)y''.\end{aligned} \tag{38}$$

The second prolongation of X is then

$$X^{[2]} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \zeta^1\frac{\partial}{\partial y'} + \zeta^2\frac{\partial}{\partial y''}. \tag{39}$$

4) Invariance :

Theorem 2:

A function $F(x, y)$ is an invariant of the group of transformations if for each point (x, y) it is constant along the trajectory determined by the totality of transformed points $(\bar{x}, \bar{y})$:

$$F(\bar{x}, \bar{y}) = F(x, y). \tag{40}$$

This requires that

$$XF = 0, \tag{41}$$

leading to the characteristic system

$$\frac{dx}{\xi} = \frac{dy}{\eta}. \tag{42}$$

Proof. Consider the Taylor series expansion of $F(\bar{\mathbf{x}})$ with respect to ϵ:

$$F(\bar{x},\bar{y}) = F(\bar{x},\bar{y})\Big|_{\epsilon=0} + \epsilon\frac{\partial \bar{F}}{\partial \epsilon}\Big|_{\epsilon=0} + \cdots. \quad (43)$$

This can be written in the form

$$F(\bar{x},\bar{y}) = F(\bar{x},\bar{y})\Big|_{\epsilon=0} + \epsilon\left(\frac{\partial \bar{x}}{\partial \epsilon}\frac{\partial \bar{F}}{\partial \bar{x}} + \frac{\partial \bar{y}}{\partial \epsilon}\frac{\partial \bar{F}}{\partial \bar{y}}\right)\Big|_{\epsilon=0} + \cdots. \quad (44)$$

That is,

$$F(\bar{x},\bar{y}) = F(x,y) + \epsilon\left(\xi\frac{\partial \bar{F}}{\partial \bar{x}} + \eta\frac{\partial \bar{F}}{\partial \bar{y}}\right)\Big|_{\epsilon=0} + \cdots, \quad (45)$$

or

$$F(\bar{x},\bar{y}) = F(x,y) + \epsilon\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)\bar{F} + \cdots. \quad (46)$$

Hence

$$F(\bar{x},\bar{y}) = F(x,y) + \epsilon X\bar{F}, \quad (47)$$

with

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}. \quad (48)$$

This means if $X\bar{F} = 0$ we have

$$F(\bar{x},\bar{y}) = F(x,y), \quad (49)$$

which concludes the theorem.

B. Modified Symmetries

The modified one-parameter point symmetries and their properties reduce to the regular one-parameter point symmetries when $\omega \to 0$. This is an infinitesimal parameter that we shall introduce and associate with them.

1) One-Parameter Point Transformations: We build our discussion on smart symmetries from the following definition on . There could be could be some confusion because at some instances they seem to resemble , at other cases the one-parameter transformation view emerges. They also seem to be wedged between the two.

Definition 4: Let

$$\breve{\mathbf{x}} = \chi(\tilde{\mathbf{x}};\delta;\epsilon) \quad (50)$$

be a family of two-parameters $\{\epsilon,\delta\} \subset \mathbb{R}$ invertible transformations, of points $\tilde{\mathbf{x}} = (\tilde{x}^1(x;\delta;\epsilon),\cdots,\tilde{x}^N(x;\delta;\epsilon)) \in \mathbf{R}^N$ into points $\breve{\mathbf{x}} = (\breve{x}^1,\cdots,\breve{x}^N) \in \mathbf{R^N}$. These we call Neo one-parameter point transformations when subjected to the conditions

$$\chi(\tilde{\mathbf{x}};\delta;\epsilon)\Big|_{\epsilon=0} = \tilde{\mathbf{x}}, \quad (51)$$

and

$$\tilde{\mathbf{x}}|_{\delta=0} = \mathbf{x}. \quad (52)$$

Furthermore,

$$\chi(\tilde{\mathbf{x}};\delta;\epsilon)\Big|_{\delta=0} = \bar{\mathbf{x}}, \quad (53)$$

so that

$$\chi(\tilde{\mathbf{x}};\delta;\epsilon)\Big|_{\delta=0,\epsilon=0} = \mathbf{x}, \quad (54)$$

for $\bar{\mathbf{x}} = (\bar{x}^1,\cdots,\bar{x}^N) \in \mathbb{R}^N$ and $\mathbf{x} = (x^1,\cdots,x^N) \in \mathbb{R}^{\mathbf{N}}$.

It should be obvious that these transformations are the regular two-parameter point transformations when the parameter both parameter ϵ and δ assume zero values. That is,

$$\chi(\tilde{\mathbf{x}};\delta;\epsilon)\Big|_{\epsilon=0,\delta=0} = \tilde{\mathbf{x}}, \quad (55)$$

or best expressed in the form

$$\chi(\tilde{\mathbf{x}};\delta;\epsilon)\Big|_{\epsilon=0,\delta=0} = \tilde{\mathbf{x}}. \quad (56)$$

They reduce to the one-parameter point transformations when the parameter δ is absent from the definition.

2) Modified local one-parameter group generators: In $\mathbb{R}^2$, we have $\chi = (\phi;\psi)$, while $\breve{\mathbf{x}} = (\breve{x},\breve{y})$ and $\tilde{\mathbf{x}}(\delta) = (\tilde{x}(\delta);\tilde{y}(\delta))$, so that

$$\breve{\tilde{x}} = \phi(\tilde{x}(\delta),\tilde{y}(\delta),\epsilon) \quad (57)$$

and

$$\breve{y} = \psi(\tilde{x}(\delta),\tilde{y}(\delta),\epsilon) \quad (58)$$

Expanding (57) and (58) about $\epsilon = 0$, in some neighborhood of $\epsilon = 0$, gives

$$\breve{\tilde{x}} = \tilde{x}(\delta) + \epsilon\frac{\partial \tilde{G}}{\partial \epsilon}\Big|_{\epsilon=0} + O(\epsilon^2). \quad (59)$$

That is,

$$\breve{\tilde{x}} = x + \delta\frac{\partial H}{\partial \epsilon}\Big|_{\delta=0} + \epsilon\left(\frac{\partial G}{\partial \epsilon}\Big|_{\delta=0,\epsilon=0} + \delta\frac{\partial^2 G}{\partial \epsilon\partial \delta}\Big|_{\delta=0,\epsilon=0}\right). \quad (60)$$

This becomes

$$\breve{x} = x + \epsilon\frac{\partial G}{\partial \epsilon}\Big|_{\delta=0,\epsilon=0} + \delta\frac{\partial H}{\partial \epsilon}\Big|_{\delta=0}. \quad (61)$$

Letting

$$\xi = \frac{\partial G}{\partial \epsilon}\Big|_{\delta=0,\epsilon=0} \quad (62)$$

and

$$\tilde{\xi} = \frac{\partial H}{\partial \epsilon}\Big|_{\delta=0}, \quad (63)$$

gives the modified local one-parameter point transformation

$$\breve{\tilde{x}} = x + \epsilon\xi + \delta\tilde{\xi}, \quad (64)$$

leading to the symmetry generator

$$\tilde{X} = \sum_{i=1}^{N}\left(\xi^i + \frac{\delta}{\epsilon}\tilde{\xi}^i\right)\frac{\partial}{\partial x^i}, \quad (65)$$

It reduces to the regular generator (20) when $\delta = 0$. In the case where the ratio δ/ϵ assumes a finite complex value, as with $\delta = i\epsilon\omega$ with $\omega \in \mathbb{R}$ being the finite value, then the operator is simply the complex symmetry generator,

$$\tilde{X} = \sum_{i=1}^{N} \left[\xi^i(x^1, \cdots, x^N) + i\omega\tilde{\xi}^i(x^1, \cdots, x^N; \omega)\right] \frac{\partial}{\partial x^i}, \tag{66}$$

otherwise it collapses into the regular symmetry generator.

3) Symmetry groups : An interesting property of symmetries $A = \{\tilde{X}_1, \tilde{X}_2, \cdots, \tilde{X}_n\}$ is that they also form a group, provided $\omega \to 0$. That is, the satisfy the following group properties:

1) **Closure**. If $\tilde{X}_1, \tilde{X}_2 \in A$, then
$$\tilde{X}_1 \circ \tilde{X}_2 \;=\; \tilde{X}_3 \in A.$$
2) **Identity**. If $\tilde{X}_0 \equiv I \in A$
$$\tilde{X}_0 \circ \tilde{X}_i \;=\; \tilde{X}_i \;=\; \tilde{X}_i \circ \tilde{X}_0, \quad i = 1, 2, \ldots, n$$
then $\tilde{X}_0$ is an identity in G.
3) **Inverses**. For $\tilde{X}_i \in G$, i=1,2, …n, there exists $\tilde{X}_a^{-1} \in G$, such that
$$\tilde{X}_i^{-1} \circ \tilde{X}_i \;=\; \tilde{X}_i \circ \tilde{X}_i^{-1}$$
with
$$\tilde{X}_i^{-1} \;=\; \tilde{X}_{i^{-1}}, \quad i = 1, 2, \ldots n,$$

where $\circ$ is a . follows from the .

4) Invariance:

Theorem 3:

A function $F(\tilde{\mathbf{x}})$ is an invariant of the group of transformations if for each point $\tilde{\mathbf{x}}$ it is constant along the trajectory determined by the totality of transformed points $\breve{\tilde{\mathbf{x}}}$:

$$F(\breve{\mathbf{x}}) = F(\tilde{\mathbf{x}}). \tag{67}$$

This requires that

$$GF = 0, \tag{68}$$

leading to the characteristic system

$$\frac{d\tilde{x}^1}{\xi^1} = \cdots = \frac{d\tilde{x}^N}{\xi^N}. \tag{69}$$

Proof. Consider the of $F(\breve{\tilde{\mathbf{x}}})$ with respect to ϵ:

$$F(\breve{\mathbf{x}}) = F(\tilde{\mathbf{x}})\Big|_{\epsilon=0} + \epsilon\frac{\partial\breve{\mathbf{F}}}{\partial\epsilon}\Big|_{\epsilon=0} + \cdots. \tag{70}$$

This can be written in the form

$$F(\breve{\mathbf{x}}) = F(\breve{\mathbf{x}})\Big|_{\epsilon=0} + \epsilon\frac{\partial\breve{\mathbf{x}}}{\partial\epsilon} \cdot \bigtriangledown\breve{F}\Big|_{\epsilon=0} + \cdots. \tag{71}$$

That is,

$$F(\breve{\mathbf{x}}) = F(\breve{\mathbf{x}})\Big|_{\epsilon=0} + \epsilon\xi \cdot \bigtriangledown\breve{F}\Big|_{\epsilon=0} + \cdots. \tag{72}$$

For $\epsilon = 0$ then we get

$$F(\breve{\mathbf{x}}) = F(\tilde{\mathbf{x}})), \tag{73}$$

thus proving the theorem.

5) Prolongations formulas : Since

$$\tilde{x} = x + \epsilon\xi + \delta\tilde{\xi} \;\text{ and }\; \tilde{y} = y + \epsilon\eta + \delta\tilde{\eta}, \tag{74}$$

then

$$\tilde{y}' = \frac{dy + \epsilon d\eta + \delta d\tilde{\eta}}{dx + \epsilon d\xi + \delta d\tilde{\xi}}. \tag{75}$$

That is,

$$\tilde{y}' = \frac{dy/dx + \epsilon d\eta/dx + \delta d\tilde{\eta}/dx}{dx/dx + \epsilon d\xi/dx + \delta d\tilde{\xi}/dx}. \tag{76}$$

Now introducing the operator D:

$$\tilde{y}' = \frac{y' + \epsilon D(\eta) + \delta D\tilde{\eta}}{1 + \epsilon D\xi + \delta D\tilde{\xi}}. \tag{77}$$

Normalising the denominator:

$$\tilde{y}' = \left(\frac{y' + \epsilon D(\eta) + \delta D\tilde{\eta}}{1 + \epsilon D\xi + \delta D\tilde{\xi}}\right)\left(\frac{1 - \epsilon D\xi - \delta D\tilde{\xi}}{1 - \epsilon D\xi - \delta D\tilde{\xi}}\right). \tag{78}$$

$$\begin{aligned}\tilde{y}' \quad &= \frac{y' + \epsilon[D(\eta) - y'D(\xi)] + \delta[D(\tilde{\eta}) - y'D(\tilde{\xi})]}{1 - \epsilon^2(D(\xi))^2 - \delta^2(D(\tilde{\xi}))^2 - 2\epsilon\delta(D\xi)(D\tilde{\xi})} \\ &+ \frac{-\epsilon^2 D(\xi)D(\eta) - \delta^2 D(\tilde{\xi})D(\tilde{\eta})}{1 - \epsilon^2(D(\xi))^2 - \delta^2(D(\tilde{\xi}))^2 - 2\epsilon\delta(D\xi)(D\tilde{\xi})}.\end{aligned} \tag{79}$$

$$\tilde{y}' = y' + \epsilon\left([D(\eta) - y'D(\xi)] + \omega[D(\tilde{\eta}) - y'D(\tilde{\xi})]\right). \tag{80}$$

That is,

$$\tilde{y}' = y' + \epsilon(D(\eta + \omega\tilde{\eta}) - y'D(\xi + \omega\tilde{\xi})), \tag{81}$$

or

$$\tilde{y}' = y' + \epsilon\tilde{\zeta}^1, \tag{82}$$

with

$$\tilde{\zeta}^1 = D(\eta + \omega\tilde{\eta}) - y'D(\xi + \omega\tilde{\xi}). \tag{83}$$

It expands into

$$\begin{aligned}\tilde{\zeta}^1 \quad &= \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)(\eta + \omega\tilde{\eta}) \\ &-y'\left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)(\xi + \omega\tilde{\xi}),\end{aligned} \tag{84}$$

so that

$$\tilde{\zeta}^1 = (\eta+\omega\tilde{\eta})_x + [(\eta+\omega\tilde{\eta})_y - \xi_x]\, y' - y'^2(\xi+\omega\tilde{\xi})_y. \quad (85)$$

The first prolongation of $\tilde{X}$ is then

$$\tilde{X}^{[1]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \tilde{\zeta}^1\frac{\partial}{\partial y'}. \quad (86)$$

For the second prolongation, we note that since

$$\tilde{x} = x + \epsilon\xi + \delta\tilde{\xi} \quad \text{and} \quad \tilde{y}' = y' + \epsilon\tilde{\zeta}^1, \quad (87)$$

then

$$\bar{y}'' = \frac{y'' + \epsilon D(\tilde{\zeta}^1)}{1 + \epsilon D(\xi) + \sigma D(\tilde{\xi})}, \quad (88)$$

$$\bar{y}'' = \left(\frac{y'' + \epsilon D(\tilde{\zeta}^1)}{1 + \epsilon D(\xi) + \sigma D(\tilde{\xi})}\right)\left(\frac{1 - \epsilon D\xi - \delta D\tilde{\xi}}{1 - \epsilon D\xi - \delta D\tilde{\xi}}\right) \quad (89)$$

$$\bar{y}'' = \frac{\left(y'' + \epsilon D(\tilde{\zeta}^1)\right)\left(1 - \epsilon D\xi - \delta D\tilde{\xi}\right)}{1 - \epsilon^2(D(\xi))^2 - \delta^2(D(\tilde{\xi}))^2 - 2\epsilon\delta(D\xi)(D\tilde{\xi})}. \quad (90)$$

$$\bar{y}'' = \left(y'' + \epsilon D(\tilde{\zeta}^1)\right)\left(1 - \epsilon D\xi - \delta D\tilde{\xi}\right). \quad (91)$$

$$\bar{y}'' = y'' - \epsilon\left[D(\tilde{\zeta}^1) - y'' D(\xi + \omega\tilde{\xi})\right]. \quad (92)$$

with

$$\tilde{\zeta}^2 = D(\tilde{\zeta}^1) - y'' D(\xi + \omega\tilde{\xi}). \quad (93)$$

This expands into

$$\begin{aligned}\tilde{\zeta}^2 \;=\; & [\eta+\omega\tilde{\eta}]_{xx} + \left(2[\eta+\omega\tilde{\eta}]_{xy} - [\xi+\omega\tilde{\xi}]_{xx}\right) y' \\ & + \left([\eta+\omega\tilde{\eta}]_{yy} - 2[\xi+\omega\tilde{\xi}]_{xy}\right) y'^2 \\ & + \left([\eta+\omega\tilde{\eta}]_y - 2[\xi+\omega\tilde{\xi}]_x - 3y'[\xi+\omega\tilde{\xi}]_y\right) y'' \\ & - y'^3[\xi+\omega\tilde{\xi}]_{yy}. \end{aligned} \quad (94)$$

The second prolongation of $\tilde{X}$ is then

$$\tilde{X}^{[2]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \tilde{\zeta}^1\frac{\partial}{\partial y'} + \tilde{\zeta}^2\frac{\partial}{\partial y''}. \quad (95)$$

C. A Simple Formula for Generating Modified Symmetries

The theory that we have just discussed in the preceeding section could be daunting to some. Fortunately, there is a simple procedure that can get one started. Consider the expression

$$bx + a, \quad (96)$$

that one usually encounters when investigating differential equations of the order two and above for symmetries. We will now show that it can be presented in the form

$$b\frac{\sin(i\omega[x + \frac{a}{b}])}{i\omega}, \quad (97)$$

for $\omega \to 0$. We will show later in the paper how it leads to the symmetries. In this section we concentrate on how it comes about.

1) Euler's ansatz: Leonhard Euler (1707–1783), investigated differential equations of the form

$$a_0\ddot{y} + b_0\dot{y} + c_0 y \;=\; 0, \quad (98)$$

using the ansatz

$$y = e^{\lambda x}, \quad (99)$$

for solutions. Here $y = y(x)$, with constant coefficients a_0, b_0 and c_0.

He concluded that

$$y = \begin{cases} e^{-\frac{b_0}{2a_0}x}\left(Ae^{-\tilde{\omega}x} + Be^{\tilde{\omega}x}\right), & b_0^2 > 4a_0c_0, \\ A + Bx, \quad b_0^2 = 4a_0c_0, \\ e^{-\frac{b_0}{2a_0}x}\left(A\cos(\tilde{\omega}x) + B\sin(\tilde{\omega}x)\right), b_0^2 < 4a_0c_0, \end{cases} \quad (100)$$

where

$$\tilde{\omega} = \frac{\sqrt{b_0^2 - 4a_0c_0}}{2a_0}, \quad (101)$$

and A and B are constants.

That is, Euler determined three solution components: y_1 for $b_0^2 > 4a_0c_0$, y_2 for $b_0^2 = 4a_0c_0$ and y_3 for the case $b_0^2 < 4a_0c_0$.

These work well in practise and still find applications today, but they are mathematically unsound.

My belief is that he allow inconsistency to pass on based on the success of the formulas. Unfortunately, this has an enormous amount of work hinged on the error, and in some cases subsequently leading to cul de sacs.

2) Continuity issues: It is hard to believe that the great Euler did not notice the discontinuity in solutions. That is,

$$\lim_{\tilde{\omega}\to 0}(y_1 - y_2) \neq 0. \quad (102)$$

Also,

$$\lim_{\tilde{\omega}\to 0}(y_3 - y_2) \neq 0. \quad (103)$$

Maybe he may have thought this to be an inconsequential little shortcoming, but in practise these cases are always avoided consciously avoided because of the catastrophes that have arisen around them in the past. The collapse of the Tacoma narrows bridge is one example. Mathematically a lot of good can result from solving equation (98) exactly, such as what I am on about in this work.

3) An exact solution : To get an exact formula, first let

$$y = \beta z,$$

with $\beta = \beta(x)$ and $z = z(x)$, so that

$$\dot{y} = \dot{\beta}z + \beta\dot{z},$$

and

$$\ddot{y} = \ddot{\beta}z + 2\dot{\beta}\dot{z} + \beta\ddot{z}.$$

These transform (98) into

$$a_0\left(\ddot{\beta}z+2\dot{\beta}\dot{z}+\beta\ddot{z}\right)+b_0\left(\dot{\beta}z+\beta\dot{z}\right)+c_0\beta z \;=\; 0.$$

That is,

$$a_0\beta\ddot{z}+\left(2a_0\dot{\beta}+b_0\beta\right)\dot{z}+\left(a_0\ddot{\beta}+b_0\dot{\beta}+c_0\beta\right)z \;=\; 0. \tag{104}$$

Choosing β to satisfy $2a_0\dot{\beta}+b_0\beta=0$ simplifies equation (104). That is,

$$\beta=C_{00}e^{\frac{-b_0}{2a_0}x},$$

for some constant C_{00}. Equation (104) assumes the form

$$\ddot{z} \;=\; -\frac{a_0\ddot{\beta}+b_0\dot{\beta}+c_0\beta}{a_0\beta}z.$$

That is,

$$\ddot{z} \;=\; \left(\frac{b_0^2-4a_0c_0}{4a_0^2}\right)z.$$

But $\ddot{z}$ can be written as $\dot{z}d\dot{z}/dx$. Therefore,

$$\dot{z}\frac{d\dot{z}}{dz} \;=\; \left(\frac{b_0^2-4a_0c_0}{4a_0^2}\right)z,$$

or

$$\dot{z}d\dot{z} \;=\; \left(\frac{b_0^2-4a_0c_0}{4a_0^2}\right)zdz.$$

That is,

$$\frac{\dot{z}^2}{2} \;=\; \left(\frac{b_0^2-4a_0c_0}{4a_0^2}\right)\frac{z^2}{2}+C_{01},$$

for some constant C_{01}. That is,

$$\dot{z} \;=\; \sqrt{\left(\frac{b_0^2-4a_0c_0}{4a_0^2}\right)\frac{z^2}{2}+C_{01}},$$

or

$$\frac{dz}{\sqrt{\left(\frac{b_0^2-4a_0c_0}{4a_0^2}\right)z^2+2C_{01}}}=dx.$$

That is,

$$\frac{dz}{\sqrt{A_{00}^2-z^2}}=\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}\;dx,$$

with $A_{00}^2=2C_{01}/\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}$. Hence,

$$z \;=\; \frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}} \times\sin\left(\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}\;x+C_{02}\right), \tag{105}$$

for some constant C_{02}. That is,

$$y \;=\; C_{00}e^{\frac{-b_0}{2a_0}x}\;\frac{2C_{01}}{\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}} \times\sin\left(\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}\;x+C_{02}\right). \tag{106}$$

Letting

$$\bar{\omega}=\sqrt{-\frac{b_0^2-4a_0c_0}{4a_0^2}}$$

we have
$y=C_{00}e^{\frac{-b_0}{2a_0}x}\;\frac{2C_{01}}{\bar{\omega}}\;\sin\left(\bar{\omega}\;x+C_{02}\right),$
or
$y=C_{00}e^{\frac{-b_0}{2a_0}x}\;2C_{01}\;[\frac{\sin(C_{02})}{\bar{\omega}}\cos\left(\bar{\omega}\;x\right)+\cos\left(C_{02}\right)\frac{\sin(\bar{\omega}\;x)}{\bar{\omega}}].$
A reduction to the trivial case $\ddot{y}=0$ requires that $\sin(C_{02})=C_{03}\sin(\bar{\omega})$ and $\cos(C_{02})=C_{04}\cos(\bar{\omega})$. That is, $C_{03}^2+C_{04}^2=1$. Hence,
$y = C_{00}e^{\frac{-b_0}{2a_0}x}\;2C_{01}\;[\frac{C_{03}\sin(\bar{\omega})}{\bar{\omega}}\cos\left(\bar{\omega}\;x\right)+C_{04}\cos\left(\bar{\omega}\right)\frac{\sin(\bar{\omega}\;x)}{\bar{\omega}}],$
or simply

$$y \;=\; C_{00}e^{\frac{-b_0}{2a_0}x}\;2C_{01}\;\frac{C_{03}\sin\left(\bar{\omega}\right)\cos\left(\bar{\omega}\;x\right)}{\bar{\omega}} +C_{00}e^{\frac{-b_0}{2a_0}x}\;2C_{01}\;\frac{C_{04}\sin\left(\bar{\omega}\;x\right)}{\bar{\omega}}. \tag{107}$$

It is very vital to indicate that if the parameters $\bar{\omega}$ in the denominator and $\sin\left(\bar{\omega}\right)$ are absorbed into the coefficients C_{01} and C_{03}, then formula (107) would reduce to one of Euler's formulas. But the consequences would be fatal, as formula (107) would not reduce to $y=A+Bx$ when $b_0=c_0=0$, that is, when $\bar{\omega}=0$.

4) The formula: The analysis of determining equations in symmetry analysis always involve equations of the form

$$\xi=bx+a, \tag{108}$$

similar to the second result in (100). The solution obtained in Section II-C3 suggests it can be written in the form

$$\xi=b\frac{\sin(i\omega[x+\frac{a}{b}])}{i\omega}, \tag{109}$$

subject to $\omega=0$. This formula provides an easier way of generating modified symmetries.

III. A TRADITIONAL LIE APPROACH TO THE MODEL

We seek here a continuous group of transformations for the two-dimensional equations (1) to (2) through a generator

$$Y \;=\; \xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial x}+\eta^1\frac{\partial}{\partial U}+\eta^2\frac{\partial}{\partial V}. \tag{110}$$

For the system (1) to (2), U and V are considered as differential variables on the space (t,x). The coordinates ξ^1,ξ^2,η^1 and η^2 of the operator (110) are sought as functions of t,x,U and V. The operator $\tilde{Y}$, is the prolongation of Y and is

$$\tilde{Y} \;=\; Y+\zeta_1^1\frac{\partial}{\partial U_t}+\zeta_2^1\frac{\partial}{\partial U_x}+\zeta_1^2\frac{\partial}{\partial V_t} + \zeta_{22}^1\frac{\partial}{\partial U_{xx}}+\zeta_{22}^2\frac{\partial}{\partial V_{xx}}, \tag{111}$$

where

$$\begin{aligned}
\zeta_1^1 &= D_t(\eta^1) - U_t D_t(\xi^1) - U_x D_t(\xi^2), && (112)\\
\zeta_2^1 &= D_x(\eta^1) - U_t D_x(\xi^1) - U_x D_x(\xi^2), && (113)\\
\zeta_1^2 &= D_t(\eta^2) - V_t D_t(\xi^1) - V_x D_t(\xi^2), && (114)
\end{aligned}$$

and ($\zeta_1^1 \equiv \zeta_t^1, \cdots, \zeta_4^1 \equiv \zeta_z^1$ and likewise for $\zeta_1^2 \equiv \zeta_t^2$ and so on) with the operators of total differentiation D_t, D_x, D_y and D_z given by

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + U_t\frac{\partial}{\partial U} + U_{tt}\frac{\partial}{\partial U_t} + U_{tx}\frac{\partial}{\partial U_x}\\
&+ V_t\frac{\partial}{\partial V} + V_{tt}\frac{\partial}{\partial V_t} + V_{tx}\frac{\partial}{\partial V_x}\\
&+ \cdots, && (115)\\
D_x &= \frac{\partial}{\partial x} + U_x\frac{\partial}{\partial U} + U_{xt}\frac{\partial}{\partial U_t} + U_{xx}\frac{\partial}{\partial U_x}\\
&+ V_x\frac{\partial}{\partial V} + V_{xt}\frac{\partial}{\partial V_t} + V_{xx}\frac{\partial}{\partial V_x}\\
&+ \cdots. && (116)
\end{aligned}$$

This gives

$$\begin{aligned}
\zeta_1^1 &= \eta_t^1 + U_t\eta_U^1 + U_{tt}\eta_{U_t}^1 + U_{tx}\eta_{U_x}^1\\
&+ V_t\eta_V^1 + V_{tt}\eta_{V_t}^1 + V_{tx}\eta_{V_x}^1\\
&- U_t(\xi_t^1 + U_t\xi_U^1 + U_{tt}\xi_{U_t}^1 + U_{tx}\xi_{U_x}^1)\\
&- U_t(\xi_t^1 + V_t\xi_V^1 + V_{tt}\xi_{V_t}^1 + V_{tx}\xi_{V_x}^1)\\
&- U_x(\xi_t^2 + U_t\xi_U^2 + U_{tt}\xi_{U_t}^2 + U_{tx}\xi_{U_x}^2)\\
&- U_x(\xi_t^2 + V_t\xi_V^2 + V_{tt}\xi_{V_t}^2 + V_{tx}\xi_{V_x}^2) && (117)\\
\zeta_2^1 &= \eta_x^1 + U_x\eta_U^1 + U_{xt}\eta_{U_t}^1 + U_{xx}\eta_{U_x}^1\\
&+ V_x\eta_V^1 + V_{xt}\eta_{V_t}^1 + V_{xx}\eta_{V_x}^1\\
&- U_t(\xi_x^1 + U_x\xi_U^1 + U_{xt}\xi_{U_t}^1 + U_{xx}\xi_{U_x}^1)\\
&- U_t(\xi_x^1 + V_x\xi_V^1 + V_{xt}\xi_{V_t}^1 + V_{xx}\xi_{V_x}^1)\\
&- U_x(\xi_x^2 + U_x\xi_U^2 + U_{xt}\xi_{U_t}^2 + U_{xx}\xi_{U_x}^2)\\
&- U_x(\xi_x^2 + V_x\xi_V^2 + V_{xt}\xi_{V_t}^2 + V_{xx}\xi_{V_x}^2), && (118)\\
\zeta_1^2 &= \eta_t^2 + U_t\eta_U^2 + U_{tt}\eta_{U_t}^2 + U_{tx}\eta_{U_x}^2\\
&+ V_t\eta_V^2 + V_{tt}\eta_{V_t}^2 + V_{tx}\eta_{V_x}^2\\
&- V_t(\xi_t^1 + U_t\xi_U^1 + U_{tt}\xi_{U_t}^1 + U_{tx}\xi_{U_x}^1)\\
&- V_t(\xi_t^1 + V_t\xi_V^1 + V_{tt}\xi_{V_t}^1 + V_{tx}\xi_{V_x}^1)\\
&- V_x(\xi_t^2 + U_t\xi_U^2 + U_{tt}\xi_{U_t}^2 + U_{tx}\xi_{U_x}^2)\\
&- V_x(\xi_t^2 + V_t\xi_V^2 + V_{tt}\xi_{V_t}^2 + V_{tx}\xi_{V_x}^2), && (119)
\end{aligned}$$

Subsequently, and in a similar manner, expressions for ζ_{22}^1 and ζ_{22}^2 follow from

$$\begin{aligned}
\zeta_{22}^1 &= D_x(\zeta_2^1) - U_{xt}D_x(\xi^1) - U_{xx}D_x(\xi^2), && (120)\\
\zeta_{22}^2 &= D_x(\zeta_2^2) - V_{xt}D_x(\xi^1) - V_{xx}D_x(\xi^2). && (121)
\end{aligned}$$

That is,

$$\begin{aligned}
\zeta_{22}^1 &= D_x[\eta_x^1 + U_x\eta_U^1 + U_{xt}\eta_{U_t}^1 + U_{xx}\eta_{U_x}^1\\
&+ V_x\eta_V^1 + V_{xt}\eta_{V_t}^1 + V_{xx}\eta_{V_x}^1\\
&- U_t(\xi_x^1 + U_x\xi_U^1 + U_{xt}\xi_{U_t}^1 + U_{xx}\xi_{U_x}^1)\\
&- U_t(\xi_x^1 + V_x\xi_V^1 + V_{xt}\xi_{V_t}^1 + V_{xx}\xi_{V_x}^1)\\
&- U_x(\xi_x^2 + U_x\xi_U^2 + U_{xt}\xi_{U_t}^2 + U_{xx}\xi_{U_x}^2)\\
&- U_x(\xi_x^2 + V_x\xi_V^2 + V_{xt}\xi_{V_t}^2 + V_{xx}\xi_{V_x}^2)]\\
&- U_{xt}(\xi_t^1 + U_t\xi_U^1 + U_{tt}\xi_{U_t}^1 + U_{tx}\xi_{U_x}^1)\\
&- U_{xt}(\xi_t^1 + V_t\xi_V^1 + V_{tt}\xi_{V_t}^1 + V_{tx}\xi_{V_x}^1)\\
&- U_{xx}(\xi_t^2 + U_t\xi_U^2 + U_{tt}\xi_{U_t}^2 + U_{tx}\xi_{U_x}^2)\\
&- U_{xx}(\xi_t^2 + V_t\xi_V^2 + V_{tt}\xi_{V_t}^2 + V_{tx}\xi_{V_x}^2), && (122)\\
\zeta_{22}^2 &= D_x[\eta_x^2 + U_x\eta_U^2 + U_{xt}\eta_{U_t}^2 + U_{xx}\eta_{U_x}^2\\
&+ V_x\eta_V^2 + V_{xt}\eta_{V_t}^2 + V_{xx}\eta_{V_x}^2\\
&- V_t(\xi_x^1 + U_x\xi_U^1 + U_{xt}\xi_{U_t}^1 + U_{xx}\xi_{U_x}^1)\\
&- V_t(\xi_x^1 + V_x\xi_V^1 + V_{xt}\xi_{V_t}^1 + V_{xx}\xi_{V_x}^1)\\
&- V_x(\xi_x^2 + U_x\xi_U^2 + U_{xt}\xi_{U_t}^2 + U_{xx}\xi_{U_x}^2)\\
&- V_x(\xi_x^2 + V_x\xi_V^2 + V_{xt}\xi_{V_t}^2 + V_{xx}\xi_{V_x}^2)]\\
&- V_{xt}(\xi_t^1 + U_t\xi_U^1 + U_{tt}\xi_{U_t}^1 + U_{tx}\xi_{U_x}^1)\\
&- V_{xt}(\xi_t^1 + V_t\xi_V^1 + V_{tt}\xi_{V_t}^1 + V_{tx}\xi_{V_x}^1)\\
&- V_{xx}(\xi_t^2 + U_t\xi_U^2 + U_{tt}\xi_{U_t}^2 + U_{tx}\xi_{U_x}^2)\\
&- V_{xx}(\xi_t^2 + V_t\xi_V^2 + V_{tt}\xi_{V_t}^2 + V_{tx}\xi_{V_x}^2). && (123)
\end{aligned}$$

It should be clear at this stage that the equations are bound to continue getting bigger and more complicated as the analysis continues. This is the nature of the pure Lie approach. It is for this reason that we opt for modified symmetries, starting in the next section.

IV. MODIFIED LIE SYMMETRIES

A Lie symmetry group analysis of the system, (1) and (2), leads to very complicated equations. To simplify it, we first combine the two

$$\frac{\partial^2}{\partial x^2}(DU + D_\omega V) = \frac{\partial V}{\partial t} + \frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(V_0 U)$$
$$-(kV + a)U - (r - k_e U)V, \qquad (124)$$

then split it into

$$\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right] w = 0, \qquad (125)$$

with $w = DU + D_\omega V$, and

$$\frac{\partial v}{\partial t} + \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(V_0 u) - (kv + a)u$$
$$-(r - k_e u)v = o, \qquad (126)$$

where $U = u$ and $V = v$ when $w = Du + D_\omega v$ satisfies (125). This simply means we only know fewer values of V and U. To get rest of the values, we use the following theorem.

Theorem 4: If $\phi = \phi(\chi)$ is defined on $\mathbf{R}$ and analytic on $\mathbf{D} \subset \mathbf{R}$, and has common zeros $\{\chi_1, \chi_2, \chi_3, \cdots\}$ with $\ddot{\phi}(\chi)$ in $\mathbf{D}$, then the differential equation

$$F(\chi, \phi(\chi), \dot{\phi}(\chi), \cdots) = 0, \qquad (127)$$

is compatible with

$$\phi^{(n)}(\chi)\phi^{(m+1)}(\chi) - \phi^{(m)}(\chi)\phi^{(n+1)}(\chi) = 0. \qquad (128)$$

The proof follows through Lipschitz's boundedness conditions and L'Hopital's principle. But first we solve (125).

A. A Lie analysis of (125)

Equation (125) is the heat equation. It has several known solutions whose exact form is known, like

$$w = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \left(C_1 + C_2 \frac{x}{t} \right), \qquad (129)$$

and

$$w = \frac{C_0}{\sqrt{t^n}} e^{-\frac{x^2}{4t}}. \qquad (130)$$

In order to generate point symmetries for equation (125), with $Du + D_\omega v = w$, we first consider a change of variables from t, x, and u to t^*, x^*, and u^* involving an infinitesimal parameter ϵ. A Taylor's series expansion in ϵ near $\epsilon = 0$ yields

$$\left.\begin{array}{rcl} t^* & \approx & t + \epsilon T(t,x,w) \\ x^* & \approx & x + \epsilon \xi(t,x,w) \\ w^* & \approx & w + \epsilon \zeta(t,x,w) \end{array}\right\} \qquad (131)$$

where

$$\left.\begin{array}{rcl} \frac{\partial t^*}{\partial \epsilon}|_{\epsilon=0} & = & T(t,x,w) \\ \frac{\partial x^*}{\partial \epsilon}|_{\epsilon=0} & = & \xi(t,x,w) \\ \frac{\partial w^*}{\partial \epsilon}|_{\epsilon=0} & = & \zeta(t,x,w) \end{array}\right\}. \qquad (132)$$

The tangent vector field (132) is associated with an operator

$$X = T\frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial w}, \qquad (133)$$

called a symmetry generator. This in turn leads to the invariance condition

$$X^{[2]} \left(w_{xx} - w_t \right) |_{\{w_{xx} = w_t\}} = 0, \qquad (134)$$

where $X^{[2]}$ is the second prolongation of X. It is obtained from the formulas:

$$\begin{array}{rcl} X^{[2]} & = & X + \zeta_t^{(1)} \frac{\partial}{\partial w_t} + \zeta_x^{(1)} \frac{\partial}{\partial w_x} + \zeta_{tt}^{(2)} \frac{\partial}{\partial w_{tt}} \\ & & + \zeta_{tx}^{(2)} \frac{\partial}{\partial w_{tx}} + \zeta_{xx}^{(2)} \frac{\partial}{\partial w_{xx}}, \end{array} \qquad (135)$$

where

$$\zeta_t^{(1)} = \frac{\partial g}{\partial t} + w\frac{\partial f}{\partial t} + \left[f - \frac{\partial T}{\partial t} \right] w_t - \frac{\partial \xi}{\partial x} w_x, \qquad (136)$$

$$\zeta_x^{(1)} = \frac{\partial g}{\partial x} + w\frac{\partial f}{\partial x} + \left[f - \frac{\partial \xi}{\partial x} \right] w_x - \frac{\partial T}{\partial t} w_t, \qquad (137)$$

$$\begin{array}{rcl} \zeta_{tt}^{(2)} & = & \frac{\partial^2 g}{\partial t^2} + w\frac{\partial^2 f}{\partial t^2} + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 T}{\partial t^2} \right] w_t - \frac{\partial^2 \xi}{\partial t^2} w_x \\ & & + \left[f - 2\frac{\partial T}{\partial t} \right] w_{tt} - 2\frac{\partial \xi}{\partial t} w_{tx}, \end{array} \qquad (138)$$

$$\begin{array}{rcl} \zeta_{xx}^{(2)} & = & \frac{\partial^2 g}{\partial x^2} + w\frac{\partial^2 f}{\partial x^2} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2} \right] w_x - \frac{\partial^2 T}{\partial x^2} w_t \\ & & + \left[f - 2\frac{\partial T}{\partial x} \right] w_{xx} - 2\frac{\partial T}{\partial x} w_{tx}, \end{array} \qquad (139)$$

and

$$\begin{array}{rcl} \zeta_{tx}^{(2)} & = & \frac{\partial^2 g}{\partial t \partial x} + w\frac{\partial^2 f}{\partial t \partial x} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 T}{\partial t \partial x} \right] w_t \\ & & + \left[2\frac{\partial f}{\partial t} - \frac{\partial^2 \xi}{\partial t \partial x} \right] w_x - \left[f - \frac{\partial T}{\partial t} - \frac{\partial \xi}{\partial x} \right] w_{tx} \\ & & - \frac{\partial T}{\partial x} w_{tt} - \frac{\partial \xi}{\partial t} w_{xx}. \end{array} \qquad (140)$$

It is to be understood here that the simplification $\zeta(t,x,w) = wf(t,x) + g(t,x)$ is adopted from the calculations that led to the old symmetries:

$$\left.\begin{array}{rcl} Y_1 & = & \frac{\partial}{\partial x}, \\ Y_2 & = & \frac{\partial}{\partial t}, \\ Y_3 & = & x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \\ Y_4 & = & xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + \left(\frac{t}{2} + \frac{x^2}{w} \right) w\frac{\partial}{\partial w}, \\ Y_5 & = & t\frac{\partial}{\partial x} - \frac{xw}{2}\frac{\partial}{\partial w}, \\ Y_6 & = & w\frac{\partial}{\partial w}, \\ Y_\infty & = & g(t,x)\frac{\partial}{\partial w}. \end{array}\right\} \qquad (141)$$

These are mentioned here to ease comparison with our own, which are at the end of this section.

The invariance condition (134) then leads to the equation $\frac{\partial^2 g}{\partial x^2} + w\frac{\partial^2 f}{\partial x^2} + \left[2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2} \right] w_x - \frac{\partial^2 T}{\partial x^2} w_t + \left[f - 2\frac{\partial T}{\partial x} \right] w_t - 2\frac{\partial T}{\partial x} w_{tx} - \frac{\partial g}{\partial t} - w\frac{\partial f}{\partial t} - \left[f - \frac{\partial T}{\partial t} \right] w_t + \frac{\partial \xi}{\partial x} w_x = 0$, called determining equation, from which follows the monomials

$$\left.\begin{array}{rcl} w_{tx} & : & T_x = 0 \\ w_t & : & T_t - 2\xi_x = 0 \\ w_x & : & 2f_x - \xi_{xx} + \xi_t = 0 \\ w & : & f_{xx} - f_t = 0 \\ 1 & : & g_{xx} - g_t = 0 \end{array}\right\} \qquad (142)$$

called the defining equations.

To begin solving these, we note that the first defining equation $T_x = 0$, suggests that T should not depend on x. The implication is that we would end with less number of symmetries if we continue this way.

That is, T depends on both t, and x near $\epsilon = 0$, but not at $\epsilon = 0$. Differentiating this defining equation with respect to t, gives

$$T_{tx} = 0. \qquad (143)$$

This can then be used to simplify the second defining equation. When the latter is differentiated with respect to x, we get

$$T_{xt} - 2\xi_{xx} = 0. \qquad (144)$$

Because the function T is analytic everywhere, Euler's mixed derivatives theorem holds, meaning $T_{xt} = T_{tx}$. This then reduces (144) into

$$\xi_{xx} = 0, \qquad (145)$$

which then integrates into

$$\xi = a + xb, \tag{146}$$

where $a = a(t)$, and $b = b(t)$. Without going into much details, which can be found in [13] and [14], that might take much of the space in this article, note that this equation can be written in the form

$$\xi = \frac{a\phi\cos(\omega x/i) + b\sin(\omega x/i)}{\omega/i}, \tag{147}$$

where $\phi = \sin(\omega/i)$. It reduces to the former when $\omega = 0$. This is where the transition to modified symmetries happen.

The second defining equation, $T_t - 2\xi_x = 0$, then leads to

$$T = \frac{-2\dot{a}\phi\sin(\omega x/i) + 2\dot{b}\cos(\omega x/i)}{\omega} + A_0, \tag{148}$$

where A_0 is a constant. Thus T now appears to also depend on x, but we know this is subject to $\omega = 0$. Substituting ξ, and T from equations (147), and (148) into the third defining equation, $2f_x = \eta_{xx} - \eta_t$, leads to

$$\begin{aligned} 2f_x &= -\frac{\dot{a}\phi}{i}\frac{w}{i}cos(\omega x/i) - \dot{b}\frac{\omega}{i}sin(\omega x/i) \\ &\quad -\ddot{a}\frac{\phi}{\omega}cos(\omega x/i) - \frac{i\ddot{b}}{\omega}sin(\omega x/i), \end{aligned} \tag{149}$$

Integrating this with respect to x gives

$$\begin{aligned} f &= -\left(\dot{a} + \ddot{a}\right)\frac{\phi}{2}sin(\omega x/i) \\ &\quad + \left(\dot{b} - \ddot{b}\right)\frac{1}{2}cos(\omega x/i) + \frac{B_0}{2}, \end{aligned} \tag{150}$$

where B_0 is a constant. We now substitute this into the fourth defining equation to establish the functions a, and b. First we differentiate (150) once with respect to t:

$$\begin{aligned} f_t &= -\left(\ddot{a} + a^{(3)}\right)\frac{\phi}{2}sin(\omega x/i) \\ &\quad + \left(\ddot{b} - b^{(3)}\right)\frac{1}{2}cos(\omega x/i), \end{aligned} \tag{151}$$

then twice with respect to x:

$$\begin{aligned} f_{xx} &= -\left(\dot{a} + \ddot{a}\right)\frac{\phi}{2}\omega^2 sin(\omega x/i) \\ &\quad + \left(\dot{b} - \ddot{b}\right)\frac{\omega^2}{2}cos(\omega x/i). \end{aligned} \tag{152}$$

The substitution leads to

$$\left(\dot{a} + \ddot{a}\right)\omega^2 = \ddot{a} + a^{(3)}, \tag{153}$$

and

$$\left(\dot{b} - \ddot{b}\right)\omega^2 = \ddot{b} - b^{(3)}. \tag{154}$$

To solve (153), we note it can be written in the form

$$\frac{\ddot{a} + a^{(3)}}{\dot{a} + \ddot{a}} = \omega^2. \tag{155}$$

That is,

$$\dot{a} + \ddot{a} = C_0 e^{\omega^2 t}. \tag{156}$$

Subsequently,

$$a = \frac{C_0}{\omega^2}\frac{1}{\omega^2 + 1}e^{\omega^2 t} + C_1 + C_2 e^{-t}. \tag{157}$$

Similarly, solving equation (154) yields

$$b = \frac{D_0}{\omega^2}\frac{1}{\omega^2 - 1}e^{\omega^2 t} + D_1 + D_2 e^{t}, \tag{158}$$

for some constants C_0, C_1, C_2, D_0, D_1, and D_2.

B. Infinitesimals

The linearly independent solutions of the defining equations (142) lead to the infinitesimals

$$\begin{aligned} T &= -2\phi\left(\frac{C_0}{\omega^4(\omega^2+1)}e^{\omega^2 t}\right)\sin(\omega x/i) \\ &\quad -2\phi\left(C_1 t - C_2 e^{-t}\right)\sin(\omega x/i) \\ &\quad +2\left(\frac{D_0}{\omega^4(\omega^2-1)}e^{\omega^2 t}\right)\cos(\omega x/i) \\ &\quad +2\left(D_1 t + D_2 e^{t}\right)\cos(\omega x/i) + A_0, \end{aligned} \tag{159}$$

$$\begin{aligned} \xi &= \frac{i\phi}{\omega}\left(\frac{C_0}{\omega^2}\frac{1}{\omega^2+1}e^{\omega^2 t}\right)\cos(\omega x/i) \\ &\quad +\frac{i\phi}{\omega}\left(C_1 + C_2 e^{-t}\right)\cos(\omega x/i) \\ &\quad +\frac{i}{\omega}\left(\frac{D_0}{\omega^2}\frac{1}{\omega^2-1}e^{\omega^2 t}\right)sin(\omega x/i) \\ &\quad +\frac{i}{\omega}\left(D_1 + D_2 e^{t}\right)sin(\omega x/i) \end{aligned} \tag{160}$$

and

$$\begin{aligned} f &= -C_0\frac{\phi e^{\omega^2 t}}{2}\sin(\omega x/i) \\ &\quad -D_0\frac{e^{\omega^2 t}}{2}\cos(\omega x/i) + \frac{B_0}{2}. \end{aligned} \tag{161}$$

C. The symmetries

According to (133), the infinitesimals: (159), (160), and (161), lead to the generators

$$\begin{aligned} X_1 &= \frac{2e^{\omega^2 t}}{\omega^4(\omega^2-1)}\cos(\omega x/i)\frac{\partial}{\partial t} \\ &\quad +\frac{ie^{\omega^2 t}}{\omega^3(\omega^2-1)}\sin(\omega x/i)\frac{\partial}{\partial x} \\ &\quad - \frac{e^{\omega^2 t}}{2}\cos(\omega x/i)w\frac{\partial}{\partial w}, \end{aligned} \tag{162}$$

$$\begin{aligned} X_2 &= -\frac{2\phi e^{\omega^2 t}}{\omega^4(\omega^2+1)}\sin(\omega x/i)\frac{\partial}{\partial t} \\ &\quad +\frac{i\phi e^{\omega^2 t}}{\omega^3(\omega^2+1)}\cos(\omega x/i)\frac{\partial}{\partial x} \\ &\quad - \frac{\phi e^{\omega^2 t}}{2}\sin(\omega x/i)w\frac{\partial}{\partial w}, \end{aligned} \tag{163}$$

$$X_3 = -2\phi\ t\ \sin(\omega x/i)\frac{\partial}{\partial t} + \frac{i\phi}{\omega}\cos(\omega x/i)\frac{\partial}{\partial x}, \quad (164)$$

$$X_4 = 2\ t\ \cos(\omega x/i)\frac{\partial}{\partial t} + \frac{i}{\omega}\sin(\omega x/i)\frac{\partial}{\partial x}, \quad (165)$$

$$X_5 = 2\phi\ e^{-t}\ \sin(\omega x/i)\frac{\partial}{\partial t} + \frac{i\phi}{\omega}\ e^{-t}\ \cos(\omega x/i)\frac{\partial}{\partial x}, \quad (166)$$

$$X_6 = 2\ e^{t}\ \cos(\omega x/i)\frac{\partial}{\partial t} + \frac{i}{\omega}\ e^{t}\ \sin(\omega x/i)\frac{\partial}{\partial x}, \quad (167)$$

$$X_7 = \frac{\partial}{\partial t}, \quad (168)$$

$$X_8 = w\frac{\partial}{\partial w}. \quad (169)$$

The last defining equation leads to an infinite symmetry generator.

$$X_\infty = g(t,x)\frac{\partial}{\partial w}. \quad (170)$$

D. Construction of invariant solutions through the symmetry X_1

The symmetries X_7, X_8, and X_∞ are not different from Y_2, Y_6, and Y_∞ obtained by Bluman, and others, as such unlikely to lead to anything not already known. We limit our construction of invariant solutions to X_1, and X_2, as they appear to be broader and more encompassing than X_3, X_4, X_5, and X_6. What is certain is that X_3, and X_4 are automatically addressed.

The characteristic equations that arise from the symmetry X_1:

$$\frac{\omega^4(\omega^2-1)e^{-\omega^2 t}dt}{2\cos(\omega x/i)} = \frac{i\omega^3(\omega^2-1)e^{-\omega^2 t}dx}{\sin(\omega x/i)} = \frac{2e^{-\omega^2 t}dw}{\cos(\omega x/i)w}, \quad (171)$$

lead to

$$\frac{\omega^4(\omega^2-1)e^{-\omega^2 t}dt}{\cos(\omega x/i)} = 2\frac{i\omega^3(\omega^2-1)e^{-\omega^2 t}dx}{\sin(\omega x/i)}, \quad (172)$$

and

$$\frac{\omega^4(\omega^2-1)e^{-\omega^2 t}dt}{2\cos(\omega x/i)} = \frac{2e^{-\omega^2 t}dw}{\cos(\omega x/i)w}. \quad (173)$$

Equation (172) becomes

$$\omega^2 dt = -2\frac{(\omega/i)\cos(\omega x/i)dx}{\sin(\omega x/i)}, \quad (174)$$

so that

$$\lambda = -\omega^2 t - 2\ln|\sin(\omega x/i)|. \quad (175)$$

Hence,

$$\eta = e^{\frac{\omega^2}{2}t}\ |\sin(\omega x/i)| \quad (176)$$

where $\eta = \exp(-\lambda/2)$.

Equation (173) becomes

$$\frac{\omega^4(\omega^2-1)dt}{4} = \frac{dw}{w}, \quad (177)$$

so that the invariant solution has the form

$$w = e^{(\omega^4(\omega^2-1))t/4}\phi(\eta). \quad (178)$$

This means

$$w_t = \frac{\omega^4(\omega^2-1)}{4}e^{(\omega^4(\omega^2-1))t/4}\phi + e^{(\omega^4(\omega^2-1))t/4}\dot{\phi}\eta_t. \quad (179)$$

That is,

$$w_t = \frac{\omega^4(\omega^2-1)}{4}e^{(\omega^4(\omega^2-1))t/4}\phi + \frac{\omega^2}{2}\eta e^{(\omega^4(\omega^2-1))t/4}\dot{\phi}. \quad (180)$$

On the other hand,

$$w_x = e^{(\omega^4(\omega^2-1))t/4}\dot{\phi}\eta_x, \quad (181)$$

so that

$$w_{xx} = e^{(\omega^4(\omega^2-1))t/4}\ddot{\phi}\ (\eta_x)^2 + e^{(\omega^4(\omega^2-1))t/4}\dot{\phi}\ \eta_{xx}. \quad (182)$$

That is,

$$w_{xx} = e^{(\omega^4(\omega^2-1))t/4}\ddot{\phi} \times \left(\pm e^{\frac{\omega^2}{2}t}\ (-\omega/i)\ \frac{\cos(\omega x/i)}{\omega}\right)^2 - e^{(\omega^4(\omega^2-1))t/4}\dot{\phi} \times \left(\mp e^{-\frac{\omega^2}{2}t}\ (-\omega/i)^2\ \frac{\sin(\omega x/i)}{\omega}\right), \quad (183)$$

or

$$w_{xx} = \omega^2 e^{(\omega^4(\omega^2-1))t/4}\ddot{\phi}\ \left(e^{\omega^2 t} - \eta^2\right) + \omega^2\eta e^{(\omega^4(\omega^2-1))t/4}\dot{\phi}. \quad (184)$$

Substituting the expression for u_t from equation (180), and the one for w_{xx} from equation (184) give

$$\omega^2\ddot{\phi}\ \left(e^{\omega^2 t} - \eta^2\right) + \omega^2\eta\dot{\phi} = \frac{\omega^4(\omega^2-1)}{4}\phi + \frac{\omega^2}{2}\eta\dot{\phi}. \quad (185)$$

In the limit ω approaching zero, this equation reduces to

$$\left(1-\eta^2\right)\ddot{\phi}+\frac{\eta}{2}\dot{\phi}=0. \tag{186}$$

That is,

$$\frac{\ddot{\phi}}{\dot{\phi}}=\frac{1}{2}\frac{\eta}{\eta^2-1}, \tag{187}$$

so that

$$\int_{\eta_1}^{\eta_2}\frac{d}{d\eta}\left(\ln\dot{\phi}\right)d\eta=\frac{1}{2}\int_{\eta_1}^{\eta_2}\frac{\tilde{\eta}}{\tilde{\eta}^2-1}d\tilde{\eta}. \tag{188}$$

The integral on the left evaluates easily. Hence,

$$\ln\dot{\phi}=\tilde{F}_0+\frac{1}{2}\int_{\eta_1}^{\eta_2}\frac{\tilde{\eta}}{\tilde{\eta}^2-1}d\tilde{\eta}, \tag{189}$$

where $\tilde{F}_0$ is a constant. The other requires letting $\eta_1=\eta$, and $\eta_2=\eta+\omega$ then invoking L'hopital's principle. That is,

$$\ln\dot{\phi}=\tilde{F}_0+\frac{\frac{\omega}{2}\frac{d\eta}{d\omega}\frac{d}{d\eta}\int_{\eta}^{\eta+\omega}\frac{\tilde{\eta}}{\tilde{\eta}^2-1}d\tilde{\eta}}{\frac{d}{d\omega}\omega}. \tag{190}$$

Evaluating $d\eta/d\omega$:

$$\begin{aligned}&\ln\dot{\phi}=\tilde{F}_0+\\&\frac{\omega}{2}\left(\frac{\omega}{2}t|\sin(\omega x/i)|\pm(x/i)\cos(\omega x/i)\right)\\&\times e^{\frac{\omega^2}{2}t}\frac{d}{d\eta}\int_{\eta}^{\eta+\omega}\frac{\tilde{\eta}}{\tilde{\eta}^2-1}d\tilde{\eta}.\end{aligned} \tag{191}$$

The fundamental theorem of calculus ensures that the derivative removes the integral, simplifying the equation to

$$\begin{aligned}&\ln\dot{\phi}=\tilde{F}_0+\\&\frac{\omega}{2}\left(e^{\frac{\omega^2}{2}t}\ |\sin(\omega x/i)|\pm(x/i)\cos(\omega x/i)\right)\\&\times e^{\frac{\omega^2}{2}t}\frac{\eta}{\eta^2-1}.\end{aligned} \tag{192}$$

A further simplification on the right gives

$$\begin{aligned}&\ln\dot{\phi}=\tilde{F}_0+\\&\frac{\omega}{2}\left(\frac{\omega}{2}t|\sin(\omega x/i)|\pm(x/i)\cos(\omega x/i)\right)\\&\times\frac{\omega e^{\frac{\omega^2}{2}t}\ \frac{|\sin(\omega x/i)|}{\omega}}{\eta^2e^{-\frac{\omega^2}{2}t}-e^{-\frac{\omega^2}{2}t}}.\end{aligned} \tag{193}$$

That is,

$$\begin{aligned}&\ln\dot{\phi}=\tilde{F}_0+\\&\frac{\omega^2}{2}\left(\frac{\omega}{2}t|\sin(\omega x/i)|\pm(x/i)\cos(\omega x/i)\right)\\&\times\frac{e^{\frac{\omega^2}{2}t}\ (\pm x/i)\cos(\omega x/i)}{\eta^2e^{-\frac{\omega^2}{2}t}-e^{-\frac{\omega^2}{2}t}},\end{aligned} \tag{194}$$

so that

$$\begin{aligned}&\ln\dot{\phi}=\tilde{F}_0+\\&\frac{\frac{\omega^2}{2}\left(\frac{\omega}{2}t|\sin(\omega x/i)|\pm(x/i)\cos(\omega x/i)\right)}{|\sin(\omega x/i)|^2e^{\frac{\omega^2}{2}t}-e^{-\frac{\omega^2}{2}t}}\\&\times e^{\frac{\omega^2}{2}t}\ (\pm x/i)\cos(\omega x/i).\end{aligned} \tag{195}$$

That is,

$$\begin{aligned}&\ln\dot{\phi}=\tilde{F}_0+\\&\frac{\frac{\omega^2}{2}\left(\frac{\omega}{2}t|\sin(\omega x/i)|\pm(x/i)\cos(\omega x/i)\right)}{\left(-\cos(\omega x/i)\right)^2e^{\frac{\omega^2}{2}t}+e^{\frac{\omega^2}{2}t}-e^{-\frac{\omega^2}{2}t}}\\&\times e^{\frac{\omega^2}{2}t}\ (\pm x/i)\cos(\omega x/i).\end{aligned} \tag{196}$$

The trigonometric, and hyperbolic identities ensure that there are further simplifications in the denominator. Hence,

$$\begin{aligned}&\ln\dot{\phi}=\tilde{F}_0+\\&\frac{\frac{\omega^2}{2}\left(\frac{\omega}{2}t|\sin(\omega x/i)|\pm(x/i)\cos(\omega x/i)\right)}{\left(-\cos(\omega x/i)\right)^2e^{\frac{\omega^2}{2}t}+2i\sin(\frac{\omega^2}{2i}t)}\\&\times e^{\frac{\omega^2}{2}t}\ (\pm x/i)\cos(\omega x/i).\end{aligned} \tag{197}$$

Evaluating the limits:

$$\ln\dot{\phi}=\tilde{F}_0+\frac{-x^2}{4t}. \tag{198}$$

That is,

$$\dot{\phi}=F_0e^{\frac{-x^2}{4t}}, \tag{199}$$

with $F_0=\exp(\tilde{F}_0)$. Hence,

$$\phi=F_0\int_{\eta_1}^{\eta_2}e^{\frac{-x^2}{4t}}d\tilde{\eta}. \tag{200}$$

The above expression then leads to

$$w=e^{(\omega^4(\omega^2-1))t/4}\left[F_0\int_{\eta_1}^{\eta_2}e^{\frac{-x^2}{4t}}d\tilde{\eta}\right]. \tag{201}$$

When $F_0=-iA/\omega$, and $\omega=0$ inside the integral in (201), we get

$$w=Ae^{(\omega^4(\omega^2-1))t/4}\int_{x_1}^{x_2}e^{\frac{-x^2}{4t}}dx. \tag{202}$$

This is a known solution, not really distinct from (129) and (130). Other solutions follow through the second symmetry X_2.

1) A first couple of solutions through X_2 :

$$w=F_1+F_0\int_{\eta_1}^{\eta_2}e^{-\frac{x^2}{2(x^2-t^2)}}d\tilde{\eta} \tag{203}$$

and

$$w=F_1+F_0\int_{\eta_1}^{\eta_2}e^{\frac{x^2}{2(x^2-t^2)}}d\tilde{\eta}. \tag{204}$$

2) A second couple of solutions through X_2 :

$$w = F_1 + Ae^{-\frac{x^2}{2(x^2-t^2)}} \tag{205}$$

and

$$w = F_1 + Ae^{\frac{x^2}{2(x^2-t^2)}}. \tag{206}$$

3) A third couple of solutions through X_2 :

$$w = F_1 + \frac{A}{\sqrt{t}}e^{-\frac{x^2}{2(x^2-t^2)}} \tag{207}$$

and

$$w = F_1 + \frac{A}{\sqrt{t}}e^{\frac{x^2}{2(x^2-t^2)}}. \tag{208}$$

4) A fourth couple of solutions through X_2 :

$$w = F_1 + \frac{A}{t^{3/2}}e^{-\frac{x^2}{2(x^2-t^2)}} \tag{209}$$

and

$$w = F_1 + \frac{A}{t^{3/2}}e^{\frac{x^2}{2(x^2-t^2)}}. \tag{210}$$

V. THE SOLUTION TO (126)

Equation (124) subsequently gives to

$$\frac{\partial}{\partial t}[Du + D_\omega v] = \frac{\partial v}{\partial t} + \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(V_0 u) \\ -(kv+a)u - (r - k_e u)v. \tag{211}$$

A. The quasilinear partial differential equation

If $w = w(t, x)$ is a solution of (124), then $v = \alpha w + \beta u$, with $\alpha = 1/D_\omega$ and $\beta = -D/D_\omega$, reduces (214) into the quasilinear partial differential equation

$$\left[\frac{\partial}{\partial t}\right](Du + D_\omega[\alpha w + \beta u]) \\ = \frac{\partial[\alpha w + \beta u]}{\partial t} + \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(V_0 u) \\ -(k[\alpha w + \beta u] + a)u \\ -(r - k_e u)[\alpha w + \beta u]. \tag{212}$$

That is,

$$Du_t + D_\omega \alpha w_t + D_\omega \beta u_t \\ = \alpha w_t + \beta u_t + u_t + V_0 u_x \\ -k\alpha w u - k\beta u^2 - au - r\alpha w \\ -[r\beta - k_e \alpha w]u + k_e \beta u^2. \tag{213}$$

This simplifies to

$$(D + D_\omega \beta - \beta - 1)u_t - V_0 u_x \\ = (k_e - k)\beta u^2 - (k\alpha w + a + r\beta - k_e \alpha w)u \\ -r\alpha w + (1 - D_\omega)\alpha w_t. \tag{214}$$

$$\frac{dt}{D + D_\omega \beta - \beta - 1} = \frac{dx}{-V_0} = \frac{du}{-\tilde{\psi}} \tag{215}$$

where $\tilde{\psi} = (k_e - k)\beta u^2 - (k\alpha w + a + r\beta - k_e \alpha w)u - r\alpha w + (1 - D_\omega)\alpha w_t$, from which we get the invariants

$$\chi = (D + D_\omega \beta - \beta - 1)x + V_0 t, \tag{216}$$

and

$$u = \lambda_0 \tan\left(C\left[\frac{t}{\lambda} + \psi(\chi)\right]\right) - \sigma, \tag{217}$$

where

$$\begin{aligned} \lambda_0 &= D + D_\omega \beta - \beta - 1, && (218)\\ C &= \frac{\sqrt{4AC - B^2}}{2}, && (219)\\ \sigma &= -\frac{k\alpha w + a + r\beta - k_e \alpha w}{k_e - k}, && (220)\\ c &= \frac{-r\alpha w + (1 - D_\omega)\alpha w_t}{k_e - k}, && (221)\\ A &= k_e - k, && (222)\\ B &= k\alpha w + a + r\beta - k_e \alpha w, && (223)\\ C &= -r\alpha w + (1 - D_\omega)\alpha w_t. && (224) \end{aligned}$$

The only outstanding quantity to evaluate now is ψ. There is no established procedure that can be followed to obtain it. In the past, mathematicians tended to use power series expansion to estimate it, and this practise is still common. Fortunately, there is an option here. There are two accepted to solving the second invariant, the first through the trigonometric function tan yielded (217). The second is through partial fractions, and it gives

$$u = \frac{r_1 - r_2 e^{\frac{\psi}{(k_e-k)(r_1-r_2)}}}{1 - e^{\frac{\psi}{(k_e-k)(r_1-r_2)}}}. \tag{225}$$

That is,

$$\psi = \frac{1}{m}\log\left(\frac{u - r_1}{u - r_2}\right), \tag{226}$$

where

$$m = \frac{1}{(k_e - k)(r_1 - r_2)}, \tag{227}$$

$$r_1 = \frac{-B - \sqrt{B^2 - 4C}}{2}, \tag{228}$$

$$r_2 = \frac{-B + \sqrt{B^2 - 4C}}{2}, \tag{229}$$

$$B = -\frac{k\alpha w + a + r\beta - k_e \alpha w}{k_e - k} \tag{230}$$

and

$$C = \frac{-r\alpha w + (1 - D_\omega)\alpha w_t}{k_e - k}, \tag{231}$$

so that

$$u = \lambda_0 \tan\left(C\left[\frac{t}{\lambda} + \frac{\log\left(\frac{u-r_1}{u-r_2}\right)}{m}\right]\right) - \sigma. \tag{232}$$

This result is implicit. The explicit case follows from $r_1 = r_2$, and is

$$u = \lambda_0 \tan\left(\frac{Ct}{\lambda}\right) - \sigma. \tag{233}$$

B. The solutions to (1) and (2)

A special case of Theorem 1, wherein $m = 1$ and $n = 1$ leads to the expression

$$\frac{(U-u)_{\chi\chi}}{(U-u)} = \frac{(U-u)_{\chi\chi\chi}}{(U-u)_\chi}, \tag{234}$$

where

$$u = \lambda \tan\left(C\left[\frac{t}{\lambda} + \psi(\chi)\right]\right) - \sigma. \tag{235}$$

The condition arising from the theorem that $U - u$ is only continuous at the points where the zeros exist, and coincides with those of its second derivative, that is

$$U - u = (U-u)_{\chi\chi} = 0, \tag{236}$$

leads to

$$U = u + \lambda\cos(\mu\chi + \phi). \tag{237}$$

The condition in (236) reduces it to

$$U = u \pm \lambda\sin(2\mu\chi), \tag{238}$$

which yields the pair:

$$U_1 = u - \lambda\sin(2\mu\chi), \tag{239}$$

and

$$U_2 = u + \lambda\sin(2\mu\chi), \tag{240}$$

where λ, μ and ϕ are constants at these zero points, but generally are functions of t and x. Trying to determine these constants where they are not constants is an impossible mathematical feat, because of the calculus. It is best then to determine them at the zero points. For that we also need

$$U_t = u_t - \lambda\mu\chi_t\sin(\mu\chi + \phi), \tag{241}$$

$$U_x = u_x - \lambda\mu\chi_x\sin(\mu\chi + \phi), \tag{242}$$

$$U_{xt} = u_{xt} - \lambda\mu^2\chi_x\chi_t\cos(\mu\chi + \phi),, \tag{243}$$

$$U_{xx} = u_{xx} - \lambda(\mu\chi_x)^2\cos(\mu\chi + \phi), \tag{244}$$

$$U_{xxx} = u_{xxx} + \lambda(\mu\chi_x)^3\sin(\mu\chi + \phi), \tag{245}$$

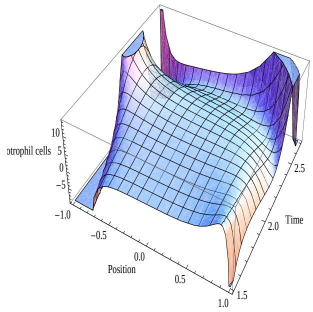

Fig. 1. Plot of the number of Neutrophil cells (U) against Time (t) and Position (x).

$$V_t = \alpha w_t + \beta U_t, \tag{246}$$

$$V_x = \alpha w_x + \beta U_x, \tag{247}$$

$$V_{xx} = \alpha w_{xx} + \beta U_{xx}. \tag{248}$$

From (234), the condition $U - u = (U-u)_{\chi\chi} = 0$ suggests $U = u$ at this point, or simply

$$[U] = u. \tag{249}$$

Similarly,

$$[U_t] = u_t - \lambda\mu\chi_t, \tag{250}$$

$$[U_x] = u_x - \lambda\mu\chi_x, \tag{251}$$

$$[U_{xt}] = u_{xt}, \tag{252}$$

$$[U_{xx}] = u_{xx}, \tag{253}$$

$$[U_{xxx}] = u_{xxx} + \lambda(\mu\chi_x)^3, \tag{254}$$

$$[V] = \alpha w + \beta[U], \tag{255}$$

$$[V_t] = \alpha w_t + \beta[U_t], \tag{256}$$

$$[V_x] = \alpha w_x + \beta[U_x], \tag{257}$$

$$[V_{xx}] = \alpha w_{xx} + \beta[U_{xx}]. \tag{258}$$

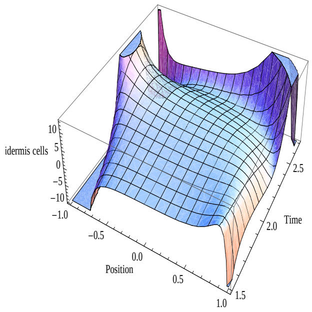

Fig. 2. Plot of the number of Epidermis cells (V) against Time (t) and Position (x).

VI. CONCLUSIONS

The objective of the study was to solve a system of second-order partial differential equations arising from the Prieto-Langraica Model for bacteria activity on a surface of a medical implant, using a slightly modified Lie symmetry group theoretical method. This was done to avoid some obstacles that one usually run into when following the normal approach.

The model is a good one, but to fully appreciate the benefits of the solutions we obtained, medically, we need to take it a step further. A step that would allow us to take advantage of the current technological advances and enable the avoiding of tedious tasks like obtaining body samples for monitoring the bacterial activity, which also require visits to medical practitioners. Such a step could require supplementing the existing model with another that interprets it microscopically at a quantum mechanical level.

Remote monitoring, when made possible this way, will not need to burden the patient with being conscious of it.

$$
\begin{aligned}
\mu \quad = \quad & -(\surd(h_{\text{tx}}\xi_t - ah_x\xi_t - kvh_x\xi_t \\
& +\text{ke}vh_x\xi_t - D_0h_{\text{xxx}}\xi_t + h_{\text{xx}}V_0\xi_t \\
& +au\xi_{\text{tx}} + kuv\xi_{\text{tx}} - h_t\xi_{\text{tx}} \\
& +\text{Dt}h_{\text{xx}}\xi_{\text{tx}} - h_xV_0\xi_{\text{tx}} \\
& -a^2u\xi_x - 2akuv\xi_x + a\text{ke}uv\xi_x - k^2uv^2\xi_x \\
& +k\text{ke}uv^2\xi_x + ah_t\xi_x + kvh_t\xi_x - \text{ke}vh_t\xi_x \\
& -a\text{Dt}h_{\text{xx}}\xi_x - \text{Dt}kvh_{\text{xx}}\xi_x \\
& +\text{Dtke}vh_{\text{xx}}\xi_x + h_{\text{tx}}V_0\xi_x \\
& -D_0h_{\text{xxx}}V_0\xi_x + h_{\text{xx}}V_0^2\xi_x \\
& -\text{Dt}h_{\text{tx}}\xi_{\text{xx}} + a\text{Dt}h_x\xi_{\text{xx}} \\
& +\text{Dt}kvh_x\xi_{\text{xx}} - \text{Dtke}vh_x\xi_{\text{xx}} \\
& +\text{Dt}D_0h_{\text{xxx}}\xi_{\text{xx}} + auV_0\xi_{\text{xx}} \\
& +kuvV_0\xi_{\text{xx}} - h_tV_0\xi_{\text{xx}} - h_xV_0^2\xi_{\text{xx}} \\
& -auD_0\xi_{\text{xxx}} - kuvD_0\xi_{\text{xxx}} + D_0h_t\xi_{\text{xxx}} \\
& -\text{Dt}D_0h_{\text{xx}}\xi_{\text{xxx}} + D_0h_xV_0\xi_{\text{xxx}})) \\
& /(\surd(auD_0\xi_x^3 \\
& +kuvD_0\xi_x^3 - D_0h_t\xi_x^3 \\
& +\text{Dt}D_0h_{\text{xx}}\xi_x^3 - D_0h_xV_0\xi_x^3)) \qquad (259)
\end{aligned}
$$

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