Mathematical and Physical modelling of the dynamic electrical impedance of a neuron

Georgios Giannoukos, Mart Min

Abstract— In this work a mathematical and a physical model of a neuron is put forward using an RLC circuit or operational amplifier circuits. The model can calculate the resistance of a healthy neuron and then by decreasing the value of the resistance we can predict how the neuron will react.

Keywords— Biompedance, neuron, Parkinson's disease, resistance, RLC circuit

I. Introduction

Bioimpedance is a method of measuring how the body impedes electrical current [1]. It can be done by applying a small electrical current using electrodes and collecting the results with other electrodes. It can be applied to the skin or to the tissues, the heart, the neurons, blood volume, calculating lean body mass, fat body mass and many other components in the body [1]. In this work a detailed mathematical and physical model for calculating the bioimpedance of an adrenergic neuron is developed. Initially the model calculates the biompedance of a healthy adrenergic neuron at different frequencies of a current and then it compares that with the alteration in the bioimpedance of an adrenergic neuron with Parkinson's disease [2]. Also this model compares the differences in the bioimpedance between the affected adrenergic neuron before and after medical treatment.

II. Problem formulation

The basic structural and functional unit of the nervous system is the nerve cell or neuron [3,4]. The nerve cells produce electrical signals transmitted from one part of the cell to another, while generating biochemical substances (acetylcholine - Ach) in order to communicate with other cells. A neuron consists of the cell body, the axon and the dendrites (see Figure 1).

The neurons exchange information through a synapse (see Figure 2).

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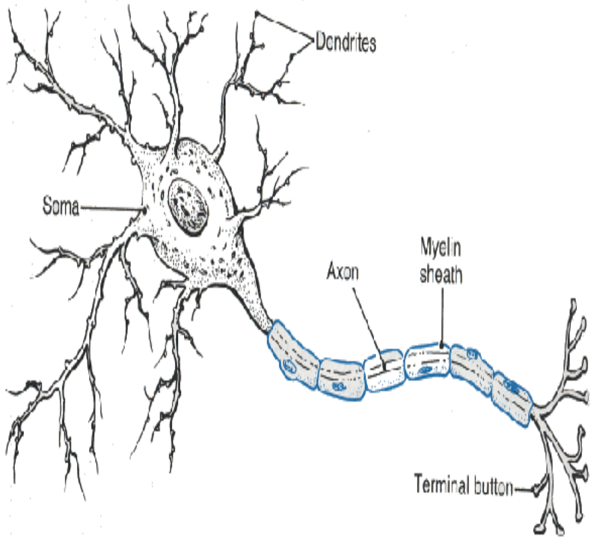

Fig.1: The structure of a neuron

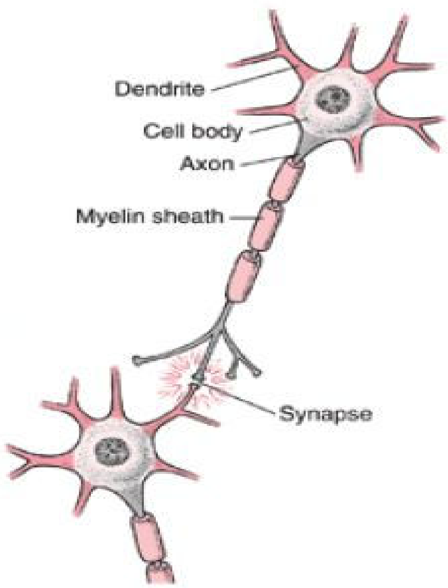

Fig.2: Synapse between two neurons

One of the functions of synapses is to determine the conduction of electrical impulses in one direction only. Therefore we can say that the synapses function as a transistor that allows the passage of electrical current in one direction [3].

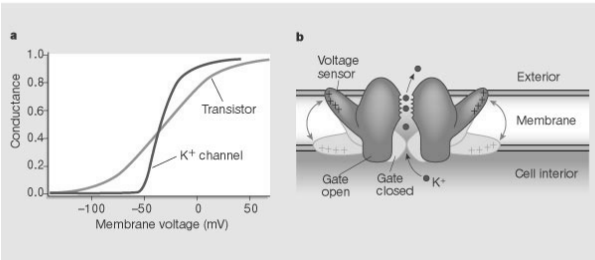

Fig.3: Comparison between a transistor's and a neuron's Conductance vs Membrane Voltage diagram

Along the surface of each neuron, potential electrical difference is due to the presence of excess negative charges on the inside and excess positive charges on the outer membrane which is why the neuron is polarized. The interior of the cell is typically 60-90mV more negative than the outside. This potential difference is called the resting potential of the neuron. When the neuron is stimulated an immediate change in the resting potential occurs. The change in the potential difference is called action potential and is transmitted along the axis.

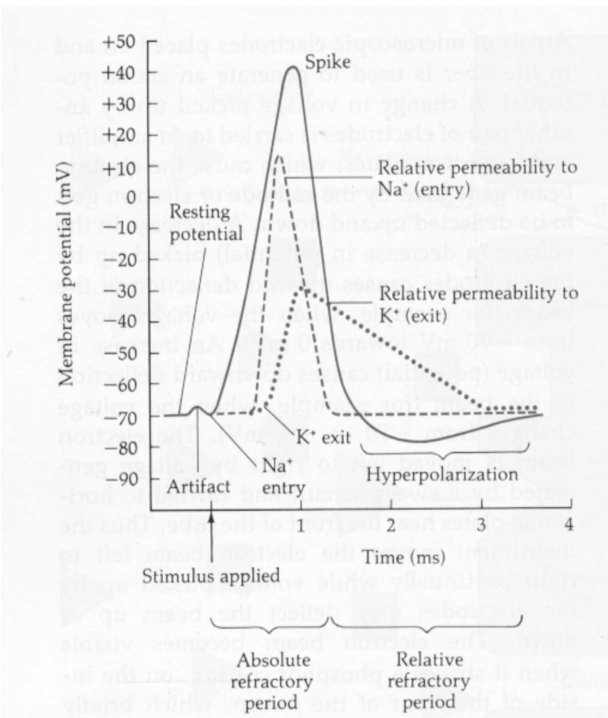

Fig.4: Membrane potential vs time

III. METHODS

The developed models were based on RLC and Operational Amplifier circuits [5]. By changing the values of the components, we can calculate the bioimpedance of a healthy neuron or one affected by Parkinson's disease. With the use of the appropriate computational software, namely Maxima [6] and Maple [7,8], the relevant equations of the models can be solved analytically. The next step is to use electronic simulation software in order to evaluate the performance of the models. Lastly, we check the biompedance of neurons which have undergone medical treatment, using the above models.

We assume the following RLC circuit:

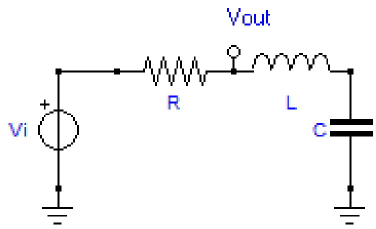

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Fig.5: RLC circuit

The differential equation in terms of the charge for the RLC circuit is the following [9]:

$$L\left(\frac{d^2}{dt^2}q(t)\right)+R\left(\frac{d}{dt}q(t)\right)+\frac{q(t)}{C}=0 \qquad (1)$$

We assume that we have an RLC circuit with resistance R which is decreasing exponentially with time, so the differential equation is:

$$L\left(\frac{d^2}{dt^2}q(t)\right)+R(t)\left(\frac{d}{dt}q(t)\right)+\frac{q(t)}{C}=0 \qquad (2)$$

If R(t)=R.e-t/τ   equation 2 becomes:

$$L\left(\frac{d^2}{dt^2}q(t)\right)+R\,e^{-\frac{t}{\tau}}\left(\frac{d}{dt}q(t)\right)+\frac{q(t)}{C}=0 \qquad (3)$$

The transfer function of the circuit is

$$V_{out}=\frac{V_i\left(1+CLs^2\right)}{1+(CR)s+(CL)s^2} \qquad (4)$$

with damping factor

$$a = \frac{R}{2L} \tag{5}$$

resonance frequency

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{6}$$

and quality factor

$$Q = \frac{\omega_0}{2a} \tag{7}$$

or

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{8}$$

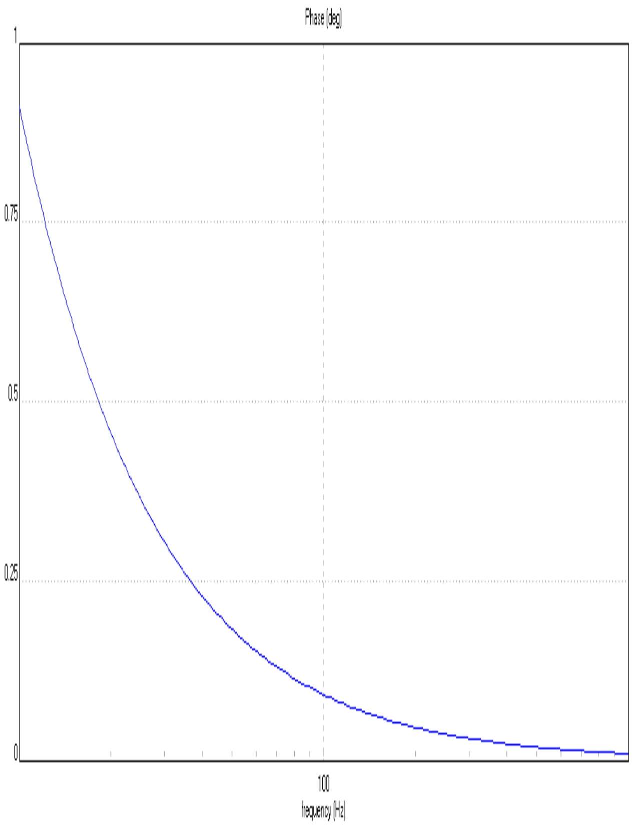

Fig.6: Phase vs frequency of the RLC circuit

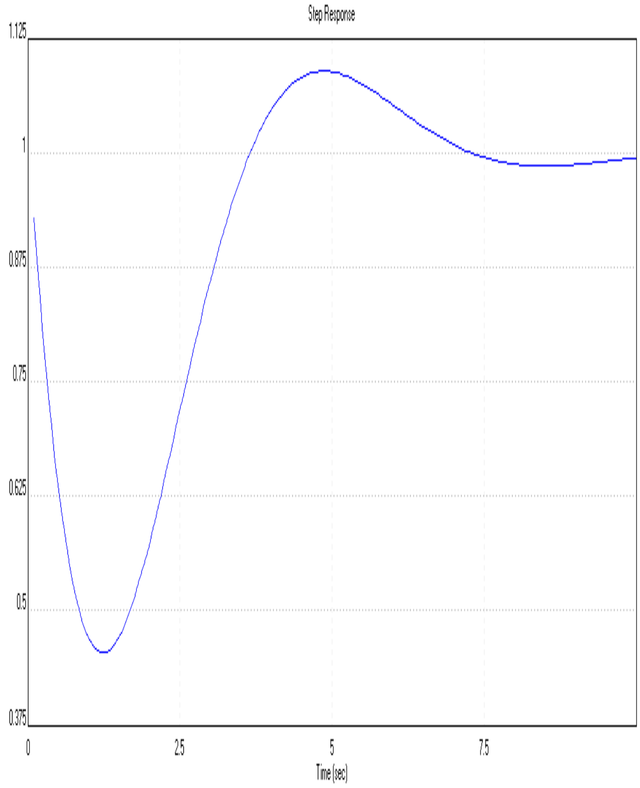

Fig.7: Step response of the RLC circuit

Instead of a typical RLC circuit we can use the following circuits [5] which behave in a manner very similar to an RLC circuit taking into account that bio inductors do not exist so in this case L in a similar differential equation is replaced with a combination of resistances and capacitors [10].

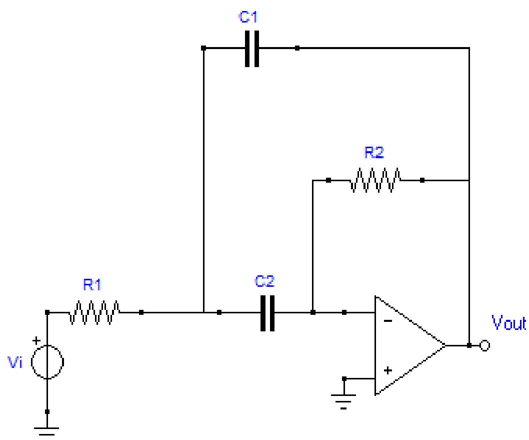

Fig.8: Circuit which behave similar to an RLC circuit

The impedance model of the above circuit is shown in the next figure

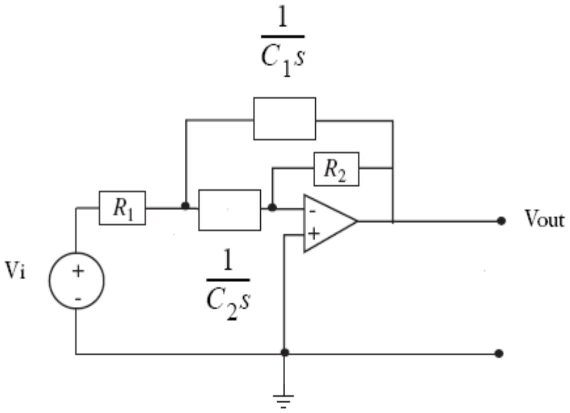

Fig.9: Impedance model of the circuit in fig.8

The transfer function of this circuit is

$$V_{out} = \frac{-V_i C_2 R_2 s}{1+(C_1R_1+C_2R_1)s+(C_1C_2R_1R_2)s^2} \tag{9}$$

with damping factor

$$a = \frac{C_1 + C_2}{2C_1C_2R_2} \tag{10}$$

resonance frequency

$$\omega_0 = \frac{1}{\sqrt{C_1C_2R_1R_2}} \tag{11}$$

and quality factor

$$Q = \frac{\sqrt{C_1C_2R_1R_2}}{C_1R_1 + C_2R_1} \tag{12}$$

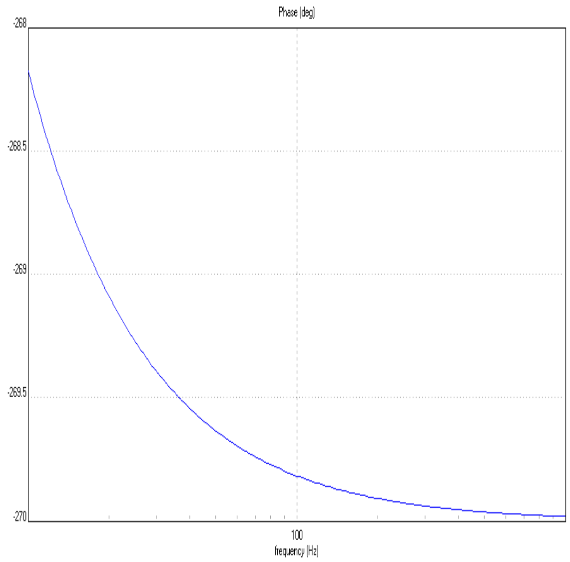

Fig.10: Phase vs frequency of the circuit in fig.8

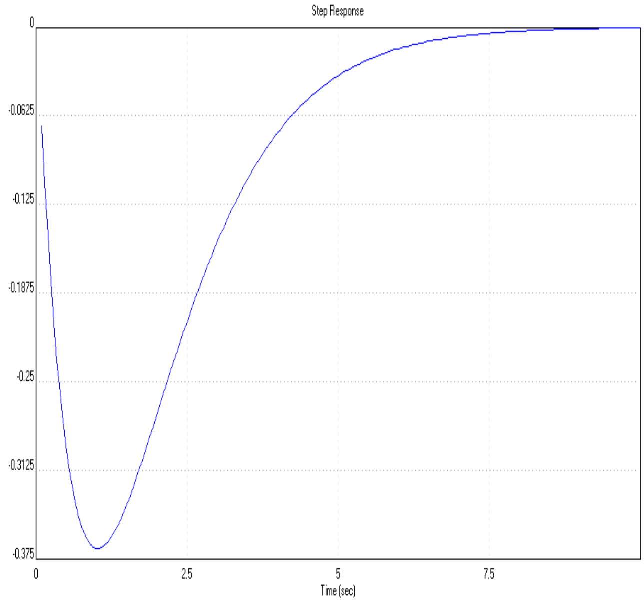

Fig.11: Step response of the circuit in fig. 8

Another similar circuit is the following

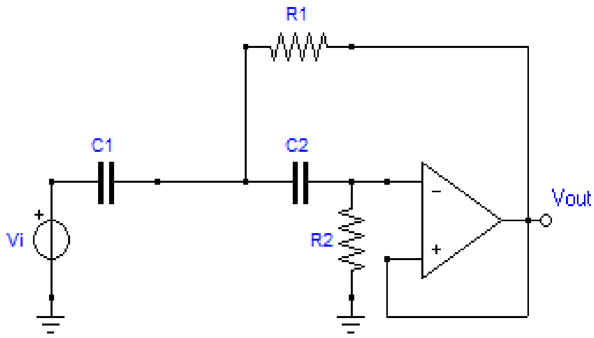

Fig.12: Circuit which behave similar to an RLC circuit

The impedance model of the above circuit is shown in the next figure

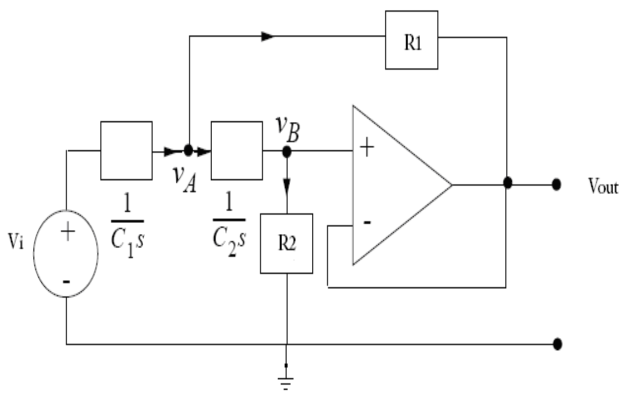

Fig.13: Impedance model of the circuit in fig, 12

The transfer function of this circuit is:

$$V_{out} = \frac{V_i C_1 C_2 R_1 R_2 s^2}{1 + (C_1 R_1 + C_2 R_1)s + (C_1 C_2 R_1 R_2)s^2} \quad (13)$$

with damping factor

$$a = \frac{C_1 + C_2}{2C_1 C_2 R_2} \quad (14)$$

resonance frequency

$$\omega_0 = \frac{1}{\sqrt{C_1 C_2 R_1 R_2}} \quad (15)$$

and quality factor

$$Q = \frac{\sqrt{C_1 C_2 R_1 R_2}}{C_1 R_1 + C_2 R_1} \quad (16)$$

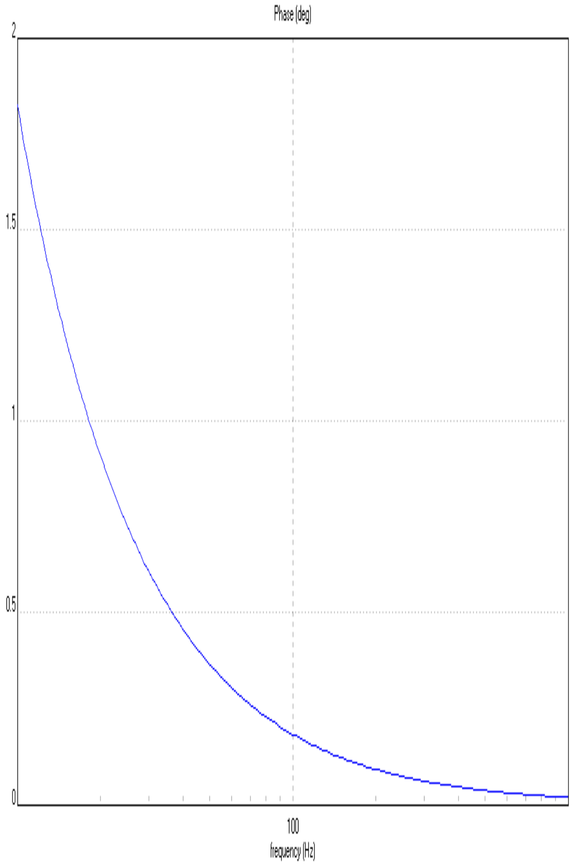

Fig.14: Phase vs frequency of the circuit in fig.6

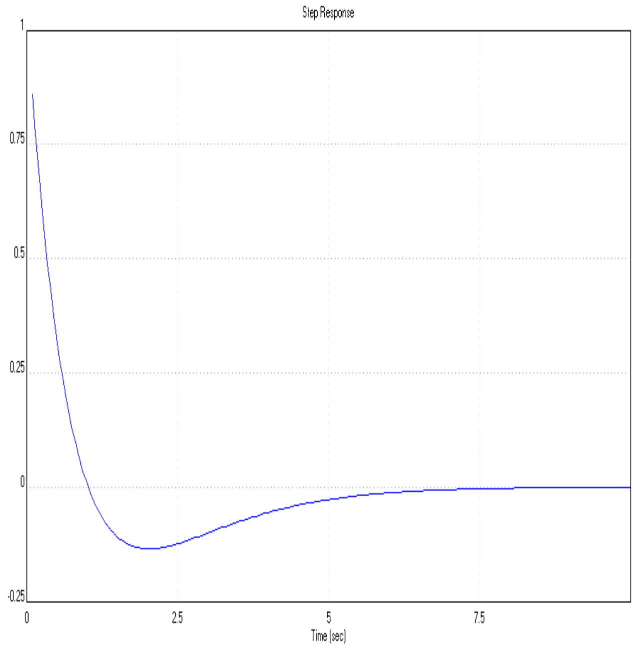

Fig.15: Step response of the circuit in fig. 6

IV. PROBLEM SOLUTION

The solution to equation (1) is:

$$q(t) = \_C1\, e^{-\frac{1}{2}\frac{\left(C R - \sqrt{C^2 R^2 - 4 C L}\right) t}{C L}} + \_C2\, e^{-\frac{1}{2}\frac{\left(C R + \sqrt{C^2 R^2 - 4 C L}\right) t}{C L}} \quad (17)$$

The condition for no oscillation which a healthy neuron satisfies but a defective neuron doesn't is:

$$0 \le C^2 R^2 - 4 C L \quad (18)$$

it is to say

$$2\sqrt{\frac{L}{C}} \le R \quad (19)$$

For initial conditions q(0)=0 and i(0)=i0 we choose _C1=1 and _C2=-1, equation 17 is now:

$$q(t) = e^{-\frac{1}{2}\frac{\left(C R - \sqrt{C^2 R^2 - 4 C L}\right) t}{C L}} - e^{-\frac{1}{2}\frac{\left(C R + \sqrt{C^2 R^2 - 4 C L}\right) t}{C L}} \quad (20)$$

Experimental results show that a neuron's response time is 10^{-4} seconds and according to equation 17, the effective time is R/2L and according to equation 19, it is possible to find appropriate values for the parameters L and C, the above give the following values: $L=10^{-3}$H and $C=10^{-6}$F [11].

With the above values equation 20 becomes:

$$q(t) = \mathrm{e}^{-500000000\left(\frac{1}{1000000}R - \sqrt{\frac{1}{1000000000000}R^2 - \frac{1}{250000000}}\right)t} - \mathrm{e}^{-500000000\left(\frac{1}{1000000}R + \sqrt{\frac{1}{1000000000000}R^2 - \frac{1}{250000000}}\right)t}$$

(21)

And equation 19 gives the following result:

$$63.24555320 \leq R$$

(22)

With the value R=70 Ohms equation 21 becomes:

$$q(t) = \mathrm{e}^{-20000.00000\,t} - 1.\,\mathrm{e}^{-50000.00000\,t}$$

(23)

According to equation 23 the following graph q=f(t), given that q is in Coulombs and t is in seconds, can be drawn

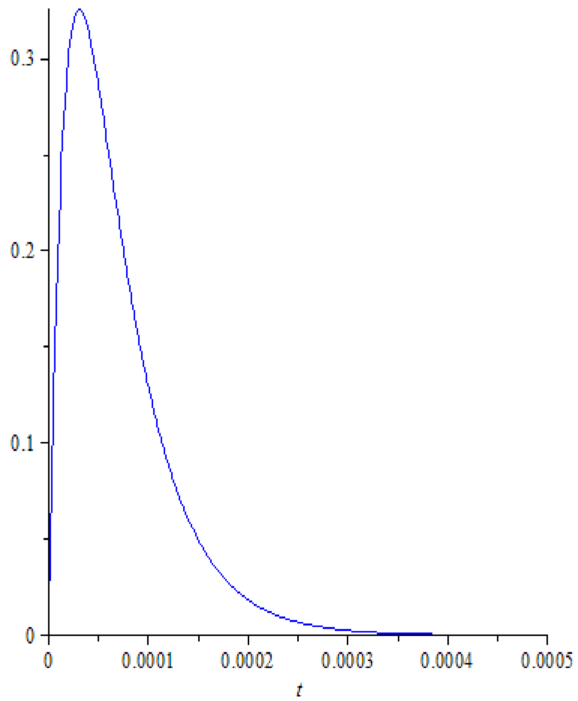

Fig.16: Charge vs time

If R is very low, close to zero then equation 21 becomes:

$$q(t) = \mathrm{e}^{31622.77660\,\mathrm{I}\,t} - 1.\,\mathrm{e}^{-31622.77660\,\mathrm{I}\,t}$$

(24)

According to equation 21 the following graph q=f(t), given that q is in Coulombs and t is in seconds, can be drawn

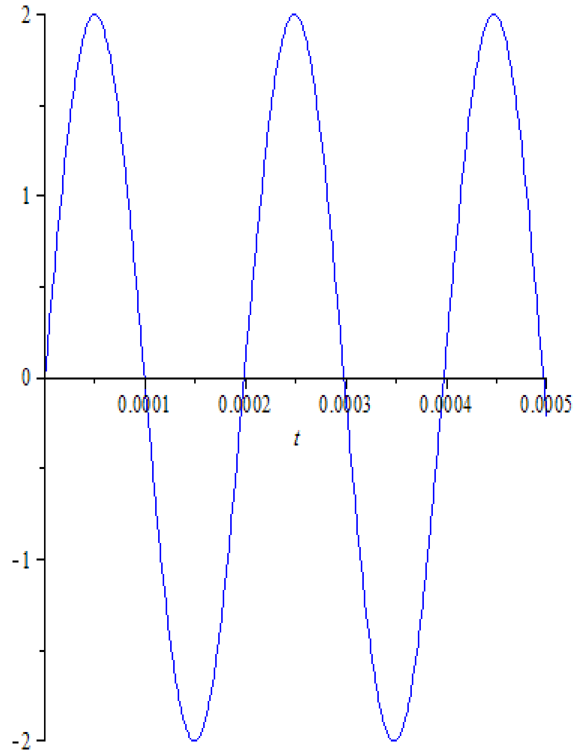

Fig.17: Charge vs time

As we can see when resistance is high (R=70 Ohms) there is no oscillation (no shaking) but when resistance is low (close to zero) there is oscillation (shaking). Thus, it appears that Parkinson’s disease is caused by decreasing resistance in the neuron.

The solution to equation 3 is given in terms of Bessel functions [12]:

$$
\begin{aligned}
q(t) = -\Bigg(& \Bigg(2\,\mathrm{BesselK}\left(\frac{1}{2}\,\frac{-\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) i_0 L^{3/2}\sqrt{C}\,\tau + \mathrm{BesselK}\left(\frac{1}{2}\,\frac{-\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) Q_0 R\sqrt{C}\sqrt{L}\,\tau \\
& - 2I\,\mathrm{BesselK}\left(\frac{1}{2}\,\frac{\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) Q_0 L\tau - 2\,\mathrm{BesselK}\left(\frac{1}{2}\,\frac{\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) i_0 L^{3/2}\sqrt{C}\,\tau \\
& + 2I\,\mathrm{BesselK}\left(\frac{1}{2}\,\frac{-\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) Q_0 L\tau \\
& - Q_0\sqrt{C}\sqrt{L}\,\mathrm{BesselK}\left(\frac{1}{2}\,\frac{3\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) R\tau \\
& - 2\,\mathrm{BesselK}\left(\frac{1}{2}\,\frac{\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) Q_0 L^{3/2}\sqrt{C}\Bigg) \\
& e^{-\frac{1}{2}\,\frac{-R\tau^2 e^{-\frac{t}{\tau}}+tL}{\tau L}} \Bigg(\mathrm{BesselI}\left(\frac{1}{2}\,\frac{-\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau e^{-\frac{t}{\tau}}}{L}\right) + \mathrm{BesselI}\left(\frac{1}{2}\,\frac{\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau e^{-\frac{t}{\tau}}}{L}\right)\Bigg)\Bigg) \Bigg/ \Bigg(\sqrt{e^{\frac{R\tau}{L}}}\sqrt{C}\sqrt{L}\,\Bigg(\\
& -R\tau\,\mathrm{BesselI}\left(\frac{1}{2}\,\frac{-\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) \mathrm{BesselK}\left(\frac{1}{2}\,\frac{\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) \\
& + 2L\,\mathrm{BesselI}\left(\frac{1}{2}\,\frac{\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) \mathrm{BesselK}\left(\frac{1}{2}\,\frac{-\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) \\
& - \mathrm{BesselI}\left(\frac{1}{2}\,\frac{3\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) R\tau\,\mathrm{BesselK}\left(\frac{1}{2}\,\frac{-\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) \\
& + \mathrm{BesselI}\left(\frac{1}{2}\,\frac{3\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) R\tau\,\mathrm{BesselK}\left(\frac{1}{2}\,\frac{\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) \\
& - R\tau\,\mathrm{BesselK}\left(\frac{1}{2}\,\frac{-\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) \mathrm{BesselI}\left(\frac{1}{2}\,\frac{\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) \\
& + \mathrm{BesselK}\left(\frac{1}{2}\,\frac{3\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) R\tau\,\mathrm{BesselI}\left(\frac{1}{2}\,\frac{-\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) \\
& + \mathrm{BesselK}\left(\frac{1}{2}\,\frac{3\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) R\tau\,\mathrm{BesselI}\left(\frac{1}{2}\,\frac{\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) \\
& + 2L\,\mathrm{BesselK}\left(\frac{1}{2}\,\frac{\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right) \mathrm{BesselI}\left(\frac{1}{2}\,\frac{-\sqrt{C}\sqrt{L}+2I\tau}{\sqrt{C}\sqrt{L}}, -\frac{1}{2}\,\frac{R\tau}{L}\right)\Bigg)\Bigg)
\end{aligned}
$$

(25)

Assume a neuron affected by Parkinson's disease, considering that the disease takes approximately 5 years (60 months) to develop and taking into account that the effective time is R/2L according to equation 20, it is possible to find appropriate values for the parameters L and C. The above give the following values: L=10H and C=20F. Also as we calculated before, values for R for the affected neuron should be under 63 Ohms approximately. We can choose any value below that, so we choose R=10 Ohms (any value in R under 63 Ohms will give similar results). The fact that the disease causes total degeneration within 10 years (120 months) after its first appearance and the life span of a person affected by Parkinson's disease after that is approximately 15 years (180 months) should be taken into account. Considering all the above as well as the fact that the time required for the neuron to loose its resistance is approximately 15 months, τ=15 months, so now equation 3 becomes:

$$10\left(\frac{d^2}{dt^2}q(t)\right) + 10\,e^{-\frac{1}{15}t}\left(\frac{d}{dt}q(t)\right) + \frac{1}{20}q(t) = 0$$

(26)

According to equation 26 the following graphs q=f(t) (Fig.18) and i=f(q) (Fig.19) can be drawn, given that q is in Coulombs, i is in Amperes and t is in months

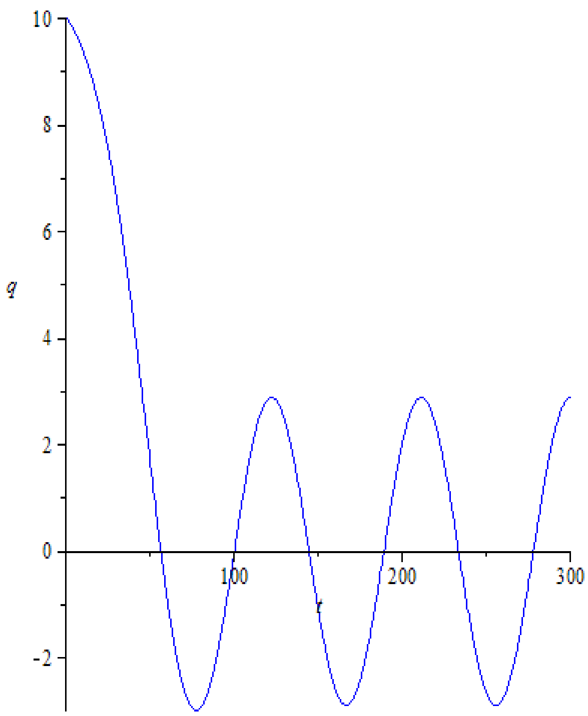

Fig.18: Charge vs time

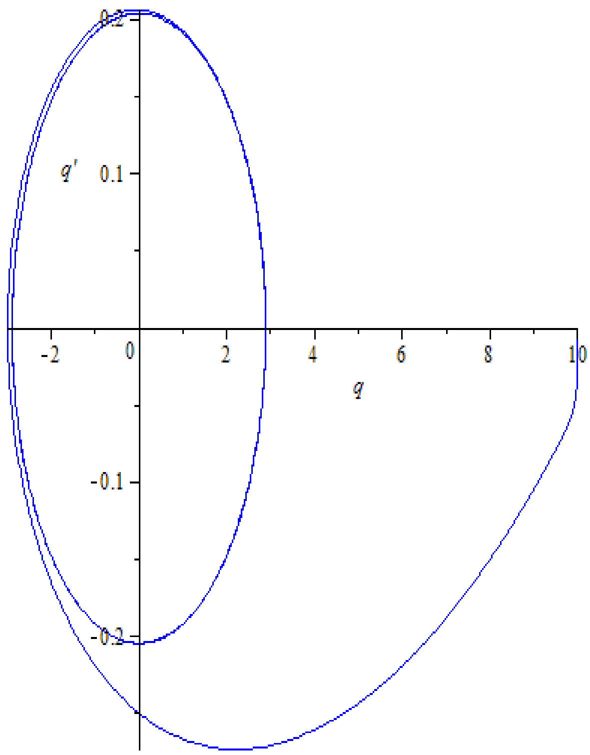

Fig.19: Current vs time

V. CONCLUSIONS

In this work biompedance both of a healthy neuron and a defective one is studied in detail through the use of a mathematical and physical model. Simulations were obtained by solving the differential equation for the RLC circuit. The model predicts the values of resistance of a healthy neuron and for a neuron which is affected by Parkinson's disease. The medication which a person affected by this disease takes aims to increase the neuron's resistance.

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