Control of discrete-time systems with state equality constraints

A. Filasová and D. Krokavec

***Abstract*—Design conditions for existence of memory-less feedback control for stabilization of discrete-time systems with equality constraints given on the state variables are presented in the paper. The design problem is addressed for linear discrete-time systems with equality constraints tying together some state variables. Using the classical memory-less feedback control principle LMI-based procedures are provided for computation of the gain matrix of state control laws, and influence of equality constraints is explained if a tracking problem be considered. The approach is successfully illustrated on simulation examples, where the validity of the proposed method is demonstrated with an equality constraint tying together all state variables of the system.**

***Keywords*—Equality constraints, linear matrix inequality, state feedback, control algorithms, singular systems, asymptotic stability.**

I. Introduction

In the last years many significant results have spurred interest in the problem of determining the control laws for the systems with constraints. For the typical case where a system state reflects a certain physical entities this class of constraints rises because of physical limits and these ones usually keep the system state in a region of the technological conditions. Some authors deal with problems of this kind that designing a control law such that states be driven to origin asymptotically while the coordinates of the command input are subject to unsymmetrical or symmetrical constraints [1], and [5], [4], [24], [26], respectively, others prefer respect of constraints by constructing a stabilizing memory-less controller with inequality defined on the control law gain matrix [3], [4], or solving problem using invariant set theory [13], [14]. Special attention is also focused on the principle of Kalman filtering with equality and inequality state constraints [8], [11], [16], where it is possible to reduce the system model, and use the reduced state equation for such systems, and for given linear state equality constraints.

However, this problem can be formulated using technique dealing with the state constraints directly, where the equations of both, the unconstraint system and the stabilized constraint relations are combined to a coupled system of equations which can be interpreted as a descriptor system [9],[28]. Because a system with state constraints generally does not satisfy the conditions under which the results of descriptor systems can be applicable this approach is limited in a realization.

In principle, it is possible and ever easy to apply a direct design method, namely to design a controller that stabilizes the systems and simultaneously forces the closed-loop systems to satisfy the constraint such that a special form of the constrained problems can be so formulated while the system state variables satisfy the equality constraints [21]. This technique for discrete-time multi-input/multi-output (MIMO) systems has been introduced in [15] and was extensively used in the state constrained control [6], as well as in the reconfigurable control design [18], [19]. Used principle can be applied in optimization of active multivariable combustion control and proportional control.

A number of problems that arise in the state feedback control can be reduced to a handful of standard convex and quasi-convex problems that involve matrix inequalities. It is known that the optimal solution can be computed by using interior point methods [20] which converge in polynomial time with respect to the problem size and efficient interior point algorithms have recently been developed for and further development of algorithms for these standard problems is an area of active research. For this approach, the stability conditions may be expressed in terms of linear matrix inequalities (LMI), which have a notable practical interest due to the existence of powerful numerical solvers. Some progres review in this field can be found in [2], [23], and the references therein.

This paper aims at providing controller design conditions with closed-loop state equality constraints for discrete time systems using quadratic in the state and linear in the parameters Lyapunov function, as well as its modification known as bounded real lemma [2]. Such a restriction does not lead to very conservative results, and design conditions are simple to be established as a set of LMIs which can be solved numerically with the help of an LMI solver. The task considered in the paper is to design state feedback control of discrete-time linear systems that forces selected state variables of a linear system to satisfy prescribed equality constraint relation and guarantees control system asymptotical stability. Based on the discrete-time linear system state description the generalized controller structure is formulated, and associated with the standard forms of the controller structure for time-invariant discrete control under defined state equality constraints.

The paper is organized as follows. Starting with problem formulation presented in Section II, then in Section III basic preliminaries are introduced together with an adapted version of discrete algebraic Riccati equation, referred to as equivalent form. These results are used in Section IV to derive a convex formulation of design conditions where closed-loop state equality constraints are considered. The proposed approaches lead to a set of LMIs to prove asymptotic stability conditions. Subsequently, in Section V a numerical example is presented to illustrate basic properties of these approaches. Section VI is finally devoted to a brief overview of the method properties demonstrating accepted conservatism of the proposed approach.

II. Problem Formulation

Through the paper the task is concerned with design of the state feedback (3) which controls a discrete-time linear dynamic system given by the set of state equations

$$\boldsymbol{q}(i+1) = \boldsymbol{F}\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i) \tag{1}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}\boldsymbol{q}(i) \tag{2}$$

where $\boldsymbol{q}(i) \in I\!R^n$, $\boldsymbol{u}(i) \in I\!R^r$, and $\boldsymbol{y}(i) \in I\!R^m$ are vectors of the state, input and objective variables, respectively, and nominal system matrices $\boldsymbol{F} \in I\!R^{n\times n}$, $\boldsymbol{G} \in I\!R^{n\times r}$, and $\boldsymbol{C} \in I\!R^{m\times n}$ are real matrices. Problem of the interest is to design an asymptotically stable closed-loop system using a linear memoryless state feedback controller of the form

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i) \tag{3}$$

while all state variables are measurable, $\boldsymbol{K} \in I\!R^{r\times n}$ is the feedback controller gain matrix, and design constraint in the next equality form

$$\boldsymbol{q}(i) \in \mathcal{N}_{\boldsymbol{D}} = \{\boldsymbol{q} : \boldsymbol{D}\boldsymbol{q} = \boldsymbol{0}\} \tag{4}$$

is considered, with $\boldsymbol{D} \in I\!R^{k\times n}$, $\text{rank}\boldsymbol{D} = k \le r$.

To optimize the state feedback controller parameters while the system state variables satisfy the equality constraints the design task is specified be singular.

III. Basic Preliminaries

Proposition 3.1: (e.g. see [16], [23]) Let $\boldsymbol{\Lambda}$ is a matrix variable and $\boldsymbol{A}$, $\boldsymbol{B}$ are known non-square matrices of appropriate dimensions such the equality

$$\boldsymbol{B}\boldsymbol{\Lambda} = \boldsymbol{A} \tag{5}$$

can be set. Then all solution to $\boldsymbol{\Lambda}$ means

$$\boldsymbol{\Lambda} = \boldsymbol{B}^{\ominus 1}\boldsymbol{A} + (\boldsymbol{I} - \boldsymbol{B}^{\ominus 1}\boldsymbol{B})\boldsymbol{\Lambda}^{\circ} \tag{6}$$

where

$$\boldsymbol{B}^{\ominus 1} = \boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-1} \tag{7}$$

is Moore-Penrose pseudoinverse of $\boldsymbol{B}$ and $\boldsymbol{\Lambda}^{\circ}$ is an arbitrary matrix of appropriate dimension.

Proof: Supposing that the product $\boldsymbol{B}\boldsymbol{B}^T$ is a regular matrix, then pre-multiplying left-hand side of (5) by the identity matrix gives

$$\boldsymbol{B}\boldsymbol{\Lambda} = \boldsymbol{B}\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-1}\boldsymbol{A} \tag{8}$$

and with (7) it yields

$$\boldsymbol{\Lambda} = \boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-1}\boldsymbol{A} = \boldsymbol{B}^{\ominus 1}\boldsymbol{A} \tag{9}$$

Let $\boldsymbol{\Lambda}^{\circ}$ is another matrix of appropriate dimension such that substituting in (5) it can be written

$$\boldsymbol{B}\boldsymbol{\Lambda}^{\circ} = \boldsymbol{B}\boldsymbol{B}^{\ominus 1}\boldsymbol{A} = \boldsymbol{B}\boldsymbol{B}^{\ominus 1}\boldsymbol{B}\boldsymbol{\Lambda}^{\circ} \tag{10}$$

Thus,

$$\boldsymbol{B}(\boldsymbol{I} - \boldsymbol{B}^{\ominus 1}\boldsymbol{B})\boldsymbol{\Lambda}^{\circ} \doteq \boldsymbol{0} \tag{11}$$

$$(\boldsymbol{I} - \boldsymbol{B}^{\ominus 1}\boldsymbol{B})\boldsymbol{\Lambda}^{\circ} \doteq \boldsymbol{0} \tag{12}$$

respectively. Therefore, for an arbitrary $\boldsymbol{\Lambda}^{\circ}$ (9), (12) implies (6). ∎

Note, matrix pseudoinverse is generalized for a singular matrix $\boldsymbol{B}\boldsymbol{B}^T$, usually written as

$$\boldsymbol{B}^{\ominus 1} = \boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{\dagger} \tag{13}$$

Proposition 3.2: Let $\boldsymbol{H} \in I\!R^{n\times n}$ is a real square matrix with non-repeated eigenvalues, satisfying the equality constraint

$$\boldsymbol{d}^T\boldsymbol{H} = \boldsymbol{0} \tag{14}$$

Then one from its eigenvalues is zero, and (normalized) $\boldsymbol{d}^T$ is the left raw eigenvector of $\boldsymbol{H}$ associated with the zero eigenvalue.

Proof: If $\boldsymbol{H} \in I\!R^{n\times n}$ is a real square matrix having non-repeated eigenvalues then the eigenvalue decomposition of $\boldsymbol{H}$ takes the form

$$\boldsymbol{H} = \boldsymbol{U}\boldsymbol{Z}\boldsymbol{V}^T \tag{15}$$

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_n \end{bmatrix}, \; \boldsymbol{V} = \begin{bmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{bmatrix} \tag{16}$$

$$\boldsymbol{Z} = \text{diag}\begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix}, \quad \boldsymbol{U}^T\boldsymbol{V} = \boldsymbol{I} \tag{17}$$

where $\boldsymbol{u}_l$, is right eigenvector, and $\boldsymbol{v}_l^T$ is left eigenvector associated with the eigenvalue z_l of $\boldsymbol{H}$, $l = 1, 2, \ldots n$. Then (14) can be rewritten as

$$\begin{aligned} &\boldsymbol{d}^T \begin{bmatrix} \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_h & \cdots \boldsymbol{u}_n \end{bmatrix} \\ &\cdot \text{diag}\begin{bmatrix} z_1 & \cdots & z_h & \cdots & z_n \end{bmatrix} \boldsymbol{V}^T = 0 \end{aligned} \tag{18}$$

If $\boldsymbol{d}^T = \boldsymbol{v}_h^T$ then orthogonal property (17) implies

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{0}_1 & \cdots & \boldsymbol{1}_h & \cdots \boldsymbol{0}_n \end{bmatrix} \\ &\cdot \text{diag}\begin{bmatrix} z_1 & \cdots & z_h & \cdots & z_n \end{bmatrix} \boldsymbol{V}^T = \boldsymbol{0} \end{aligned} \tag{19}$$

and it is evident that (19) can be satisfied only if $z_h = 0$. ∎

Proposition 3.3: (Schur complement) Let $\boldsymbol{Q} > 0$, $\boldsymbol{R} > 0$, $\boldsymbol{S}$ are real matrices of appropriate dimensions, then the next inequalities are equivalent

$$\begin{aligned} \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R} \end{bmatrix} > 0 \;&\Leftrightarrow\; \begin{bmatrix} \boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} > 0 \\ &\Updownarrow \\ \boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}&\boldsymbol{S}^T > 0, \quad \boldsymbol{R} > 0 \end{aligned} \tag{20}$$

Proof: Let a linear matrix inequality takes the form

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R} \end{bmatrix} > 0 \tag{21}$$

then using Gauss elimination it yields

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{I} & -\boldsymbol{S}\boldsymbol{R}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{R}^{-1}\boldsymbol{S}^T & \boldsymbol{I} \end{bmatrix} = \\ &= \begin{bmatrix} \boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} \end{aligned} \tag{22}$$

$$\det \begin{bmatrix} \boldsymbol{I} & \boldsymbol{S}\boldsymbol{R}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} = 1 \tag{23}$$

and it is evident that this transform doesn‘t change positivity of (21), and so (22) implies (20). ∎

Proposition 3.4: Let for given matrices $\boldsymbol{M}$, $\boldsymbol{N}$ and $\boldsymbol{\Theta} = \boldsymbol{\Theta}^T > 0$ of appropriate dimension a matrix $\boldsymbol{X}$ has to satisfy the inequality

$$\boldsymbol{M}\boldsymbol{X}\boldsymbol{N}^T + \boldsymbol{N}\boldsymbol{X}^T\boldsymbol{M}^T - \boldsymbol{\Theta} < 0 \tag{24}$$

then any solution of $\boldsymbol{X}$ can be generated using a solution of inequality

$$\begin{bmatrix} -\boldsymbol{M}\boldsymbol{H}^{-1}\boldsymbol{M}^T-\boldsymbol{\Theta} & \boldsymbol{M}\boldsymbol{H}^{-1}+\boldsymbol{N}\boldsymbol{X}^T \\ * & -\boldsymbol{H}^{-1} \end{bmatrix} < 0 \qquad (25)$$

where $\boldsymbol{H} = \boldsymbol{H}^T > 0$ is a free design parameter.
Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof: If (24) yields then there exists a matrix $\boldsymbol{H} = \boldsymbol{H}^T > 0$ such that

$$\boldsymbol{M}\boldsymbol{X}\boldsymbol{N}^T + \boldsymbol{N}\boldsymbol{X}^T\boldsymbol{M}^T - \boldsymbol{\Theta} + \boldsymbol{N}\boldsymbol{X}^T\boldsymbol{H}\boldsymbol{X}\boldsymbol{N}^T < 0 \qquad (26)$$

Completing the square in (26) it can be obtained

$$\begin{aligned} &(\boldsymbol{M}\boldsymbol{H}^{-1}+\boldsymbol{N}\boldsymbol{X}^T)\boldsymbol{H}(\boldsymbol{M}\boldsymbol{H}^{-1}+\boldsymbol{N}\boldsymbol{X}^T)^T - \\ &-\boldsymbol{M}\boldsymbol{H}^{-1}\boldsymbol{M}^T - \boldsymbol{\Theta} < 0 \end{aligned} \qquad (27)$$

and using Schur complement (27) implies (25). ■

IV. Constrained Control Design

A. Constrained Control

The above format of the stabilization problem with the pure matrix algebraic equation constraint of the form

Using control law (3) the equilibrium control equation takes the form

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) \qquad (28)$$

$$\boldsymbol{z}(i) = \boldsymbol{C}\boldsymbol{q}(i) \qquad (29)$$

The format of the stabilization problem with the pure matrix algebraic equation constraint is prescribed by a matrix $\boldsymbol{D} \in \mathbb{R}^{k \times n}$, $\text{rank}\boldsymbol{D} = k < r$ to give the design constraint

$$\boldsymbol{q}(i) \in \mathcal{N}_{\boldsymbol{D}} = \{\boldsymbol{q} : \boldsymbol{D}\boldsymbol{q} = \boldsymbol{0}\} \qquad (30)$$

implying that the state-variable vectors have to satisfy equalities

$$\boldsymbol{D}\boldsymbol{q}(i+1) = \boldsymbol{D}(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) = \boldsymbol{0} \qquad (31)$$

for $i = 1, 2, \ldots$. It is supposed the matrix $\boldsymbol{D}$ is chosen by such way that

$$\boldsymbol{D}(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}) = \boldsymbol{0} \qquad (32)$$

$$\boldsymbol{D}\boldsymbol{F} = \boldsymbol{D}\boldsymbol{G}\boldsymbol{K} \qquad (33)$$

respectively, as well as that the closed-loop system matrix $(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})$ is stable (all its eigenvalues lie in the unit circle in the complex plane $\mathcal{Z}$).

Therefore, $\mathcal{N}_{\boldsymbol{D}}$ is the constrain subspace, and the states be constrained in this subspace (the null space of $\boldsymbol{D}$). Under these conditions the system state stays within the constrain subspace, i.e. $\boldsymbol{q}(i), \boldsymbol{F}\boldsymbol{q}(i) \in \mathcal{N}_{\boldsymbol{D}}$.

Equality (14) implies that constrain control design condition (32) results in a singular matrix form. Because such system with state constraints generally does not satisfy the conditions under which the results of descriptor systems can be applicable, special design methods have to be proposed so solve this design task.

Solving (33) with respect to $\boldsymbol{K}$ then (6) implies all solutions of $\boldsymbol{K}$ as follows

$$\boldsymbol{K} = (\boldsymbol{D}\boldsymbol{G})^{\ominus 1}\boldsymbol{D}\boldsymbol{F} + (\boldsymbol{I} - (\boldsymbol{D}\boldsymbol{G})^{\ominus 1}\boldsymbol{D}\boldsymbol{G})\boldsymbol{K}^{\circ} \qquad (34)$$

where $\boldsymbol{K}^{\circ}$ is an arbitrary matrix with appropriated dimension and

$$(\boldsymbol{D}\boldsymbol{G})^{\ominus 1} = (\boldsymbol{D}\boldsymbol{G})^T\big(\boldsymbol{D}\boldsymbol{G}(\boldsymbol{D}\boldsymbol{G})^T\big)^{\dagger} \qquad (35)$$

Thus, it is possible express (34) as

$$\boldsymbol{K} = \boldsymbol{J} + \boldsymbol{L}\boldsymbol{K}^{\circ} \qquad (36)$$

where

$$\boldsymbol{J} = (\boldsymbol{D}\boldsymbol{G})^{\ominus 1}\boldsymbol{D}\boldsymbol{F} \qquad (37)$$

and

$$\boldsymbol{L} = \boldsymbol{I} - (\boldsymbol{D}\boldsymbol{G})^T\big(\boldsymbol{D}\boldsymbol{G}(\boldsymbol{D}\boldsymbol{G})^T\big)^{\dagger}\boldsymbol{D}\boldsymbol{G} \qquad (38)$$

is the projection matrix (the orthogonal projector onto the null space $\mathcal{N}_{\boldsymbol{D}\boldsymbol{G}}$ of $\boldsymbol{D}\boldsymbol{G}$) (e.g. see [16], [17]).

B. Control Parameter Design

Theorem 4.1: For the system (1) the sufficient condition for the stable control (3) with constrain (4) is that there exist a positive definite symmetric matrix $\boldsymbol{Y} > 0, \boldsymbol{Y} \in \mathbb{R}^{n \times n}$, and a matrix $\boldsymbol{Z} \in \mathbb{R}^{r \times n}$ such that

$$\boldsymbol{Y} = \boldsymbol{Y}^T > 0 \qquad (39)$$

$$\begin{bmatrix} -\boldsymbol{Y} & \boldsymbol{Y}(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{J})^T - \boldsymbol{Z}^T\boldsymbol{L}^T\boldsymbol{G}^T \\ * & -\boldsymbol{Y} \end{bmatrix} < 0 \qquad (40)$$

where $\boldsymbol{J}$, $\boldsymbol{L}$ are defined in (37), (38), respectively.

Thus, $\boldsymbol{K}^{\circ}$ can be computed as

$$\boldsymbol{K}^{\circ} = \boldsymbol{Z}\boldsymbol{Y}^{-1} \qquad (41)$$

and the control law gain matrix $\boldsymbol{K}$ is given as in (36).

Proof: Defining Lyapunov function as follows

$$v(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) > 0 \qquad (42)$$

where $\boldsymbol{P} = \boldsymbol{P}^T > 0$, $\boldsymbol{P} \in \mathbb{R}^{n \times n}$, then the forward diference along a solution of the system (1) is

$$\Delta v(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i+1)\boldsymbol{P}\boldsymbol{q}(i+1) - \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) < 0 \qquad (43)$$

$$\Delta v(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i)\big(\boldsymbol{F}_c^T\boldsymbol{P}\boldsymbol{F}_c - \boldsymbol{P}\big)\boldsymbol{q}(i) < 0 \qquad (44)$$

respectively, where

$$\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{J} - \boldsymbol{G}\boldsymbol{L}\boldsymbol{K}^{\circ} \qquad (45)$$

and (44) implies

$$\boldsymbol{P}_c^{\circ} = \boldsymbol{F}_c^T\boldsymbol{P}\boldsymbol{F}_c - \boldsymbol{P} < 0 \qquad (46)$$

Therefore, using Schur complement property it yields

$$\boldsymbol{P}_c^{\circ} = \begin{bmatrix} -\boldsymbol{P} & (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{J} - \boldsymbol{G}\boldsymbol{L}\boldsymbol{K}^{\circ})^T \\ * & -\boldsymbol{P}^{-1} \end{bmatrix} < 0 \qquad (47)$$

Defining the congruence transform matrix

$$\boldsymbol{T}_{c1} = \text{diag}\begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{I}_n \end{bmatrix} \qquad (48)$$

and multiplying right-hand and left-hand side of (47) by $\boldsymbol{T}_{c1}$ it can be obtained

$$\begin{bmatrix} -\boldsymbol{P}^{-1} & \boldsymbol{P}^{-1}(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{J} - \boldsymbol{G}\boldsymbol{L}\boldsymbol{K}^{\circ})^T \\ * & -\boldsymbol{P}^{-1} \end{bmatrix} < 0 \qquad (49)$$

and with notation

$$\boldsymbol{P}^{-1}=\boldsymbol{Y}=\boldsymbol{Y}^T>0, \quad \boldsymbol{K}^{\circ}\boldsymbol{P}^{-1}=\boldsymbol{Z} \tag{50}$$

(49) implies (40). ■

Analogously, as above the problem concerning with non-expansive conditions can be also formulated as a pure matrix algebraic-equation based constrained task with appropriated modification of Lyapunov function. The objective is to assign stabile eigenvalues to the system, and simultaneously force the constraint equation to be satisfied.

Theorem 4.2: (Bounded real lemma) For the system (1), (2) the sufficient condition for a stable control (3) with constrain (4) is that there exist a positive definite symmetric matrix $\boldsymbol{Q}>0$, $\boldsymbol{Q}\in I\!R^{n\times n}$, a matrix $\boldsymbol{K}^{\circ}\in I\!R^{r\times n}$, and a scalar $\gamma>0$, $\gamma\in I\!R$ such that

$$\boldsymbol{Q}=\boldsymbol{Q}^T>0 \tag{51}$$

$$\begin{bmatrix} -\boldsymbol{Q} & \boldsymbol{F}^{\bullet T} & \boldsymbol{F}^{\bullet T}\boldsymbol{Q}\boldsymbol{G}^{\circ} & \boldsymbol{C}^T \\ * & -\boldsymbol{Q}^{-1} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & \boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ}-\gamma\boldsymbol{I}_r & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix}<0 \tag{52}$$

where

$$\boldsymbol{F}^{\bullet}=\boldsymbol{F}^{\circ}-\boldsymbol{G}^{\circ}\boldsymbol{K}^{\circ} \tag{53}$$

$$\boldsymbol{F}^{\circ}=\boldsymbol{F}-\boldsymbol{G}\boldsymbol{J}, \qquad \boldsymbol{G}^{\circ}=\boldsymbol{G}\boldsymbol{L} \tag{54}$$

The control law gain matrix $\boldsymbol{K}$ is given as in (36).

Proof: Inserting (36) into (1) gives

$$\boldsymbol{q}(i+1)=\boldsymbol{F}^{\circ}\boldsymbol{q}(i)+\boldsymbol{G}^{\circ}\boldsymbol{u}(i) \tag{55}$$

$$\boldsymbol{z}(i)=\boldsymbol{C}\boldsymbol{q}(i) \tag{56}$$

Then, defining Lyapunov function for the system (55), (56) as follows

$$\begin{aligned} v(\boldsymbol{q}(i)) &= \boldsymbol{q}^T(i)\boldsymbol{Q}\boldsymbol{q}(i)+ \\ &+\sum_{l=0}^{i-1}(\boldsymbol{z}^T(l)\boldsymbol{z}(l)-\gamma\boldsymbol{u}^T(l)\boldsymbol{u}(l))>0 \end{aligned} \tag{57}$$

the forward diference along a solution of the system (55), (56) is

$$\begin{aligned} \Delta v(\boldsymbol{q}(i)) &= \boldsymbol{q}^T(i+1)\boldsymbol{Q}\boldsymbol{q}(i+1)- \\ &-\boldsymbol{q}^T(i)\boldsymbol{Q}\boldsymbol{q}(i)+\boldsymbol{z}^T(i)\boldsymbol{z}(i)-\gamma\boldsymbol{u}^T(i)\boldsymbol{u}(i)<0 \end{aligned} \tag{58}$$

$$\begin{aligned} &\Delta v(\boldsymbol{q}(i)) = \\ &= \boldsymbol{q}^T(i)(\boldsymbol{C}^T\boldsymbol{C}-\boldsymbol{Q}+\boldsymbol{F}^{\circ T}\boldsymbol{Q}\boldsymbol{F}^{\circ})\boldsymbol{q}(i)+ \\ &+\boldsymbol{u}^T(i)\boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{F}^{\circ}\boldsymbol{q}(i)+\boldsymbol{q}^T(i)\boldsymbol{F}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ}\boldsymbol{u}(i)+ \\ &+\boldsymbol{u}^T(i)(\boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ}-\gamma\boldsymbol{I}_r)\boldsymbol{u}(i)<0 \end{aligned} \tag{59}$$

respectively, where (2) was inserted. Thus, using vector notation

$$\boldsymbol{q}_c^T(i)=\begin{bmatrix} \boldsymbol{q}^T(i) & \boldsymbol{u}^T(i) \end{bmatrix} \tag{60}$$

it can be obtained

$$\Delta v(\boldsymbol{q}_c(i))=\boldsymbol{q}_c^T(i)\boldsymbol{P}_c^{\bullet}\boldsymbol{q}_c(i)<0 \tag{61}$$

where

$$\boldsymbol{P}_c^{\bullet}=\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ * & \boldsymbol{A}_{22} \end{bmatrix}<0 \tag{62}$$

$$\boldsymbol{A}_{11}=\boldsymbol{F}^{\circ T}\boldsymbol{Q}\boldsymbol{F}^{\circ}+\boldsymbol{C}^T\boldsymbol{C}-\boldsymbol{Q} \tag{63}$$

$$\boldsymbol{A}_{12}=\boldsymbol{F}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ}, \quad \boldsymbol{A}_{22}=\boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ}-\gamma\boldsymbol{I}_r \tag{64}$$

Using Schur complement property then (62) can be rewritten as follows

$$\begin{bmatrix} \boldsymbol{F}^{\circ T}\boldsymbol{Q}\boldsymbol{F}^{\circ}-\boldsymbol{Q} & \boldsymbol{F}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ} & \boldsymbol{C}^T \\ * & \boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ}-\gamma\boldsymbol{I}_r & \boldsymbol{0} \\ * & * & -\boldsymbol{I}_m \end{bmatrix}<0 \tag{65}$$

It is obvious that (65) gives the stability condition of (55) in the bounded real lemma form.

Replacing $\boldsymbol{F}^{\circ}$ by $\boldsymbol{F}^{\bullet}$ it can be written

$$\begin{bmatrix} \boldsymbol{F}^{\bullet T}\boldsymbol{Q}\boldsymbol{F}^{\bullet}-\boldsymbol{Q} & \boldsymbol{F}^{\bullet T}\boldsymbol{Q}\boldsymbol{G}^{\circ} & \boldsymbol{C}^T \\ * & \boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ}-\gamma\boldsymbol{I}_r & \boldsymbol{0} \\ * & * & -\boldsymbol{I}_m \end{bmatrix}<0 \tag{66}$$

and it is obvious that (66) implies (52). ■

Since of unknown $\boldsymbol{K}^{\circ}$ it is evident that (52) is not an LMI, and only a conservative solution can by obtained generally solving (52).

Another way to solve this problem is Finsley lemma application [10] within unified algebraic approach. Note this method doesn't improve solutions conservatism.

C. Unified Algebraic Approach

Using a similar reasoning provided in the proof of Theorem 4.2 it can be arrived the following result.

Theorem 4.3: Let are given $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{J}$, and $\boldsymbol{L}$. Then the constrained control is stable if there exits a matrix $\boldsymbol{Q}>0$ such that

$$\boldsymbol{Q}=\boldsymbol{Q}>0 \tag{67}$$

$$\boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ}-\gamma\boldsymbol{I}_r<0 \tag{68}$$

Then the gain matrix $\boldsymbol{K}^{\circ}$ exists if for obtained $\boldsymbol{Q}$ there exist a symmetric matrix $\boldsymbol{H}>0$, $\boldsymbol{H}\in I\!R^{r\times r}$, and a matrix $\boldsymbol{\Phi}$ such that

$$\begin{bmatrix} -\boldsymbol{M}\boldsymbol{H}^{-1}\boldsymbol{M}^T-\boldsymbol{\Theta} & \boldsymbol{M}\boldsymbol{H}^{-1}+\boldsymbol{N}^T\boldsymbol{K}^{\circ T} \\ * & -\boldsymbol{H}^{-1} \end{bmatrix}<0 \tag{69}$$

where

$$\begin{aligned} &\boldsymbol{\Theta}=\boldsymbol{\Phi}- \\ &-\begin{bmatrix} \boldsymbol{F}^{\circ T}\boldsymbol{Q}\boldsymbol{F}^{\circ}-\boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{F}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ} & \boldsymbol{C}^T \\ * & -\boldsymbol{Q}^{-1} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & \boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ}-\gamma\boldsymbol{I}_r & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} \end{aligned} \tag{70}$$

$$\boldsymbol{M}=\begin{bmatrix} -\boldsymbol{F}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ} \\ -\boldsymbol{G}^{\circ} \\ \boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ} \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{N}=\begin{bmatrix} \boldsymbol{I}_n \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{71}$$

and $\boldsymbol{\Phi}>0$, $\boldsymbol{\Phi}\in I\!R^{(2n+m+r)\times(2n+m+r)}$ is an arbitrary positive definite matrix such that $\boldsymbol{\Theta}$ be positive definite. The control law gain matrix $\boldsymbol{K}$ is given as in (36).

Proof: Inserting (53) inequality (52) can be rewritten as

$$\begin{bmatrix} -\boldsymbol{Q} & \boldsymbol{F}^{\circ T} & \boldsymbol{F}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ} & \boldsymbol{C}^T \\ * & -\boldsymbol{Q}^{-1} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & \boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ}-\gamma\boldsymbol{I}_r & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} + \\ + \begin{bmatrix} \boldsymbol{0} \\ -\boldsymbol{G}^{\circ} \\ \boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{K}^{\circ} \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} + \\ + \begin{bmatrix} \boldsymbol{I}_n \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{K}^{\circ T} \begin{bmatrix} \boldsymbol{0} & -\boldsymbol{G}_h^{\circ T} & \boldsymbol{G}_h^{\circ T}\boldsymbol{Q}_h\boldsymbol{G}_h^{\circ} & \boldsymbol{0} \end{bmatrix} < 0 \tag{72}$$

Introducing the congruence transform matrix

$$\boldsymbol{T}_{c2} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{F}^{\circ T}\boldsymbol{Q} & & \\ & \boldsymbol{I}_n & & \\ & & \boldsymbol{I}_r & \\ & & & \boldsymbol{I}_m \end{bmatrix} \tag{73}$$

then multiplying left-hand side of (72) by (73), and right-hand side of (72) by the transposition of (73) gives

$$\begin{bmatrix} \boldsymbol{F}^{\circ T}\boldsymbol{Q}\boldsymbol{F}^{\circ}-\boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{F}^{\circ T}\boldsymbol{Q}_h\boldsymbol{G}^{\circ} & \boldsymbol{C}^T \\ * & -\boldsymbol{Q}^{-1} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & \boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ}-\gamma\boldsymbol{I}_r & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} + \\ +\boldsymbol{M}\boldsymbol{K}^{\circ}\boldsymbol{N}^T + \boldsymbol{N}\boldsymbol{K}^{\circ T}\boldsymbol{M}^T < 0 \tag{74}$$

Since the orthogonal complement to $\boldsymbol{N}$ is

$$\boldsymbol{N}^{\perp} = \begin{bmatrix} \boldsymbol{I}_n \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}^{\perp} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_n & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix} \tag{75}$$

multiplying left-hand side of (74) by (75), and right-hand side of (74) by the transposition of (75) gives

$$\text{diag}\begin{bmatrix} -\boldsymbol{Q}^{-1} & \boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ}-\gamma\boldsymbol{I}_r & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{76}$$

and subsequently, (76) implies (68).

To obtain a solution the orthogonal complement $\boldsymbol{M}^{\perp}$ can be defined as follows

$$\boldsymbol{M}^{\perp} = \begin{bmatrix} -\boldsymbol{F}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ} \\ -\boldsymbol{G}^{\circ} \\ \boldsymbol{G}^{\circ T}\boldsymbol{Q}\boldsymbol{G}^{\circ} \\ \boldsymbol{0} \end{bmatrix}^{\perp} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{G}^{\circ\perp} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix} \tag{77}$$

Then multiplying left-hand side of (74) by (77), and right-hand side of (74) by the transposition of (77) gives

$$\text{diag}\begin{bmatrix} -\boldsymbol{G}^{\circ\perp}\boldsymbol{Q}^{-1}\boldsymbol{G}^{\circ\perp T} & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{78}$$

It is evident, that (78) is satisfied for all positive definite $\boldsymbol{Q}$ and so the design condition be given by (68).

Subsequently, writing (74) in the form

$$\boldsymbol{M}\boldsymbol{K}^{\circ}\boldsymbol{N}^T + \boldsymbol{N}\boldsymbol{K}^{\circ T}\boldsymbol{M}^T - \boldsymbol{\Theta} < 0 \tag{79}$$

then comparing with (24), (25) it is obvious that (79) implies (69). ∎

The additive design parameters $\boldsymbol{H}$, $\boldsymbol{\Phi}$ bring about a more conservative solution, potentially using for control properties tuning.

D. Constrained tracking problem

Considering the tracking problem defined by the control policy

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i) + \mathbf{W}\mathbf{w}(i) \tag{80}$$

where $\mathbf{w}(i) \in I\!R^r$ is the tracking control desired vector signal, and $\mathbf{W} \in I\!R^{r\times r}$ is the gain matrix of tracking signal, and a forced motion of the system (1), (2) can be written in the form

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) + \boldsymbol{G}\mathbf{W}\mathbf{w}(i) \tag{81}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}\boldsymbol{q}(i) \tag{82}$$

If $\mathbf{q}(0) = \mathbf{0}$ and $m = r$ it is possible to write

$$\tilde{\boldsymbol{q}}(z) = (z\boldsymbol{I} - \boldsymbol{F}_c)^{-1}\boldsymbol{G}\boldsymbol{W}\tilde{\boldsymbol{w}}(z) \tag{83}$$

where

$$\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K} \tag{84}$$

Then the state space description (1), (2) is related by the matrix transfer function

$$\boldsymbol{G}_c(z) = \frac{\tilde{\boldsymbol{y}}(z)}{\tilde{\boldsymbol{w}}(z)} = \boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{F}_u)^{-1}\boldsymbol{G}\boldsymbol{W} \tag{85}$$

This function is said to be coupled if any individual input influences more than one output. If $m = r$ the matrix transfer function $\boldsymbol{G}_c(z)$ be a square matrix, and considering

$$\lim_{z\to 1}(z-1)\tilde{\boldsymbol{y}}(z) = \lim_{z\to 1}(z-1)\boldsymbol{G}_c(z)\tilde{\boldsymbol{w}}(z) \tag{86}$$

it is possible to set

$$\lim_{z\to 1}\boldsymbol{G}_c(z) = \boldsymbol{C}(\boldsymbol{I} - \boldsymbol{F}_c)^{-1}\mathbf{G}\mathbf{W} = \mathbf{I}_m \tag{87}$$

Thus, if $\mathbf{G}_c(1)$ is non-singular and $\boldsymbol{A}_c$ is stable then

$$\boldsymbol{W} = \big(\boldsymbol{C}(\boldsymbol{I} - \boldsymbol{F}_c)^{-1}\boldsymbol{G}\big)^{-1} \tag{88}$$

and (88) results static system decoupling. The static decoupling problem by state feedback is solvable if and only if [25] $(\boldsymbol{F}, \boldsymbol{G})$ is stabilizable and

$$\text{rank}\begin{bmatrix} \boldsymbol{F} & \boldsymbol{G} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = n + m \tag{89}$$

Theorem 4.4: Using state control satisfying equality constrain (30) in forced mode then state constraints given on the system state variables attains the state steady-state value

$$\boldsymbol{q}_d = \boldsymbol{D}\boldsymbol{W}\boldsymbol{w}_s \tag{90}$$

where $q_d(i) = \boldsymbol{D}\boldsymbol{q}(i)$ is a common state variable.

Proof: Pre-multiplying left-hand side of (81) by $\boldsymbol{D}$ gives

$$\boldsymbol{D}\boldsymbol{q}(i+1) = \boldsymbol{D}(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) + \boldsymbol{D}\boldsymbol{G}\mathbf{W}\mathbf{w}(i) \tag{91}$$

It is evident that using (30) the equality (91) implies (90) ∎

In comparison, using the state control satisfying equality constrains on state variables in unforced (autonomous) mode there the state variables hold constrain conditions in all time instant $i = 1, 2, \ldots$. (in an initial time instant only if $\boldsymbol{D}\boldsymbol{q}(0) = \mathbf{0}$.)

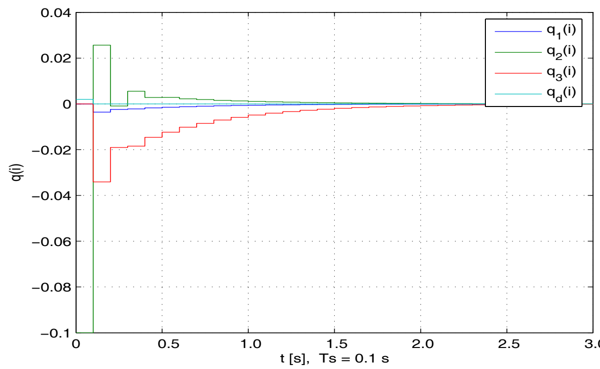

Fig. 1. Step response of the closed-loop system (AM)

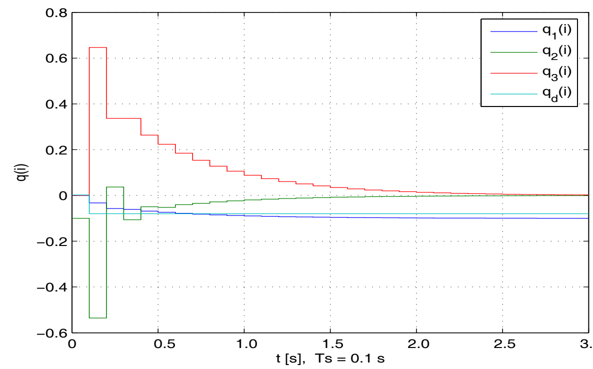

Fig. 2. Step response of the closed-loop system (FM)

V. Illustrative Example

To demonstrate properties of the proposed approach, the system with two-inputs and two-outputs is used in the example. The parameters of this system were

$$\boldsymbol{F} = \begin{bmatrix} 0.9993 & 0.0987 & 0.0042 \\ -0.0212 & 0.9612 & 0.0775 \\ -0.3875 & -0.7187 & 0.5737 \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} 0.0010 & 0.0010 \\ 0.0206 & 0.0197 \\ 0.0077 & -0.0078 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix}$$

respectively, for sampling period $\Delta t = 0.1$ s. The state constraint was specified as

$$\frac{0.8\, q_1(t) - 0.02\, q_2(t)}{q_3(t)} = 0.1$$

which implies

$$\boldsymbol{D} = \begin{bmatrix} 0.8 & -0.02 & -0.1 \end{bmatrix}$$

and subsequently it yields

$$(\boldsymbol{D}\boldsymbol{G})^{\ominus 1} = \begin{bmatrix} -48.2558 \\ 154.2200 \end{bmatrix}, \quad \boldsymbol{L} = \begin{bmatrix} 0.9108 & 0.2850 \\ 0.2850 & 0.0892 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} -40.4680 & -6.3507 & 2.6811 \\ 129.3311 & 20.2963 & -8.5685 \end{bmatrix}$$

Solving (39), (40) with respect to the LMI matrix variables $\boldsymbol{Y}$ and $\boldsymbol{Z}$ using Self–Dual–Minimization (SeDuMi) package for Matlab [22], the feedback gain matrix design problem in the constrained control was solved as feasible with the matrices

$$\boldsymbol{Y} = \begin{bmatrix} 0.0226 & -0.0732 & 0.0827 \\ -0.0732 & 1.0788 & -0.3398 \\ 0.0827 & -0.3398 & 1.2889 \end{bmatrix}$$

$$\boldsymbol{Z} = \begin{bmatrix} -0.3038 & 0.0329 & 0.5665 \\ -0.0951 & 0.0103 & 0.1773 \end{bmatrix}$$

Inserting $\boldsymbol{Y}$ and $\boldsymbol{Z}$ into (41) there were computed the feedback gain matrices as follows

$$\boldsymbol{K}^{\circ} = \begin{bmatrix} -22.5626 & -0.9876 & 1.6276 \\ -7.0599 & -0.3090 & 0.5093 \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} -63.0305 & -7.3384 & 4.3087 \\ 122.2712 & 19.9872 & -8.0592 \end{bmatrix}$$

The closed loop is stable with the system matrix eigenvalue spectrum

$$\rho(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}) = \{\ 0.0000 \quad 0.8311 \quad -0.2838\ \}$$

It is evident that one eigenvalue of $\boldsymbol{F}$–$\boldsymbol{G}\boldsymbol{K}$ is zero (rank($\boldsymbol{D}$) = 1)) since the constrain control design task is a singular problem [18].

To demonstrate the forced regime properties the control policy (80) the gain matrix of tracking signal $\boldsymbol{W}$ was computed as

$$\boldsymbol{W} = \begin{bmatrix} -4.5931 & -53.5173 \\ 6.8725 & 110.6048 \end{bmatrix}$$

and simulation was done using

$$\boldsymbol{q}^T(0) = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}, \qquad \boldsymbol{w}^T(i) = \begin{bmatrix} -0.1 & -0.1 \end{bmatrix}$$

In autonomous mode there was used the same initial system state vector. In the presented figures the examples are shown of the closed-loop system step response, where Figure V represents the autonomous mode step response and Figure V is concerned with the forced mode, respectively. The control law parameters are designed to satisfy Lyapunov inequality (40).

Solving (67), (68) with respect to the LMI matrix variables $\boldsymbol{Q}$, γ, and subsequently (69) with respect to $\boldsymbol{K}^{\circ}$ the problem was solved also as feasible with

$$\gamma = 0.7809$$

$$\boldsymbol{Q} = \begin{bmatrix} 0.6820 & -0.0001 & -0.0000 \\ -0.0001 & 0.6807 & -0.0003 \\ -0.0000 & -0.0003 & 0.6820 \end{bmatrix}$$

$$\boldsymbol{K}^{\circ} = \begin{bmatrix} -12.9664 & -0.6628 & 2.1789 \\ -4.0572 & -0.2074 & 0.6818 \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} -53.4344 & -7.0136 & 4.8600 \\ 125.2738 & 20.0889 & -7.8867 \end{bmatrix}$$

where design parameters were set as $\boldsymbol{H} = 0.01\boldsymbol{I}_2$, $\boldsymbol{\Phi} = 3.7\boldsymbol{I}_{10}$, respectively. Resulting control is stable with the closed-loop system matrix eigenvalue spectrum

$$\rho(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}) = \{\ 0.0000 \quad 0.6799 \quad -0.2547\ \}$$

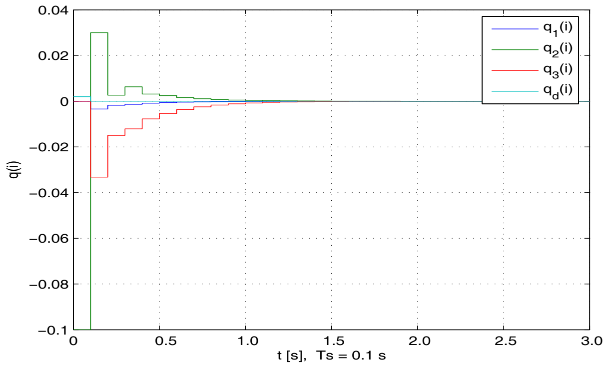

Fig. 3. Step response of the closed-loop system (AM)

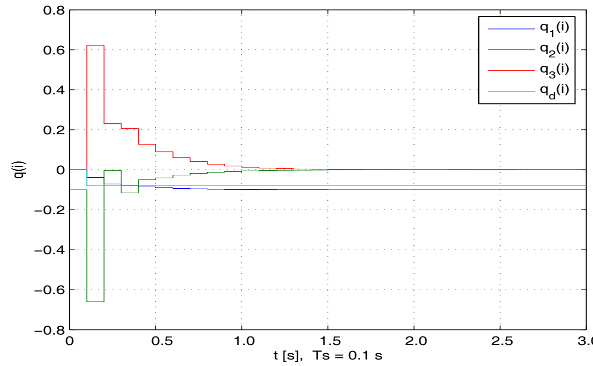

Fig. 4. Step response of the closed-loop system (FM)

Analogously, the gain matrix of tracking signal $\boldsymbol{W}$ was computed as

$$\boldsymbol{W} = \begin{bmatrix} -4.8845 & -43.6091 \\ 6.7814 & 113.7051 \end{bmatrix}$$

and simulation done using the same initial conditions as well as the desired input tracking signal as was used above. Two examples are shown of the closed-loop system step response also, where Figure V represents the autonomous mode step response and Figure 15 is concerned with the forced mode, respectively, where the control policy parameters are designed to satisfy inequality (68), (69).

It is evident, that constrain (31) is satisfied at all time instant (common variable $q_d(i)$) in autonomous mode for both control policy.

VI. Concluding Remarks

The paper describes a technique for discrete-time systems state feedback control design with equality constraints given on state variables. The proposed method poses the problem as a stabilization problem with a static output feedback controller, while the design principle exploits stability conditions to obtain implementation of the constraint control concept, and its limitations. The stability of the control scheme is established under a non-expansive condition given on the closed-loop system to have a more efficient control law, where sufficient conditions in the sense of bounded real lemma inequality, as well as Lyapunov inequality are derived. The validity of the proposed method is verified by a numerical example to demonstrate the role of an equality constraint tying together the state variables.

Acknowledgement

The work presented in this paper was supported by VEGA, Grant Agency of Ministry of Education and Academy of Science of Slovak Republic under Grant No. 1/0328/08, as well by Research & Development Operational Programme Grant No. 26220120030 realized in Development of Centre of Information and Communication Technologies for Knowledge Systems. This supports are very gratefully acknowledged.

References

[1] A. Benzaouia and C. Gurgat, "Regulator problem for linear discrete-time systems with nonsymmetrical constrained control," *International Journal of Control*, vol. 48, no. 6, pp. 2441-2451, 1988.

[2] D. Boyd, L. El Ghaoui, E. Peron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Philadelphia: SIAM 1994.

[3] E.B. Castelan and J.C. Hennet, "Eigenstructure assignment for state constrained linear continuous time systems," *Automatica*, vol. 28, no. 3, pp. 605-611, 1992.

[4] C.E.T. Dórea and B.E.A. Milani, "Design of L-Q regulators for state constrained continuous-time systems," *IEEE Transactions on Automatic Control*, vol. AC-40, no. 3, pp. 544-548, 1995.

[5] A. Filasová and D. Krokavec, "State estimate based control design using the unified algebraic approach," *Archives of Control Sciences*, vol. 20, no. 1, pp. 5-18, 2010.

[6] A. Filasová and D. Krokavec, "Stable in state constrained control of discrete-time systems," in *Proc. 10th WSEAS International Conference on Systems Theory ans Scientific Computation ISTASC'10*, Taipei, Taiwan, pp. 153-158, 2010.

[7] P. Gahinet, A. Nemirovski, A.J. Laub, and M. Chilali, *LMI Control Toolbox User's Guide*, Natick: The MathWorks, Inc., 1995.

[8] N. Gupta, "Kalman filtering in the presence of state space equality constraints," in *Proc. 26^{th} Chinese Control Conference*, Zhangjiajie, China, 2007, pp. 107-113.

[9] H. Hahn, "Linear systems controlled by stabilized constraint relations," in *Proc. 31^{st} IEEE Conference on Decision and Control*, Tucson, Arizona, 1992, pp. 840-848.

[10] G. Herrmann, M.C. Turner, and I. Postlethwaite, "Linear matrix inequalities in control," in *Mathematical Methods for Robust and Nonlinear Control*, M. C. Turner and D. G. Bates, Eds. Berlin: Springer, 2007, pp. 123-142.

[11] R.K. Jatoth And T. K. Kumar, "Particle swarm optimization based tuning of extended Lalman filter for manoeuvring target tracking," in *International Journal of Circuits, Systems and Signal Processing*, vol. 3, no.3, pp. 127-136, 2009.

[12] C.D. Johnson, "New results in subspace-stabilization control theory," *Mathematical Problems in Engineering*, vol. 6, no. 2-3, pp. 99-124, 2000.

[13] B. Kahng, "The invariant set theory of multiple valued iterative dynamical systems," in *Proc. 7^{th} WSEAS International Conference on System Science and Simulation in Engineering ICOSSSE '08*, Corfu, Greece, pp. 19-24, 2008.

[14] B. Kahng, "Maximal invariant sets of multiple valued iterative dynamics in disturbed control systems," *International Journal of Circuits, Systems and Signal Processing*, vol. 2, no. 2, pp. 113-120, 2008.

[15] S. Ko and R.R. Bitmead, "Optimal control for linear systems with state equality constraints," *Automatica*, vol. 43, no. 9, pp. 1573-1582, 2007.

[16] S. Ko and R.R. Bitmead, "State estimation for linear systems with state equality constraints," *Automatica*, vol. 43, no. 9, pp. 1363-1368, 2007.

[17] D. Krokavec and A. Filasová, *Discrete-time Systems.* Košice: Elfa, 2008. (in Slovak)

[18] D. Krokavec and A. Filasová, "Performance of reconfiguration structures based on the constrained control", in *Proc. 17^{th} IFAC World Congress 2008*, Seoul, Korea, pp. 1243-1248, 2008.

[19] D. Krokavec and A. Filasová, "Control reconfiguration based on the constrained LQ control algorithms," in *Preprints of 7th IFAC Symposium on Fault Detection, Supervision and Safety of Technical Processes SAFEPROCESS 2009*, Barcelona, Spain, pp. 686-691, 2009.

[20] Y. Nesterov and A. Nemirovsky, *Interior Point Polynomial Methods in Convex Programming. Theory and Applications*, Philadelphia: SIAM, 1994.

[21] H. Oloomi and B. Shafai, "Constrained stabilization problem and transient mismatch phenomenon in singularity perturbed systems," *International Journal of Control*, vol. 67, no. 2, pp. 435-454, 1997.

[22] D. Peaucelle, D. Henrion, Y. Labit, and K. Taitz, *User's Guide for SeDuMi Interface 1.04*, Toulouse: LAAS-CNRS, 2002.

[23] R.E. Skelton, T. Ivasaki, and K. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, London: Taylor & Francis, 1998.

[24] S. Tarbouriech and E.B. Castelan, "An eigenstructure assignment approach for constrained linear continuous-time singular systems," *Systems & Control Letters*, vol. 24, no. 5, pp. 333-343, 1995.

[25] Q.G. Wang, *Decoupling Control*, Berlin: Springer, 2003.

[26] Y. Xue, Y. Wei and G. Duan, "Eigenstructure assignment for linear systems with constrained input via state feedback. A parametric approach," in *Proc. 25th Chinese Control Conference*, Harbin, China, pp. 108-113, 2006.

[27] T.J. Yu, C.F. Lin, and P.C. Müller: "Design of LQ regulator for linear systems with algebraic-equation constraints," in *Proc. 35th Conference on Decision and Control*, Kobe, Japan, pp. 4146-4151, 1996.

[28] L. Zhang, Y. Chen, and N. E. Mastorakis, "Uniformly ultimate boundedness control for switched linear systems with parameter uncertainties," *International Journal of Mathematical Models and Methods in Applied Sciences*, vol. 1, no. 4, pp. 243-246, 2007.

Anna Filasová graduated in technical cybernetics and received M.Sc. degree in 1975, and Ph.D. degree in 1993 both from the Faculty of Electrical Engineering and Informatics, Technical University of Košice, Slovakia. In 1999 she was appointed Associated Professor from the Technical University in Košice in technical cybernetics.

She is with the Department of Cybernetics and Artificial Intelligence, Faculty of Electrical Engineering and Informatics, Technical University of Košice, and she has been working there as an Assistant Professor from 1975 to 1999. Her main research interests are in robust and predictive control, decentralized control, large-scale system optimization, and control reconfiguration.

Dušan Krokavec received M.Sc. degree in automatic control in 1967 and Ph.D. degree in technical cybernetics in 1982 from the Faculty of Electrical Engineering, Slovak University of Technology in Bratislava, Slovakia. In 1984 he was promoted Associated Professor from the Technical University in Košice, Slovakia, and in 1999 he was appointed Full Professor in automation and control.

From 1968 to 1971 he was a Research Assistant at the Research Institute of Automation and Mechanization in Nové Mesto n/Váhom, Slovakia, since 1971 he is with the Department of Cybernetics and Artificial Intelligence, Faculty of Electrical Engineering and Informatics, Technical University of Košice. In the long term, he specializes in stochastic processes in dynamic systems, digital control systems and digital signal processing, and in dynamic system fault diagnosis.

Prof. Krokavec is a member of the IFAC Technical Committee on Stochastic Systems.