Controllability and Observability of Matrix Differential Algebraic Equations

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Abstract**—**Controllability and observability of a class of matrix Differential Algebraic Equation (DAEs) are studied in this paper. The structure of a closed-form solution for the system is sought via two one-sided sub-systems. The solution is then used to derive necessary and sufficient conditions for the controllability and observability of the time-varying matrix DAE systems. More straightforward conditions on the controllability and observability of linear time-invariant matrix DAE systems that only depend on the state matrices are also obtained.

Keywords**—** Controllability, Differential Algebraic Equations, Gramian, Observability.

2010 Mathematics Subject Classification: Primary 93B05 and 93B07, Secondary 93C15

I. Introduction

MANY engineering systems, such as mechanical system, electrical circuits and chemical reaction kinetics, are modeled by coupled differential and algebraic equations (DAEs) that cannot be transformed into ordinary differential equations. Such DAEs also referred to as singular systems have been studied extensively in the view of numerical simulation. The most commonly studied linear differential algebraic equations are the vector DAEs like the following,

$$E(t)\dot{x}(t) + A(t)x(t) = f(t) \qquad (1.1)$$

where $x \in \mathbb{R}^n$, $E(t)$ is singular for all t in the associated time interval, $E, A \in \mathbb{R}^{n\times n}$, and $f \in \mathbb{R}^n$. A characterization of solvability for (1.1) and a general canonical form representation for solvable DAEs in the form of (1.1) can be found in [1]. The DAE system (1.1) is called linear time-invariant (LTI) if E and A are constant matrices. For such LTI systems, the problems of feedback pole placement [2] and optimal control [3] through state feedback have been studied. The analysis and control results for linear time-invariant systems have also been generalized to time-varying [4] and discrete-time [5] systems.

Controllability and observability are of important and fundamental properties of control systems. In [6], the controllability and observability Gramians are devised in the frequency domain for controller reduction design. The controllability and observability properties are required for minimal realization, such as the multidimensional hybrid systems introduced in [7]. Structural properties of generalized linear systems are studied in [8] using the Perron-Stieltjes integral to obtain the input-output map. Necessary and sufficient conditions are derived for complete controllability and observability. Different control methods are applicable to control systems that meet the controllability and observability criteria for state feedback and output feedback designs, such as the thermosyphon system in [9].

The controllability, observability and realizability of first-order matrix Lyapunov systems are first introduced in [10]. In this paper, we study the differential algebraic matrix Lyapunov systems over its solvability and control perspectives. The rest of the paper is organized as follows. A closed form solution for the proposed matrix DAE system is presented in section 2. This solution is used to derive necessary and sufficient criteria for controllability and observability of the matrix DAE system for both linear time-varying and linear time-invariant systems, all presented in section 3. Conclusion and remarks on future work are found in section 4.

II. Closed form solution of the matrix DAE systems

The class of matrix differential algebraic equation system considered in this paper with input and output structures is defined as

$$E\dot{X} = AX + EXB + DU\,,\ X(t_0) = X_0 \qquad (2.1)$$

and

$$Y = FX \qquad (2.2)$$

where $\dot{X} = \dfrac{dX}{dt}$, the state coefficient matrices $E,\ A,\ B \in \mathbb{R}^{n\times n}$, the input structure matrix $D \in \mathbb{R}^{n\times m}$, the control input $U \in \mathbb{R}^{m\times n}$, and the output structure matrix $F \in \mathbb{R}^{q\times n}$. It is also considered that the state coefficient matrices as well as the input/output structure matrices are time-dependent. If that is the case, we assume the matrix functions are continuously differentiable over $\Im = [t_0, t_1]$. Moreover, E is singular for all

$t \in \Im$. In this section, our main objective is to obtain a general form of solution for (2.1). The closed-form solution plays an important role in deriving the conditions on the controllability of (2.1), and observability of (2.1) and (2.2) in section 3.

The solution of (2.1) is constructed via the solution of two one-sided subsystems of (2.1). To set the stage, we begin with some basic definitions associated with the solutions of linear first-order matrix differential equations.

Definition 2.1 A matrix function $Z(t) \in \mathbb{R}^{n \times n}$ is a fundamental solution of a linear first-order matrix differential equation

$$L(\dot{X}, X, A_1, A_2, \ldots A_r) = DU \,,\; t \in \Im \tag{2.3}$$

where $A_i \in \mathbb{R}^{n \times n}, i = 1,2,\ldots,r$, $D \in \mathbb{R}^{n \times m}$ known as the input structure matrix, and $U \in \mathbb{R}^{m \times n}$ is the input matrix, if Z satisfies (2.3) with $D = 0$, $\det(Z(t)) \neq 0$ for all $t \in \Im$, and every solution to (2.3) can be written as $X_h = Z\Upsilon + X_p$, where Υ is an arbitrary n by n constant matrix and X_p is a particular solution to (2.3).

Obviously, the matrix differential algebraic equation (2.1) is a special case of (2.3). The concept of state transition matrix associated with a system of ordinary differential equations was introduced in [11]. We extend the state transition matrix to (2.3) via the fundamental solution of (2.3). First, rewrite (2.3) as an initial value problem,

$$L(\dot{X}, X, A_1, A_2, \ldots A_r) = 0 \,,\; X(t_0) = X_0 \,,\; t \in \Im \tag{2.4}$$

According to Definition 2.1, the unique solution to (2.4) is

$$X = Z(t)Z^{-1}(t_0)X_0 = \Phi(t, t_0)X_0 \tag{2.5}$$

The matrix function $\Phi(t, t_0)$ can be generalized as $\Phi(t,s) = Z(t)Z^{-1}(s)$, which is known as the state transition matrix associated with (2.4).

Definition 2.2 The state transition matrix associated with (2.4) is defined as

$$\Phi(t,s) = Z(t)Z^{-1}(s) \tag{2.6}$$

where $Z(t)$ is a fundamental solution of (2.4).

It is easy to check that Φ satisfies the matrix differential equation (2.4) in t, and it satisfies the following three properties:

Lemma 2.1 The state transition matrix satisfies the following properties

(i) $\Phi(t,t) = I$, $t \in \Im$

(ii) $\Phi(t_1, t_2)\Phi(t_2, t_3) = \Phi(t_1, t_3)$, $t_1, t_2, t_3 \in \Im$

(iii) $\Phi^{-1}(t_1, t_2) = \Phi(t_2, t_1)$, $t_1, t_2 \in \Im$

We will explore two special linear matrix differential equations of (2.3), which lead to the solution of (2.1). The first one is the standard linear matrix differential equation as follows,

$$\dot{X} = AX \tag{2.7}$$

where $A, X \in \mathbb{R}^{n \times n}$, and (2.7) is a linear time-invariant (LTI) matrix differential system if A is a constant matrix; otherwise, it is known as linear time-varying (LTV) system. If (2.7) is LTI, the fundamental solution of (2.7) is $Z(t) = e^{tA}$. The matrix exponential e^{tA} is formally defined by the convergent power series,

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \ldots + \frac{t^n}{n!}A^n + \ldots$$

There are many numerical methods available for computing e^{tA}. A nice tutorial review can be found in [12]. The state transition matrix associated with (2.7) is

$$\Phi(t,s) = Z(t)Z^{-1}(s) = e^{(t-s)A}$$

This is because e^{tA} and e^{-sA} commute with each other and $(e^{sA})^{-1} = e^{-sA}$.

The solution to (2.7) is more complicated if it is LTV. It is written as an initial value problem,

$$\dot{X} = A(t)X \,,\; X(t_0) = X_0 \,,\; t \in \Im \tag{2.8}$$

The general solution of (2.8) is of the form

$$X(t) = \Phi(t, t_0)X(t_0) \tag{2.9}$$

where Φ is the state transition matrix associated with (2.8). In the time-varying case, the state transition matrix Φ has an analogous form similar to the LTI case, i.e.

$$\Phi(t, t_0) = e^{\int_{t_0}^{t} A(\tau)d\tau} \tag{2.10}$$

if $A(t)$ and $\int_{t_0}^{t} A(\tau)d\tau$ commute. Otherwise, the matrix exponential (2.10) is generalized to the so-called Peano-Baker formula as an extension to the power series expansion for Φ,

$$\Phi(t,t_0) = I + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t}\int_{t_0}^{\tau_1} A(\tau_1)A(\tau_2)d\tau_2 d\tau_1$$
$$+ ... + \int_{t_0}^{t}\int_{t_0}^{\tau_1} ... \int_{t_0}^{\tau_{n-1}} A(\tau_1)A(\tau_2)...A(\tau_n)d\tau_n...d\tau_1 + ...$$

In spite of the lack of closed-form expressions for Φ from time-varying systems, the state transition matrix is a useful tool for studying the properties of solutions of (2.1), which leads to the exploration of controllability and observability properties of the matrix DAE system.

The second class of matrix differential equation off (2.3) to be considered here is the so-called matrix differential algebraic equation as follows

$$E\dot{X} = AX + DU(t)\,,\ X(t_0) = X_0\,,\ t \in \Im \qquad (2.11)$$

Recall that *E* is a singular matrix. When system (2.11) is time-invariant, i.e. *E*, *A* and *D* are constant matrices, the solution of (2.11) can be determined by the associated matrix pencil, $sE - A$, as a result of Laplace transform.

Definition 2.3 A matrix pencil $sE - A$ is regular if $\det(sE - A) \neq 0$ for some $s \in \mathbb{C}$.

The matrix DAE (2.11) is solvable if and only if the pencil $sE - A$ is regular, which is analogous to the result from [13]. In order to obtain an explicit form of solution for (2.11), one needs to transform (2.11) into a canonical form. To be more specific, suppose the rank of matrix *E* satisfies $r(E) = r < n$, and $\det(sE - A)$ is a nonzero polynomial of degree *m*, $0 \le m \le r$, then there exist non-singular matrices, $P, Q \in \mathbb{C}^{n\times n}$, such that (2.11) is transformed into the following canonical form by premultiplying (2.11) by *P* and a coordinate change with *Q*,

$$\begin{aligned} \dot{\tilde{X}}_1 &= A_1\tilde{X}_1 + D_1 U \\ N\dot{\tilde{X}}_2 &= \tilde{X}_2 + D_2U(t) \end{aligned} \qquad (2.12)$$

where $\tilde{X}_1 \in \mathbb{C}^{m\times n}$, $\tilde{X}_2 \in \mathbb{C}^{(n-m)\times n}$, $\begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} = QX$, and N is an $(n-m)\times(n-m)$ matrix of nilpotency index κ, which means $N^i \neq 0$ for $i < \kappa$ and $N^\kappa = 0$. An algorithm for constructing the similarity transformation matrices *P* and *Q* is developed in [14]. In the case of $m = r$, matrix *N* is identically zero, and the second equation in (2.12) becomes an algebraic matrix equation. Meanwhile, if $m < r$, matrix *N* is in the Jordan canonical form with zeros on the main diagonal. In any event, the ODE subsystem in $\tilde{X}_1$ is totally decoupled from the DAE subsystem in $\tilde{X}_2$, and they can be solved separately. The solutions are given as follows,

$$\tilde{X}_1(t) = e^{A_1(t-t_0)}X_1(t_0) + \int_{t_0}^{t} e^{A_1(t-\tau)}D_1U(\tau)d\tau\,,\ t \in \Im \qquad (2.13)$$

and

$$\tilde{X}_2(t) = -\sum_{i=0}^{\kappa-1} N^i D_2 U^{(i)}(t)\,,\ t \in \Im \qquad (2.14)$$

where $U^{(i)}(t)$ denotes the i_{th} derivative of the input matrix function $U(t)$. The DAE system (2.11) has smooth solutions if the initial condition $\tilde{X}_2(t_0)$ satisfies (2.14). Furthermore, if the nilpotency index κ is greater than one, then the input matrix function $U(t)$ is continuously differentiable up to the order of $\kappa - 1$.

For linear time-varying DAEs, we rewrite (2.11) as

$$E(t)\dot{X} = A(t)X + G(t)\,,\ X(t_0) = X_0\,,\ t \in \Im \qquad (2.15)$$

where $G(t) = D(t)U(t)$. The necessary and sufficient conditions for the solution of (2.15) can be deduced from [1]. First, for any $\hat{t} \in \Im$, we obtain Taylor series of the matrix functions, *E*, *A*, *G* and *X*. Due to their similarity in Taylor series expansions, we use the following expression

$$H(t) = \sum_k H_k(t - \hat{t})^k$$

where $H_k = \dfrac{H^{(k)}(\hat{t})}{k!}$, $H = E, A, G, \text{and}\, X$. The Taylor series expansions are substituted in (2.15), then for each $j > 0$, but less than the order of smoothness of *E*, *A*, *G* and *X*, and $t \in \Im$, the DAE (2.15) is transformed into a system of algebraic equations,

$$\psi_j\zeta_j = \varphi_j X_0 + g_j \qquad (2.16)$$

where

$$\psi_j = \begin{bmatrix} E_0 & 0 & ... & 0 \\ E_1 - A_0 & 2E_0 & ... & . \\ E_2 + A_1 & 2E_1 - A_0 & 3E_0 ... & . \\ \vdots & \vdots & & \vdots \\ E_{j-1} - A_{j-2} & 2E_{j-2} - A_{j-3} & ... & jE_0 \end{bmatrix}$$

$$\zeta_j = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_j \end{bmatrix}, \varphi_j = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{j-1} \end{bmatrix}, \text{and } g_j = \begin{bmatrix} G_0 \\ G_1 \\ \vdots \\ G_{j-1} \end{bmatrix}.$$

Definition 2.4 The matrix ψ_j is called smoothly 1-full if there exists a smooth non-singular matrix function $P(t)$ on $\Im$ such that

$$P(t)\psi_j(t) = \begin{bmatrix} I_{n\times n} & 0 \\ 0 & R(t) \end{bmatrix}.$$

With $X_0(t) = X(t)$, the algebraic system can be used to derive an equation for $\dot{X}(t)$ explicitly when ψ_j is 1-full, i.e. $\dot{X}(t) = \tilde{A}(t)X(t) + \tilde{G}(t)$, which can be solved accordingly.

Theorem 2.1 The system (2.15) with *E*, *A* real analytic is solvable if and only if there is an integer $\ell \in [1, n+1]$ such that (i) $r(\psi_\ell)$ is constant on $\Im$; (ii) ψ_ℓ is 1-full on $\Im$; and (iii) $\Re(\psi_\ell) + \Re(\varphi_\ell) = \mathbb{R}^{jn}$ on $\Im$, where $\Re(\cdot)$ is the column space of the matrix.

The proof of Theorem 2.1 is similar to the one in [1], as such omitted here.

In what follows, we assume solvability for each of the two one-sided matrix differential equations based on the previous discussions. Our goal is to construct a closed-form solution for (2.1). Let X_+ be the fundamental matrix solution of right-side differential algebraic system $E\dot{X} = AX$ and X_- be the fundamental solution of the left-side system $\dot{X} = B^T X$. We have the following result.

Theorem 2.2 The complementary solution of the homogeneous equation

$$E\dot{X} = AX + EXB \tag{2.17}$$

has the unique form $X = X_+ C X_-^*$, where * represents the operation of complex conjugate transpose and *C* is an *n* by *n* constant matrix.

Proof: Notice that $\dot{X}_-^* = X_-^* B$ since B is a real matrix. Hence, it is straightforward to show that $X = X_+ C X_-^*$ is a general solution. To show every solution of the homogeneous solution is of the form $X = X_+ C X_-^*$. Let $Z(t)$ be a solution and $Z(t) = YX_-^*$. Insert $\dot{Z} = \dot{Y}X_-^* + Y\dot{X}_-^*$ in (2.17), one has

$$E\dot{Z} = E\dot{Y}X_-^* + EYX_-^*B = AYX_-^* + EYX_-^*B$$

Since $Z(t)$ is a solution, it requires $E\dot{Y} = AY$. Also, because X_+ is a fundamental solution of the right-side system, we have $Y = X_+ C$, where *C* is an arbitrary constant matrix (*n* by *n*). This leads to $Z(t) = X_+ C X_-^*$, hence the uniqueness .

■

Our next step is to construct a solution for the non-homogeneous system (2.1).

Theorem 2.3 Every solution of (2.1) is of the form of $X(t) = X_+ C X_-^* + X_p$, where X_p is a particular solution of (2.1).

Proof: It is straightforward to show that $X(t) = X_+ C X_-^* + X_p$ is a solution of (2.1). It is also easy to see that $X - X_p$ satisfies the homogeneous equation (2.17). Therefore, $X - X_p$ can only be written as $X_+ C X_-^*$ according to Theorem 2.2.

■

Similar to solving non-homogeneous ODEs, the particular solution X_p can be found via variation of parameters. Assume $X_p = X_+ C X_-^*$, where *C* depends on t. Then,

$$\dot{X}_p = \dot{X}_+ C X_-^* + X_+ \dot{C} X_-^* + X_+ C \dot{X}_-^*$$

after substituting in (2.1), one has

$$E\dot{X}_+ C X_-^* + EX_+ \dot{C} X_-^* + EX_+ C \dot{X}_-^* = AX_+ C X_-^* + EX_+ C X_-^* B + DU$$

Hence, $EX_+ \dot{C} X_-^* = DU$. Consider $EP = DU$, let Ω_D represent the set of all *m* by *n* matrices such that for any $U \in \Omega_D$, the columns of DU is in the column space of *E*, i.e. $\Re(E)$. Since U is a control input matrix, and with the assumption of solvability of (2.1), Ω_D is non-empty. The set Ω_D is indeed guaranteed to be non-empty if $\Re(E) \supseteq \Re(D)$. However, this condition is not necessary. Therefore, $P = E^+ DU$, where E^+ is the pseudo-inverse of E. Now, we obtain an explicit equation for $\dot{C}$ as follows,

$$\dot{C} = X_+^{-1} E^+ DU X_-^{*^{-1}}.$$

After integrating the equation over $\Im$, and applying $C(t)$ in X_p, we obtain the expression for the particular solution of (2.1),

$$X_p(t) = X_+(t)\left[\int_{t_0}^{t} X_+^{-1} E^+ DUX_-^{*^{-1}} d\tau\right] X_-^*(t) \qquad (2.18)$$

The final solution of (2.1) with an initial condition over $\Im$ is summarized in the following theorem.

Theorem 2.4 The solution of (2.1) with initial condition $X(t_0) = X_0$ is given by

$$X(t) = \Phi_+(t,t_0) X_0 \Phi_-^*(t,t_0) + \int_{t_0}^{t} \Phi_+(t,\tau) E^+ DU \Phi_-^*(t,\tau) d\tau \qquad (2.19)$$

where Φ_+ and Φ_- are state transition matrices associated with the right-side and left-side subsystems, respectively.

Proof: According to Theorem 2.3, the general solution of the matrix DAE (2.1) is given by, $X(t) = X_+ C X_-^* + X_p$, where X_p is given by (2.18). It is obvious $X_p(t_0) = 0$. Therefore, after using the initial condition, the solution of the IVP (2.1) is

$$\begin{aligned} X(t) = X_+(t) X_+^{-1}(t_0) X_0 X_-^{*^{-1}}(t_0) X_-^*(t) + \\ X_+(t)\left[\int_{t_0}^{t} X_+^{-1} E^+ DUX_-^{*^{-1}} d\tau\right] X_-^*(t) \end{aligned}.$$

Hence the result (2.19). ■

Since the state transition matrices satisfy Lemma 2.1, the solution can also be written in an alternative form,

$$\begin{aligned} X(t) = \Phi_+(t,t_0) X_0 \Phi_-^*(t,t_0) + \\ \Phi_+(t,t_0) \int_{t_0}^{t} \Phi_+(t_0,\tau) E^+ DU \Phi_-^*(t_0,\tau) d\tau \Phi_-^*(t,t_0) \end{aligned} \qquad (2.20)$$

which will be used when we explore the controllability properties of (2.1).

III. CONTROLLABILITY AND OBSERVABILITY OF MATRIX DAE SYSTEMS

In this section, we explore the controllability and observability properties of the matrix DAE system (2.1) and (2.2), while (2.2) is considered when the observability is concerned. In the case of time-varying systems, the controllability and observability are examined over the time interval $\Im$.

When working with control systems, the first step is to determine whether a prescribed control objective can be achieved by manipulating the control input. A direct approach is to construct a control input that will drive the system state trajectory to the desired state. More convenient criteria can also be obtained to test the controllability of the system as shown in this section.

Definition 3.1 The matrix DAE system (2.1) and (2.2) is said to be completely controllable if for any t_0, any arbitrary initial state $X(t_0) = X_0$, and any arbitrary final state X_e, there exists a finite time interval $\Im = [t_0, t_1]$ and a control $U(t)$, $t \in \Im$, such that $X(t_1) = X_e$.

The controllability property is important for a control system. If the system is not controllable, a control $U(t)$ may not exist to achieve the control objective. There are other types of controllability. Complete state controllability only requires (2.1); on the other hand, complete output controllability requires attainment of arbitrary output, where (2.2) has to be considered. The control input $U(t)$ can be piecewise continuous over $\Im$.

The controllability of a control system usually boils down to checking whether a set of functions associated with the state transition matrix and the input structure matrix are linearly independent over $\Im$. A well-know and more convenient test for linear independence of a set of functions is by way of the Gramian matrix [11].

Definition 3.2 The Gramian matrix associated with a set of functions $\{f_i(t)\}_{i=1}^n$ over $\Im$ is defined as $G = [g_{ij}]$, where

$$g_{ij} = \int_{\Im} f_i(\tau) \overline{f}_j(\tau) d\tau .$$

As such, a set of functions $\{f_i(t)\}_{i=1}^n$ are linearly independent over $\Im$ if the Gramian matrix G is non-singular or positive definite since G is a non-negative Hermitian matrix.

Theorem 3.1 The matrix DAE system (2.1) is completely state controllable on $\Im$ if and only if the controllability Gramian matrix

$$\aleph_c(t_0,t_1) = \int_{\Im} \Phi_+(t_0,\tau) E^+ DD^T E^{+T} \Phi_+^*(t_0,\tau) d\tau \qquad (3.1)$$

is positive definite.

Proof: Suppose $\aleph_c(t_0,t_1)$ is positive definite. Let $X(t_0) = X_0$ and $X(t_1) = X_e$ be two arbitrary initial and final state over the interval $\Im = [t_0, t_1]$. Need to show that there is a control input $U(t)$ over $\Im$ that will drive X_0 to X_e in $\Im$. We choose

$$U(t) = -D^T E^{+T} \Phi_+^*(t_0, t) \aleph_c^{-1}(t_0, t_1)[X_0 - \Phi_+(t_0, t_1) \\ X_e \Phi_-^*(t_0, t_1)] \Phi_-^*(t, t_0) \tag{3.2}$$

It suffices to show that the right-side of (2.20) yields X_e if $t = t_1$ and substitute (3.2) for U in (2.20), using Lemma 2.1 to simplify the expressions involving the state-transition matrix,

$$\Phi_+(t_1, t_0) X_0 \Phi_-^*(t_1, t_0) + \Phi_+(t_1, t_0) \cdot \\ \int_{\Im} \Phi_+(t_0, \tau) E^+ D U(\tau) \Phi_-^*(t_0, \tau) d\tau \Phi_-^*(t_1, t_0)$$

$$= \Phi_+(t_1, t_0) X_0 \Phi_-^*(t_1, t_0) - \Phi_+(t_1, t_0) \cdot \\ \int_{\Im} \Phi_+(t_0, \tau) E^+ D D^T E^{+T} \Phi_+^*(t_0, \tau) d\tau \cdot \\ \aleph_c^{-1}(t_0, t_1) X_0 \Phi_-^*(t_1, t_0) + \Phi_+(t_1, t_0) \cdot \\ \int_{\Im} \Phi_+(t_0, \tau) E^+ D D^T E^{+T} \Phi_+^*(t_0, \tau) d\tau \cdot \\ \aleph_c^{-1}(t_0, t_1) \Phi_+(t_0, t_1) X_e \Phi_-^*(t_0, t_1) \Phi_-^*(t_1, t_0)$$

$$= X_e$$

Therefore, the matrix DAE system (2.1) is completely state controllable on $\Im$.

Conversely, assume the system (2.1) is completely state controllable on $\Im$, we need to show that the controllability Gramian $\aleph_c(t_0, t_1)$ is positive definite. It is obvious that the matrix $\aleph_c(t_0, t_1)$ is symmetric and non-negative, all we need to show is that the matrix is invertible. Assume the matrix $\aleph_c(t_0, t_1)$ is not invertible, or equivalently, its null space is non-empty. As such, there exists a non-zero vector $\theta \in \mathbb{R}^n$ such that

$$\theta^T \aleph_c(t_0, t_1) \theta = \int_{\Im} \theta^T \Phi_+(t_0, \tau) E^+ D D^T E^{+T} \Phi_+^*(t_0, \tau) \theta d\tau \\ = \int_{\Im} \left\| \theta^T \Phi_+(t_0, \tau) E^+ D \right\|^2 d\tau = 0$$

Hence,

$$\theta^T \Phi_+(t_0, t) E^+(t) D(t) \equiv 0, \ t \in \Im \tag{3.3}$$

It is known that the system (2.1) is completely state controllable. Therefore, there exists a control U such that the initial state $X(t_0) = \theta\theta^T$ is driven to the final state $X(t_1) = 0$. Hence, from (2.20),

$$0 = \Phi_+(t_1, t_0) \theta\theta^T \Phi_-^*(t_1, t_0) + \\ \Phi_+(t_1, t_0) \int_{t_0}^{t_1} \Phi_+(t_0, \tau) E^+ D U \Phi_-^*(t_0, \tau) d\tau \Phi_-^*(t_1, t_0)$$

which can be simplified as

$$\theta\theta^T = -\int_{t_0}^{t_1} \Phi_+(t_0, \tau) E^+ D U \Phi_-^*(t_0, \tau) d\tau$$

Pre-multiply both sides by θ^T, one has

$$\|\theta\|^2 \theta^T = -\int_{t_0}^{t_1} \theta^T \Phi_+(t_0, \tau) E^+ D U \Phi_-^*(t_0, \tau) d\tau = 0$$

due to (3.3), which implies $\theta = 0$, hence the contradiction. ■

From the proof of Theorem 3.1, the controllability of matrix DAE system from an arbitrary initial state to an arbitrary final state is equivalent to the controllability from an arbitrary initial state to the origin.

It is usually cumbersome to establish controllability of a control system through the controllability Gramian. Our next step is to develop an alternative criterion for testing controllability without integration. We would assume differentiability of the matrix coefficient functions in (2.1). Eventually, this new criterion will be applied to test the controllability of linear time-invariant matrix DAEs.

Definition 3.3 Consider the matrix DAE system (2.1), with the assumption that all coefficient matrices in (2.1) are differentiable, a sequence of $n \times n$ matrix functions $P_k(t)$ over $\Im$ are defined recursively as follows,

$$\text{(i) } P_0(t) = E^+ D(t) D^T(t) \tag{3.4}$$

$$\text{(ii) } \dot{P}_{k-1}(t) = P_k(t) + E^+ A(t) P_{k-1}(t) + P_{k-1}(t) B(t) \\ k = 1, 2, ... \tag{3.5}$$

The following lemmas will be used to prove a derivative formula involving the matrix functions $P_k(t)$ and state-transition matrix functions Φ_+ and Φ_-.

Lemma 3.1 Let $X(t)$ be an invertible matrix function over $\Im$. Then,

$$\dot{X}^{-1}(t) = -X^{-1}(t) \dot{X}(t) X^{-1}(t) \tag{3.6}$$

Proof: Since $X(t) X^{-1}(t) = I$, differentiate both sides with respect to t to get $\dot{X}(t) X^{-1}(t) + X(t) \dot{X}^{-1}(t) = 0$. ■

Lemma 3.2 The following holds for all $t,s \in \Im$

$$\frac{\partial^k}{\partial s^k}\left[\Phi_+(t,s)P_0(s)\Phi_-^*(t,s)\right]=\Phi_+(t,s)P_k(s)\Phi_-^*(t,s),\ k=1,2,... \quad (3.7)$$

Proof: Use mathematical induction. With $k=1$,

$$\frac{\partial}{\partial s}\left[\Phi_+(t,s)P_0(s)\Phi_-^*(t,s)\right]=(\frac{\partial}{\partial s}\Phi_+)P_0\Phi_-^*+\Phi_+\dot{P}_0(s)\Phi_-^* + \Phi_+P_0(\frac{\partial}{\partial s}\Phi_-^*) \quad (3.8)$$

Recall that $\Phi_+(t,s)=X_+(t)X_+^{-1}(s)$, where X_+ is the fundamental matrix of the right-side subsystem $E\dot{X}=AX$. According to Lemma 3.1, $\frac{\partial}{\partial s}\Phi_+(t,s)=-\Phi_+(t,s)E^+A$. On the other hand, $\Phi_-^*(t,s)=X_-^{*^{-1}}(s)X_-^*(t)$, where X_- is the fundamental matrix of the left-side subsystem $\dot{X}=B^*X$. According to Lemma 3.1, it is easy to see

$$\dot{X}_-^{*^{-1}}(s)=-X_-^{*^{-1}}(s)\dot{X}_-^*(s)X_-^{*^{-1}}(s)$$

Hence, $\frac{\partial}{\partial s}\Phi_-^*(t,s)=-B\Phi_-^*(t,s)$. After substituting these expressions in (3.8), one has

$$\frac{\partial}{\partial s}\left[\Phi_+(t,s)P_0(s)\Phi_-^*(t,s)\right]=\Phi_+[-E^+AP_0+\dot{P}_0-P_0B]\Phi_-$$
$$=\Phi_+(t,s)P_1(s)\Phi_-^*(t,s)$$

from (3.5). Now, assume

$$\frac{\partial^n}{\partial s^n}\left[\Phi_+(t,s)P_0(s)\Phi_-^*(t,s)\right]=\Phi_+(t,s)P_n(s)\Phi_-^*(t,s)$$

Continue to differentiate both sides,

$$\frac{\partial^{n+1}}{\partial s^{n+1}}\left[\Phi_+(t,s)P_0(s)\Phi_-^*(t,s)\right]=\frac{\partial}{\partial s}[\Phi_+(t,s)P_n(s)\Phi_-^*(t,s)]$$
$$=(\frac{\partial}{\partial s}\Phi_+)P_n\Phi_-^*+\Phi_+\dot{P}_n(s)\Phi_-^*+\Phi_+P_n(\frac{\partial}{\partial s}\Phi_-^*)$$
$$=\Phi_+[-E^+AP_n+\dot{P}_n-P_nB]\Phi_-$$
$$=\Phi_+(t,s)P_{n+1}(s)\Phi_-^*(t,s)$$

due to (3.5) with $k=n+1$. ■

The matrices $P_k(t)$ introduced in Definition 3.3 are useful for finding alternative criterion for state controllability of (2.1) along with additional smoothness condition on the system matrices all due to Lemma 3.2 and Definition 3.3.

Theorem 3.2 Suppose the matrix functions in the linear time-varying matrix DAE system (2.1) satisfy the smoothness condition, i.e., let ℓ be a positive integer, $A(t),B(t)\in C_{\Im}^{\ell-1}$ and $D(t),E^+(t)\in C_{\Im}^{\ell}$. Then, system (2.1) is completely state controllable if there exists $t_\alpha \in \Im$ such that the controllability matrix

$$\Omega_c(t_\alpha)=[P_0(t_\alpha)\quad P_1(t_\alpha)\quad ...\quad P_\ell(t_\alpha)] \quad (3.9)$$

is full row rank, i.e. $r(\Omega_c(t_\alpha))=n$.

Proof: Assume the DAE system (2.1) is not completely state controllable. According to Theorem 3.1, the associated controllability Gramian $\aleph_c(t_0,t_1)$ must be singular. Hence, there exists a non-zero constant vector $\upsilon\in\mathbb{R}^n$, such that $\upsilon^T\aleph_c(t_0,t_1)\upsilon=0$, where $\aleph_c(t_0,t_1)$ is given by (3.1), which implies that $\upsilon^T\Phi_+(t_0,t)E^+(t)D(t)=0$ for all $t\in\Im$. Hence, $\upsilon^T\Phi_+(t_0,t)E^+DD^T\Phi_-^*(t_0,t)=0$, or from (3.4),

$$\upsilon^T\Phi_+(t_0,t)P_0(t)\Phi_-^*(t_0,t)=0,\ t\in\Im \quad (3.10)$$

Take *k*-derivatives of both sides of (3.10) with respect to *t*, according to Lemma 3.2, we have

$$\upsilon^T\Phi_+(t_0,t)P_k(t)\Phi_-^*(t_0,t)=0,\ t\in\Im,\ k=1,2,...,\ell \quad (3.11)$$

Let t_β be an arbitrary point in $\Im$, i.e. $t_0<t_\beta<t_1$, due to Lemma 2.1, (3.10) and (3.11) can be combined as

$$u^T\Phi_+(t_\beta,t)P_k(t)\Phi_-^*(t_0,t)=0,\ t\in\Im,\ k=0,1,2,...,\ell \quad (3.12)$$

where $u^T=\upsilon^T\Phi_+(t_0,t_\beta)$. u is a non-zero vector because $\upsilon\neq 0$ and $\Phi_+(t_0,t_\beta)$ is invertible. Now, substitute $t=t_\beta$ in (3.12) to get $u^TP_k(t_\beta)\Phi_-^*(t_0,t_\beta)=0$ or, since $\Phi_-^*(t_0,t_\beta)$ is non-singular, $u^TP_k(t_\beta)=0,\ k=0,1,...,\ell$. This is written as

$$u^T[P_0(t_\beta)\quad P_1(t_\beta)\quad ...\quad P_\ell(t_\beta)]=u^T\Omega_c(t_\beta)=0$$

which implies that $\Omega_c(t_\beta)$ cannot be full row rank, i.e. $r(\Omega_c(t_\beta)) < n$, which contradicts the rank condition of the controllability matrix Ω_c. ■

Theorem 3.2 can be used to derive controllability criterion for linear time-invariant (LTI) matrix DAEs (2.1), in which all coefficient matrices are constant matrices. To this end, an explicit formula is derived for the P_k matrices from the recursive relation (3.4) and (3.5) as follows,

$$P_k = (-1)^k \sum_{i=0}^{k} \binom{k}{i} (E^+A)^{k-i} E^+DD^T B^i \,, \; k = 0,1,2... \tag{3.13}$$

with the understanding that $A^0 = I$, the identity matrix. After P_k being substituted in the controllability matrix (3.9), with the help of elementary column block operations on Ω_c, we obtain a controllability test for the LTI matrix DAE (2.1) in the following theorem.

Theorem 3.3 The linear time-invariant matrix DAE system

$$E\dot{X} = AX + EXB + DU \,, \; X(t_0) = X_0$$

is completely state controllable over $\Im$ if either of the following controllability matrices is full rank, i.e. $r(\Omega_{lc}) = n$ or $r(\Omega_{rc}) = n$, where

$$\Omega_{lc} = \begin{bmatrix} DD^T & E^+AE^+DD^T & \dots & (E^+A)^{n-1}E^+DD^T \end{bmatrix}$$

$$\Omega_{rc} = \begin{bmatrix} DD^T & E^+DD^TB & \dots & E^+DD^TB^{n-1} \end{bmatrix}. \tag{3.14}$$

The observability of a control system is often considered as a dual problem of controllability for linear ODE systems. However, this may not be true for DAEs [16]. Observability is important for a control system because, if the system is observable, the outputs of the system can completely determine the states of the system. On the other hand, if a system is not observable, it means some of the current states cannot be determined by the measurement of the outputs through sensors. As such, a controller constructed based on these outputs do not fulfill the control specifications related to those unobservable states. A formal definition for observability is given below.

Definition 3.4 The matrix DAE systems (2.1) and (2.2) is said to be completely observable if for any t_0 and any initial state $X(t_0) = X_0$, there exists a finite time $t_1 > t_0$ such that, let $\Im = [t_0, t_1]$, the control $U(t)$ and output $Y(t)$ for $t \in \Im$ suffice to determine the initial state X_0.

Without loss of generality, it can be assumed that the control $U(t)$ is identically zero throughout the time interval $\Im$ [15]. We have the following result on the observability of matrix DAE (2.1) and (2.2), with zero control input, via the observability Gramian matrix.

Theorem 3.4 The matrix DAE system (2.1) and (2.2) is completely observable on $\Im$ if and only if the observability Gramian matrix

$$\aleph_o(t_0,t_1) = \int_{\Im} \Phi_+^*(\tau,t_0) F^T(\tau) F(\tau) \Phi_+(\tau,t_0) d\tau \tag{3.15}$$

is positive definite.

Proof: Assume the observability Gramian matrix is positive definite, consider (2.1) and (2.2) with $U(t) = 0$ over $\Im$. According to (2.20), $X(t) = \Phi_+(t,t_0) X_0 \Phi_-^*(t,t_0)$ over $\Im$. Then, the output $Y(t) = F(t)\Phi_+(t,t_0) X_0 \Phi_-^*(t,t_0)$. Since $\Phi_-^{*-1}(t,t_0) = \Phi_-^*(t_0,t)$, one has

$$F(t)\Phi_+(t,t_0) X_0 = Y(t)\Phi_-^*(t_0,t) \,.$$

Pre-multiply both sides of the above equation by $\Phi_+^*(t,t_0)F^T(t)$ and integrate over $\Im$, we have

$$X_0 = \aleph_o^{-1}(t_0,t_1) \int_{\Im} \Phi_+^*(\tau,t_0) F^T(\tau) Y(\tau) \Phi_-^*(t_0,\tau) d\tau \,.$$

Therefore, the matrix DAE system (2.1) and (2.2) is completely observable.
Conversely, assume the system is completely observable, we need to show that the observability Gramian matrix $\aleph_o(t_0,t_1)$ is positive definite. Obviously, $\aleph_o(t_0,t_1)$ is symmetric. Assume, however, $\aleph_o(t_0,t_1)$ is singular. Then, there exists a non-zero vector $v_1 \in \mathbb{C}^n$ such that $v_1^* \aleph_o(t_0,t_1) v_1 = 0$. From (3.15),

$$v_1^* \aleph_o(t_0,t_1) v_1 = \int_{\Im} \left\| F(\tau)\Phi_+(\tau,t_0) v_1 \right\|^2 d\tau = 0$$

Hence, $F(\tau)\Phi_+(\tau,t_0)v_1 \equiv 0$ over $\Im$. If we choose the initial condition $X_0 = v_1 v_1^*$, which is a non-zero matrix. The output is

$$Y(t) = F(t)\Phi_+(t,t_0) v_1 v_1^* \Phi_-^*(t,t_0) \equiv 0 \text{ in } \Im \,.$$

Hence, the initial state $X_0 = v_1 v_1^*$ cannot be uniquely determined from the above equation. This contradicts the condition that the system is completely observable. ■

Similar to the treatment of controllability, we will derive algebraic conditions on the observability of (2.1) and (2.2).

Definition 3.5 A sequence of $n \times n$ matrix functions $Q_k(t)$ over $\Im$ associated with the matrix DAE system (2.1) and (2.2) are defined recursively as follows, with the assumption that all coefficient matrices in (2.1) and (2.2) are differentiable

$$\text{(i)}\; Q_0(t) = F^T(t)F(t) \tag{3.16}$$

$$\text{(ii)}\; \dot{Q}_{k-1}(t) = Q_k(t) - Q_{k-1}(t)E^+A(t) - B(t)Q_{k-1}(t) \quad k = 1,2,... \tag{3.17}$$

Lemma 3.3 For all $t, s \in \Im$, the following holds,

$$\frac{\partial^k}{\partial t^k}\left[\Phi_-^*(t,s)Q_0(t)\Phi_+(t,s)\right] = \Phi_-^*(t,s)Q_k(t)\Phi_+(t,s), \quad k = 1,2,... \tag{3.18}$$

Notice that the recursive relation (3.16), (3.17) along with the derivative relation (3.18) are different from those for controllability, i.e. (3.4), (3.5), and (3.7). The proof of Lemma 3.3 is similar to that of Lemma 3.2 with the exception that Lemma 3.1 is not needed in the proof. Lemma 3.3 can be used to prove the following observability theorem for time-varying matrix DAE systems (2.1) and (2.2).

Theorem 3.5 Suppose the matrix functions in the linear time-varying matrix DAE system (2.1) and (2.2) satisfy the smoothness condition, i.e., $A(t), B(t), E^+(t) \in C_{\Im}^{\ell-1}$ and $F(t) \in C_{\Im}^{\ell}$. Then, system (2.1) and (2.2) is completely observable if there exists $t_\beta \in \Im$ such that the observability matrix

$$\Omega_o(t_\beta) = \begin{bmatrix} Q_0(t_\beta) \\ Q_1(t_\beta) \\ \vdots \\ Q_l(t_\beta) \end{bmatrix} \tag{3.19}$$

is full column rank, i.e. $r(\Omega_o(t_\beta)) = n$.

For linear time-invariant DAE system (2.1) and (2.2), a binomial formula for Q_m is obtained from (3.16) and (3.17),

$$Q_m = \sum_{k=0}^{m} \binom{m}{k} B^k F^T F (E^+ A)^{m-k}.$$

Finally, we obtain the observability criteria for the LTI DAE system (2.1) and (2.2).

Theorem 3.6 The linear time-invariant matrix DAE system with I/O structures is completely observable over $\Im$ if either of the following observability matrices is full rank, i.e. $r(\Omega_{lo}) = n$ or $r(\Omega_{ro}) = n$, where

$$\Omega_{lo} = \begin{bmatrix} F^T F \\ F^T F E^+ A \\ \vdots \\ F^T F (E^+ A)^{n-1} \end{bmatrix} \text{ and } \Omega_{ro} = \begin{bmatrix} F^T F \\ B F^T F E^+ \\ \vdots \\ B^{n-1} F^T F E^+ \end{bmatrix}.$$

IV. Conclusion

Controllability and observability criteria for the matrix differential algebraic systems (2.1) and (2.2) are derived for linear time-varying and linear time-invariant cases, respectively. Due to the unique structure of the system, closed form solutions are obtained to construct the Gramians and recursive relations for the controllability and observability matrices. The problem becomes more complicated if the descriptor matrix does not appear on the right side of (2.1) as a closed form solution may not be available for the new system. This is an ongoing research and our preliminary results indicate that certain systems can be transformed into the form of (2.1).

Acknowledgment

The author would like to thank the reviewers for their helpful comments.

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