Sampling - Reconstruction Procedures of Non Gaussian Processes by Two Algorithms

Vladimir Kazakov, and Yair Olvera

Abstract—Two algorithms are investigated for the Sampling - Reconstruction Procedures of non Gaussian processes. The optimal algorithm is analyzed on the basis of the conditional mean rule and cumulant functions. The non optimal algorithm is based on the covariance function of the output process. Using this algorithm we obtain the total approximate reconstruction error function. We investigate the Rayleigh processes and the non Gaussian processes on the output of exponential and polynomial converters driven by the Gaussian Markov process. Comparison of both algorithms is given.

Keywords—Conditional mean rule, Non Gaussian process, Non linear converter, Sampling – Reconstruction Procedure.

I. Introduction

The classical sampling theorem, usually associated with the names of Whittaker, Kotelnikov and Shannon, is valid for deterministic functions with a restricted amplitude and phase density spectrum. This theorem has been generalized by A. Balakrishnan for stationary random processes with a restricted power spectrum [1]. Balakrishnan's theorem (BT) is characterized by some drawbacks: the probability density function (pdf) of a sampled process is not used; the model of the sampled process is no realizable; the number of samples is equal to infinity; the reconstruction procedure is linear and the same for all types of random processes; the reconstruction error is equal to zero for all types of processes.

In order to overcome these drawbacks, we use the conditional mean rule (CMR) [2, 3]. This rule has been applied to the statistical description of the Sampling – Reconstruction Procedure (SRP) of various types of random processes [4-12]. On the basis of CMR one can analyze the SRP of random processes with different types of pdf, taking into account the following aspects: the process can be stationary or non stationary; the number of samples is arbitrary and limited; the intervals between neighbor samples can be arbitrary or periodical; etc. Generally, one can declare: any random process has its own optimal reconstruction algorithm and reconstruction error function. In the case of Gaussian processes the reconstruction function is a linear function of samples and the reconstruction error function does not depend on the samples. If the sampled process is non Gaussian then the reconstruction function is a non linear function of samples and the reconstruction error function depends on the samples.

The present paper is devoted to the statistical SRP description of non Gaussian stationary processes. First, we investigate the SRP of a Rayleigh process, and then we give the SRP analysis of some output processes of non linear converters, driven by the Gaussian Markov process. In this part we designate the output process $\eta(t)$ of a non linear converter driven by the given input process $\xi(t)$. The types of non linearity are exponential and polynomial.

We analyze the SRP of non Gaussian processes by using two different reconstruction algorithms. The first algorithm is optimal. Using CMR, we obtain the reconstruction function and the reconstruction error function. The CMR estimation provides the minimum of the mean square error automatically. The second algorithm is non optimal, but is simple. The reconstruction function is formed by using the covariance function $K(\tau)$ and the mean $m(t)$ of the output process. It means that we apply the reconstruction algorithm which is optimal for Gaussian processes. We compare both algorithms.

II. The Optimal Reconstruction Algorithm

A. Generals Remarks

The methodology of this work is based on the conditional mean rule. The principal idea of the application of this algorithm for the SRP has been proposed in [4]. We consider a stochastic process $\xi(t)$ characterized by its multidimensional probability functions $w_m[\xi(t_1),\xi(t_2),\ldots,\xi(t_m)]$. One realization of this process is discretized in time instants $T=\{T_1,T_2,\ldots,T_N\}$. Therefore, we form a set of samples $\Xi,T=\xi(T_1),\xi(T_2),\ldots,\xi(T_N)$, where the number of samples N and their times of occurrence T are arbitrary. This means that the initial and central moment functions and their probability densities are modified. Now, they are conditional, and depend on the value of each sample $\xi(T_1),\xi(T_2),\ldots,\xi(T_N)$ [3]. In this way, the probability density function, the reconstruction function, and the reconstruction error function are:

$$w[\xi(t)|\Xi,T]=w[\xi(t)|\xi(T_1),\xi(T_2),\ldots,\xi(T_N)], \quad (1)$$

$$\tilde{m}_\xi(t)=\langle \xi(t)|\Xi,T\rangle=\langle \xi(t)|\xi(T_1),\xi(T_2),\ldots,\xi(T_N)\rangle = \int_{-\infty}^{\infty}\xi(t)w[\xi(t)|\Xi,T]d\xi(t), \quad (2)$$

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$$\tilde{\sigma}_{\xi}^{2}(t) = \left\langle (\xi(t) - \tilde{m}(t))^2 \,|\Xi, T \right\rangle = \int_{-\infty}^{\infty} (\xi(t) - \tilde{m}(t))^2 \, w\left[\xi(t)|\Xi, T\right] d\xi(t) \quad (3)$$

It is evident that the SRP depends on the set of samples Ξ, T and the pdf w_m. One cannot know exactly the sampled realization, but it can get a statistical approach for each moment of time t. We use the conditional mean rule, because it provides the minimum estimation error for random variables with an arbitrary pdf.

In order to obtain both principal SRP characteristics $\tilde{m}(t)$ and $\tilde{\sigma}^2(t)$, it is necessary to know a multidimensional pdf. However it is impossible for the majority of non Gaussian processes. Below we investigate the SRP of the realizations of two types of non Gaussians processes. The first type is a non Gaussian Markov process. In this case, the interpolation SRP depends on two neighbor samples only. So, we need to express the required conditional pdf $w(\xi, t|\xi(T_1), \xi(T_2))$ by the given conditional pdf $w(\xi_i, t_i|\xi_j, t_j)$. The second type of non Gaussian process is formed on the output of non linear converters driven by a Gaussian process. In this case, it is not necessary to know the multidimensional pdf of the output sampled process. There is a method for the recalculation of the output conditional characteristics by the conditional characteristics of the input process [4]. Let us consider the general expressions for both methods.

B. *The SRP of Non Gaussian Markov Process Realizations*

Taking into account the main property of the Markov processes, we have to use one $\xi(T_1)$ or two $\xi(T_1), \xi(T_2)$ samples only for shaping the optimal reconstruction function in the Markov case. It is possible to fully describe the conditional process between two given samples knowing the conditional pdf $w(\xi, t|\xi(T_1), \xi(T_2))$. At this point highlights the fact that the process $\tilde{\xi}(t)$ between two samples of a Markov process $\xi(t)$ is non Markov, but it is correct to represent its conditional pdf on the basis of the transitional pdf $w(\xi_i, t_i|\xi_j, t_j)$ of the given Markov process. If there are three sections of the process $\xi(t)$ at the times $t_1 < t < t_2$, in the non Markov general case the three-dimensional pdf is presented by [6]:

$$w(\xi_1, t_1; \xi, t; \xi_2, t_2) = w(\xi_1, t_1) \, w(\xi, t|\xi_1, t_1) \times \times w(\xi_2, t_2|\xi_1, t_1; \xi, t) \quad (4)$$

In the last equation there are three different types of pdf. The most difficult to obtain is the third. In the Markov variant this three-dimensional pdf could be represented as a product of an one-dimensional pdf and a bi- transitional pdf:

$$w(\xi_1, t_1; \xi, t; \xi_2, t_2) = w(\xi_1, t_1) w(\xi, t|\xi_1, t_1) \times \times w(\xi_2, t_2|\xi, t) \quad (5)$$

Thus, the required conditional pdf can be expressed by the transitional pdf $w(\xi_i, t_i|\xi_j, t_j)$ of the given Markov process. Namely, one can see that:

$$w(\xi, t|\xi_1, t_1; \xi_2, t_2) = \frac{w(\xi, t|\xi_1, t_1) w(\xi_2, t_2|\xi, t)}{w(\xi_2, t_2|\xi_1, t_1)}. \quad (6)$$

With this expression, one can obtain the conditional moments of *i-th* order [3]:

$$\tilde{m}_i^{\xi}(t) = \left\langle \xi^j(t) | \xi(T_1), \xi(T_2) \right\rangle. \quad (7)$$

When $i = 1$ we have the reconstruction function or conditional mathematical expectation $\tilde{m}_1^{\xi}(t)$, when $i = 2$ the second moment function $\tilde{m}_2^{\xi}(t)$. Then using:

$$\tilde{\sigma}_{\xi}^2(t) = \tilde{m}_2^{\xi}(t) - \left[\tilde{m}_1^{\xi}(t)\right]^2, \quad (8)$$

one can find the conditional variance or reconstruction error function $\tilde{\sigma}_{\xi}^2(t)$.

C. *The SRP of Non Gaussian Markov Process Realizations on the Output of Non Linear Converters*

Let us consider a non linear non inertial converter with the direct function $\eta(t) = g[\xi(t)]$. This function has its inverse function $\xi(t) = h[\eta(t)]$. There is a restriction: the inverse function must be simple. We form an arbitrary set of samples $\eta, T = \{\eta(T_1), \eta(T_2), \ldots, \eta(T_N)\}$. Using the inverse function, we find the corresponding set of samples of the input process $\Xi, T = \{\xi(T_1), \xi(T_2), \ldots, \xi(T_N)\}$. Then one can express the conditional moment function $\tilde{m}_i^{\eta}(t|\eta, T) = \tilde{m}_i^{\eta}(t)$ of the output process by the conditional moment functions of the input process $\tilde{m}_i^{\xi}(t|\Xi, T) = \tilde{m}_i^{\xi}(t)$ [4]:

$$\tilde{m}_i^{\eta}(t) = \left\langle g\left[\xi^i(t)\right] \Xi(T) \right\rangle. \quad (9)$$

Using (2) and (9) we apply the statistical conditional average operation to both parts of the non linear function for obtain the reconstruction function or the first conditional moment $\tilde{m}_1^{\eta}(t)$:

$$\tilde{m}_1^{\eta}(t) = \left\langle g[\xi(t)] \Xi(T) \right\rangle. \quad (10)$$

Based on (9), we obtain the second conditional moment $\tilde{m}_2^{\eta}(t)$. Now, by (3) it is possible to find the conditional variance or the reconstruction error function $\tilde{\sigma}_{\eta}^2(t)$:

$$\tilde{\sigma}_{\eta}^2(t) = \tilde{m}_2^{\eta}(t) - \left[\tilde{m}_1^{\eta}(t)\right]^2. \quad (11)$$

It is clear that the reconstruction function and the reconstruction error function depend on the samples.

III. The Non Optimal Reconstruction Algorithm

We describe a non optimal reconstruction algorithm by using a Gaussian approximation. It means that we determine the reconstruction function and the reconstruction error function on the basis of the mathematical expectation, the variance and the covariance function of the non Gaussian process. So, we form the reconstruction function $\hat{m}_1^{\xi}(t)$ for the non Gaussian process as a conditional mathematical expectation function of the Gaussian process [13]:

$$\hat{m}_1^{\xi}(t) = m^{\xi}(t) + \sum_{i=1}^{N}\sum_{j=1}^{N} K_{\xi}(t, T_i) a_{ij}\left[\xi(T_j) - m^{\xi}(T_j)\right]. \quad (12)$$

This choice generates a special deterministic part of the reconstruction error function:

$$\varepsilon_d^2(t) = \left[\hat{m}_1^{\xi}(t) - \tilde{m}_1^{\xi}(t)\right]^2. \quad (13)$$

We call it as the first part of the reconstruction error.

Besides this, there is the second part of the reconstruction error function. This is a random component of the reconstruction error. We describe this second part of error on the basis of the Gaussian approximation:

$$\hat{\sigma}_{\xi}^2(t) = \sigma_{\xi}^2(t) - \sum_{i=1}^{N}\sum_{j=1}^{N} K_{\xi}(t, T_i) a_{ij} K_{\xi}(T_j, t). \quad (14)$$

Here $m^{\xi}(t)$ is the mathematical expectation, $\sigma_{\xi}^2(t)$ is the variance, $K_{\xi}(\tau)$ is the covariance function and, a_{ij} is an element of the inverse covariance matrix $A = K_{\xi}^{-1}(T_i, T_j)$ of the Gaussian process. Then, the total approximate reconstruction error function $\varepsilon_t^2(t)$ is determined by:

$$\varepsilon_t^2(t) = \varepsilon_d^2(t) + \hat{\sigma}_{\xi}^2(t). \quad (15)$$

This error depends on the samples. It is important to mention that if we consider a non Gaussian process on the output of a non linear converter, we must change all these parameters according the output process $\eta(t)$. It means that now we have $\hat{m}_1^{\eta}(t)$ as the reconstruction function.

IV. SRP of a Rayleigh Process

We start with a popular case of all non Gaussian processes, a Rayleigh process. The probability density function is:

$$w(\xi) = \frac{\xi}{\sigma^2}\exp\left(-\frac{\xi^2}{2\sigma^2}\right), \quad (16)$$

where $\sigma^2 = 1$ is a parameter. Fig. 1 shows this pdf (16). SRP analysis of Rayleigh random processes is based on expressions of the transitional pdf. From (4) – (6) and considering a Rayleigh Markov process, the transitional pdf is [13]:

$$w(\xi_i, t_i | \xi_j, t_j) = \frac{\xi_i}{\sigma^2(1-Q^2)}\exp\left\{-\frac{\xi_i^2 + Q^2\xi_j^2}{2\sigma^2(1-Q^2)}\right\} \times \times I_0\left(\frac{Q}{1-Q^2}\frac{\xi_i\xi_j}{\sigma^2}\right), \quad (17)$$

where I_0 is the modified Bessel function of zero order, and $Q = \exp(-\alpha|\tau|)$ is the auxiliary function used as covariance function. Having the transitional pdf, we can obtain the reconstruction function for the optimal algorithm $\tilde{m}_1^{\xi}(t)$:

$$\tilde{m}_1^{\xi}(t) = \int_{-\infty}^{\infty} \xi w(\xi, t | \Xi, T) d\xi. \quad (18)$$

With (7) we find the second conditional moment $\tilde{m}_2^{\xi}(t)$. The reconstruction error function $\tilde{\sigma}_{\xi}^2(t)$ is given in (8).

The covariance function $K_{\xi}(\tau)$ is [13]:

$$K_{\xi}(\tau) = \frac{\pi\sigma^2}{2}\left\{\left(\frac{1}{2}\right)^2 Q^2(\tau) + \left(\frac{1}{8}\right)^2 Q^4(\tau) + \ldots\right\}. \quad (19)$$

The covariance function (19) is presented in Fig. 2. Substituting (19) in (12) and (14) we obtain the reconstruction function $\hat{m}_1^{\xi}(t)$, and the second part of the reconstruction error $\hat{\sigma}_{\xi}^2(t)$ for the non optimal algorithm respectively.

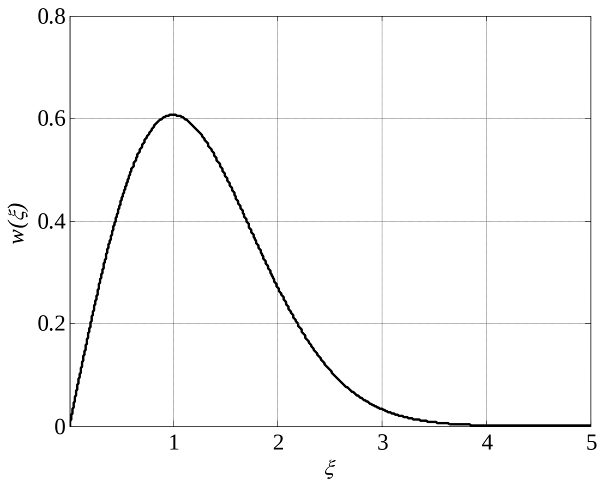

Fig. 1 Pdf of the Rayleigh process

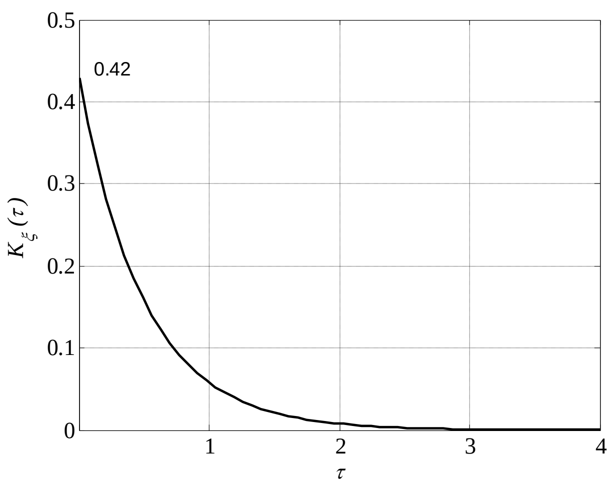

Fig. 2 Covariance function of the Rayleigh process

As example we consider two samples with a separation of 0.5 seconds. The reconstruction succeeds in the interpolation region. The values of the samples are presented in Table I. We show two cases (A and B) for both algorithms.

TABLE I
SAMPLES FOR THE RAYLEIGH PROCESS

	$\xi\ (T_1)$	$\xi\ (T_2)$
A	*0.5*	*1*
B	*1*	*2*

The results of the reconstruction functions are presented in Fig. 3. It is clear that the curves in the optimal algorithm (continuous lines) and the non optimal algorithm (dotted lines) are almost equals. The reason is that the Rayleigh conditional pdf is closely linked to the Gaussian conditional pdf.

The difference is the reconstruction error function, which is illustrated in Fig 4. There is one error curve for each case or pair the samples in both algorithms. However, the total approximate reconstruction errors for A and B have the same behavior. That is because the difference between the reconstruction functions of both algorithms is minimal. Also, there is a set of samples in the optimal algorithm with a smaller error than the curves in the non optimal algorithm. It is necessary to emphasize that we use the same covariance function, the same values of samples, and the same sampled interval in both algorithms.

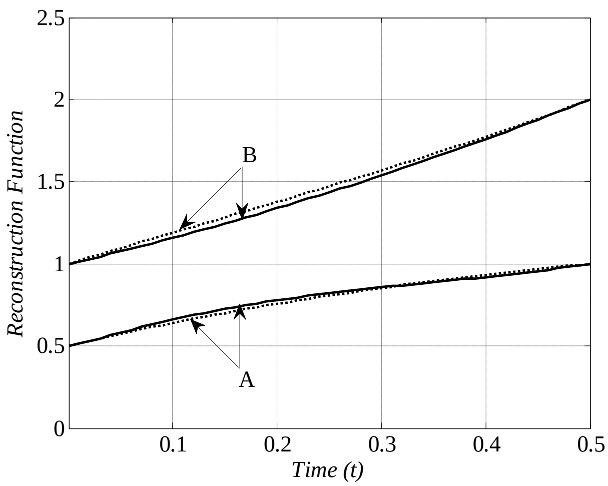

Fig. 3 Reconstruction functions for the Rayleigh process

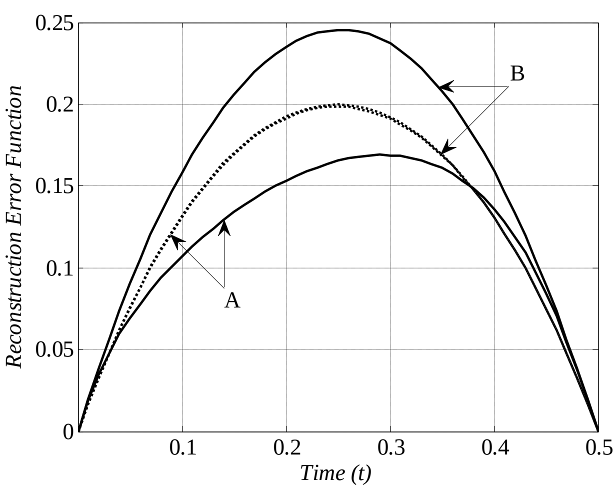

Fig. 4 Reconstruction error functions for the Rayleigh process

V. SRP on the Output of an Exponential Converter

Now we can explain the non Gaussian processes which are formed by a non linear converter driven by a Markov Gaussian process. The exponential non linearity is:

$$\eta(t) = g[\xi(t)] = a_0 \exp[\beta \xi(t)], \tag{20}$$

where a_0 y β are constants. Let us assume that the input process $\xi(t)$ is Gaussian Markov with characteristics $\langle \xi(t) \rangle = 0$, $\sigma_\xi^2 = 1$ and $K_\xi(\tau) = \exp(-\alpha|\tau|)$. Putting $\alpha = 1$ one can find the expressions for the output process $\eta(t)$.

The one dimensional pdf $w(\eta)$ is found in [10]:

$$w(\eta) = \frac{1}{a_0 \beta \sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{1}{\beta} \ln \frac{\eta}{a_0} \right)^2 - \ln \frac{\eta}{a_0} \right). \tag{21}$$

According to *"C"*, we establish the expressions for the reconstruction function $\tilde{m}_1^\eta(t)$ and the second conditional moment $\tilde{m}_2^\eta(t)$ for the optimal algorithm [13]:

$$\tilde{m}_1^\eta(t) = a_0 \exp\left\{ \beta \tilde{m}_1^\xi(t) + \frac{1}{2}\beta^2 \right\}, \tag{22}$$

$$\tilde{m}_2^\eta(t) = a_0^2 \exp\left\{ 2\beta \tilde{m}_1^\xi(t) + \beta^2 + \beta^2 \tilde{\sigma}_\xi^2(t) \right\}, \tag{23}$$

where $\tilde{m}_1^\xi(t)$ is the conditional mathematic expectation and $\tilde{\sigma}_\xi^2(t)$ is the conditional variance:

$$\tilde{m}_1^\xi(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} K_\xi(t - T_i) a_{ij} \xi(T_j), \tag{24}$$

$$\tilde{\sigma}_\xi^2(t) = 1 - \sum_{i=1}^{N} \sum_{j=1}^{N} K_\xi(t - T_i) a_{ij} K_\xi(T_j - t). \tag{25}$$

The reconstruction error function $\tilde{\sigma}_\eta^2(t)$ is obtained substituting (22) and (23) in (11).

Now we need to know the covariance function $K_\eta(\tau)$ [14]:

$$K_\eta(\tau) = a_0^2 \exp(\beta^2) \left[\exp\left(\beta^2 R_\xi(\tau)\right) - 1 \right]. \tag{26}$$

Taking the non optimal algorithm, the reconstruction function $\hat{m}_1^\xi(t)$ and the second part of the approximate reconstruction error function $\hat{\sigma}_\xi^2(t)$ are obtained substituting (26) in (12) and (14).

As example, we use the next type of exponential non linear function, where $a_0 = \beta = 1$:

$$\eta(t) = \exp[\xi(t)]. \tag{27}$$

The non linearity, pdf and covariance function for (27) are presented in Fig. 5, Fig. 6, and Fig. 7 respectively.

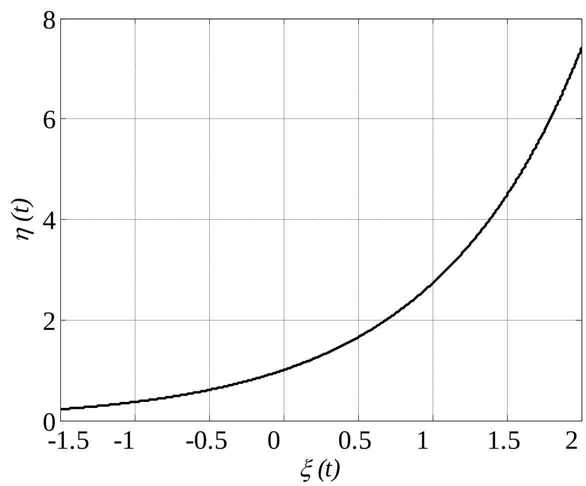

Fig. 5 Non linearity for $\eta(t) = \exp[\xi(t)]$

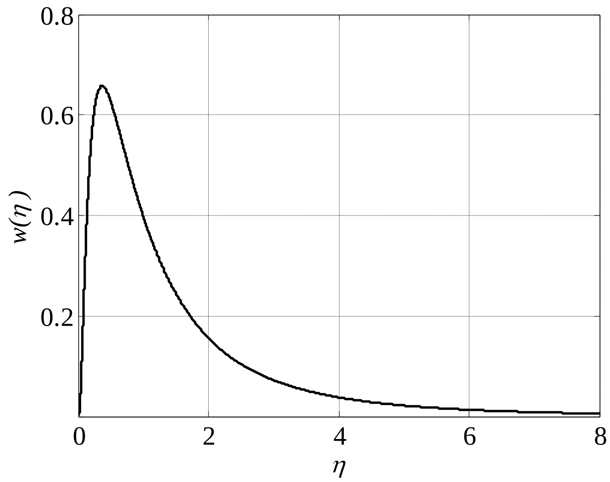

Fig. 6 Pdf of the process on the output of the exponential converter $\eta(t) = \exp[\xi(t)]$

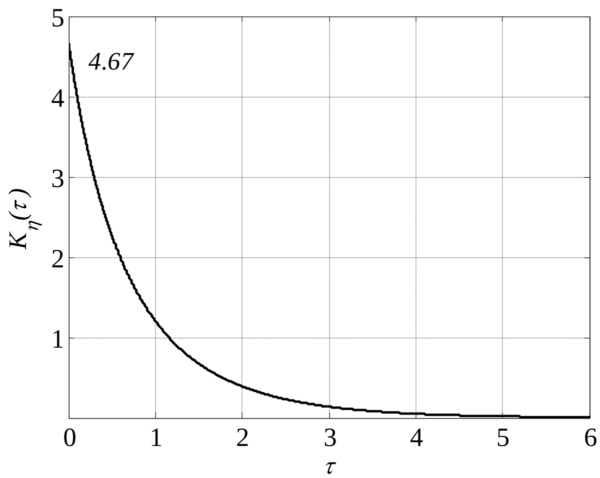

Fig. 7 Covariance function of the process on the output of the exponential converter $\eta(t) = \exp[\xi(t)]$

We consider the case with two samples separated 0.1 seconds. The values are given in Table II for A and B.

TABLE II

SAMPLES FOR THE EXPONENTIAL NON LINEARITY

	$\xi\ (T_1)$	$\xi\ (T_2)$	$\eta\ (T_1)$	$\eta\ (T_2)$
A	*1.5*	*0*	*4.4*	*1*
B	*0.5*	*1*	*1.6*	*2.7*

The calculation results of the reconstruction functions are illustrated in Fig. 8. It is clear that the curves of the optimal algorithm (continuous lines) are more precise in comparison with the curves of the non optimal algorithm (dotted lines). The difference is more evident by increasing the value of samples or the sampling interval.

In Fig. 9 we can see the calculation results of the reconstruction error functions. There are various types of curves. As in the Rayleigh examples, the error reconstruction function of both algorithms depends on the samples values. In other words, any couple of samples has its own error function. This effect occurred owing to the non Gaussian character of the sampled process $\eta(t)$. In the non optimal algorithm the total error curves reflect the approximated error. These curves are bigger than the error curves in the optimal algorithm because there is a considerable difference in the reconstruction function. Equally, some set of samples in the optimal algorithm have a lower error than the non optimal algorithm.

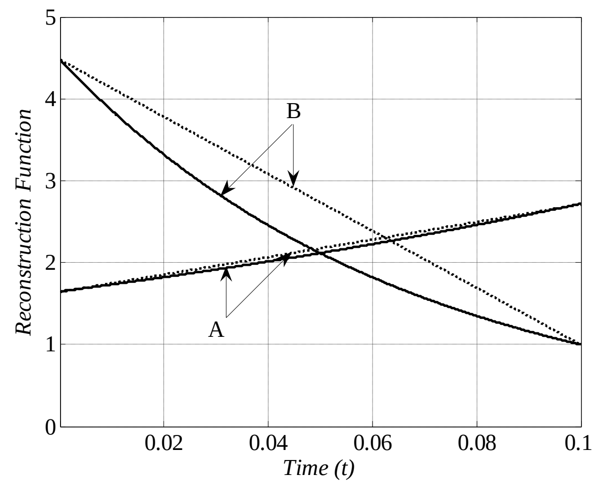

Fig. 8 Reconstruction functions for $\eta(t) = \exp[\xi(t)]$

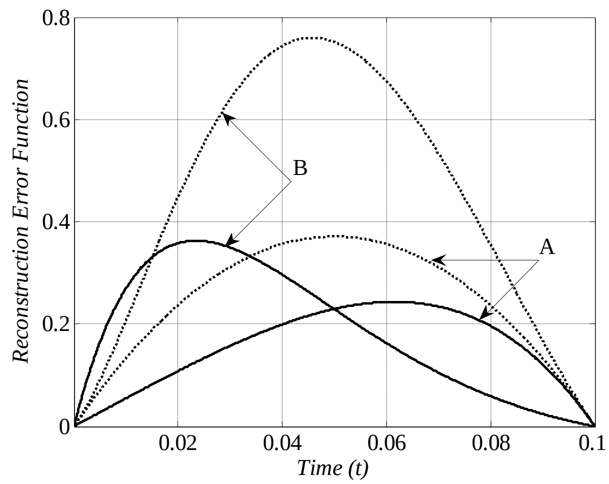

Fig. 9 Reconstruction error functions for $\eta(t) = \exp[\xi(t)]$

VI. SRP on the Output of a Polynomial Converter

Finally we describe the SRP of the process on the output of a polynomial converter. The non linearity is:

$$\eta(t) = g[\xi(t)] = a_0 + a_1\xi(t) + a_2\xi^2(t) + \ldots + a_n\xi^n(t), \quad (28)$$

where $a_i\,(i = 0,1,2,\ldots,n)$ are constant. In the same form that in the exponential converter, the input process $\xi(t)$ is Gaussian Markov with $\langle\xi(t)\rangle = 0$, $\sigma_\xi^2 = 1$ and $K_\xi(\tau) = \exp(-\alpha|\tau|)$.

The expression for the one dimensional pdf $w(\eta)$ can be determined by:

$$w(\eta) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\xi^2\right)\frac{1}{\left|\frac{d\eta}{d\xi}\right|}. \quad (29)$$

Following “C” we obtain the reconstruction function $\tilde{m}_1^\eta(t)$

and the second conditional moment $\tilde{m}_2^{\eta}(t)$ for the optimal algorithm [4]:

$$\tilde{m}_1^{\eta}(t) = a_0 + a_1\tilde{m}_1^{\xi}(t) + a_2\tilde{m}_2^{\xi}(t) + \ldots + a_n\tilde{m}_n^{\xi}(t), \qquad (30)$$

$$\begin{aligned}\tilde{m}_2^{\eta}(t) = {} & a_0^2 + a_1^2\tilde{m}_2^{\xi}(t) + \ldots + a_n^2\tilde{m}_{2n}^{\xi}(t) + \\ & + 2a_0a_1\tilde{m}_1^{\xi}(t) + \ldots + 2a_{n-1}a_n\tilde{m}_{2n-1}^{\xi}(t)\end{aligned}. \qquad (31)$$

These equations show that the conditional output moments require the knowledge of the input conditional moments of high orders. The Gaussian input process $\xi(t)$ is only characterized by the cumulant functions of the first and second orders: $\tilde{k}_1^{\xi}(t)$, $\tilde{k}_2^{\xi}(t)$ and its covariance function. The cumulants of higher order are equal to zero. Following [10] we write the relations between conditional moment functions $\tilde{m}_i^{\xi}(t)\,(i = 1,2,\ldots,N)$ and conditional cumulant functions $\tilde{k}_i^{\xi}(t)\,(i = 1,2)$ [13]:

$$\begin{aligned}\tilde{m}_1^{\xi} &= \tilde{k}_1 \\ \tilde{m}_2^{\xi} &= \tilde{k}_2 + \tilde{k}_1^2 \\ \tilde{m}_3^{\xi} &= 3\tilde{k}_2\tilde{k}_1 + \tilde{k}_1^3 \\ \tilde{m}_4^{\xi} &= 3\tilde{k}_2^2 + 6\tilde{k}_2\tilde{k}_1^2 + \tilde{k}_1^4 \\ \tilde{m}_5^{\xi} &= 15\tilde{k}_2^2\tilde{k}_1 + 10\tilde{k}_2\tilde{k}_1^3 + \tilde{k}_1^5 \\ &\vdots\end{aligned}, \qquad (32)$$

As the input process $\xi(t)$ is stationary, both input conditional cumulant functions are [9]: $\tilde{k}_1^{\xi}(t) = \tilde{m}_1^{\xi}(t)$ expressed by (24) and $\tilde{k}_2^{\xi}(t) = \tilde{\sigma}_{\xi}^2(t)$ by (25). The reconstruction error function $\tilde{\sigma}_{\eta}^2(t)$ is obtained substituting (30) and (31) in (11)

The covariance function $K_{\eta}(\tau)$ is expressed by [14]:

$$K_{\eta}(\tau) = \nu_1^2 K_{\xi}(\tau) + \frac{\nu_2^2}{2!}K_{\xi}^2(\tau) + \ldots + \frac{\nu_n^2}{n!}K_{\xi}^n(\tau), \qquad (33)$$

where ν_n is determined by $\nu_n\left(m_{\xi}, \sigma_{\xi}^2\right) = n!a_n$. The reconstruction function $\hat{m}_1^{\xi}(t)$ and the second part of the approximate reconstruction error function $\hat{\sigma}_{\xi}^2(t)$ for the non optimal algorithm are obtained substituting (33) in (12) and (14) respectively.

As one example, we consider a polynomial transfer function of third order:

$$\eta(t) = -\xi^3(t). \qquad (34)$$

The non linearity, pdf and covariance function for (34) are represented in Fig. 10, Fig. 11, and Fig 12 respectively.

We use two samples in the output realization. The values are in Table III for A and B. They are separated 0.1 seconds.

TABLE III

SAMPLES FOR THE POLYNOMIAL NONLINEARITY

	$\xi\,(T_1)$	$\xi\,(T_2)$	$\eta\,(T_1)$	$\eta\,(T_2)$
A	*1.6*	*0*	*-4.1*	*0*
B	*1.9*	*0*	*-6.8*	*0*

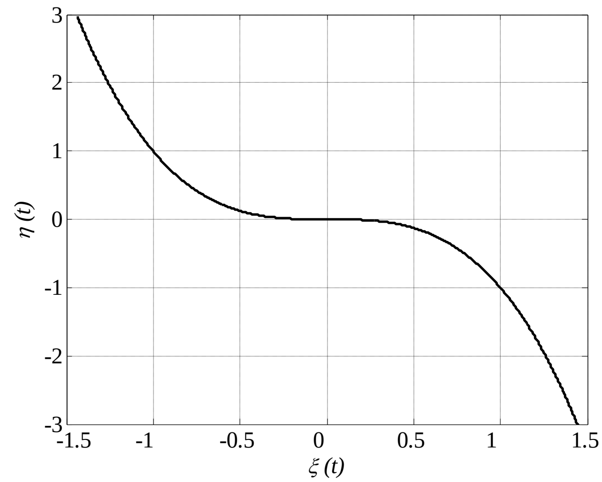

Fig. 10 Non linearity for $\eta(t) = -\xi^3(t)$

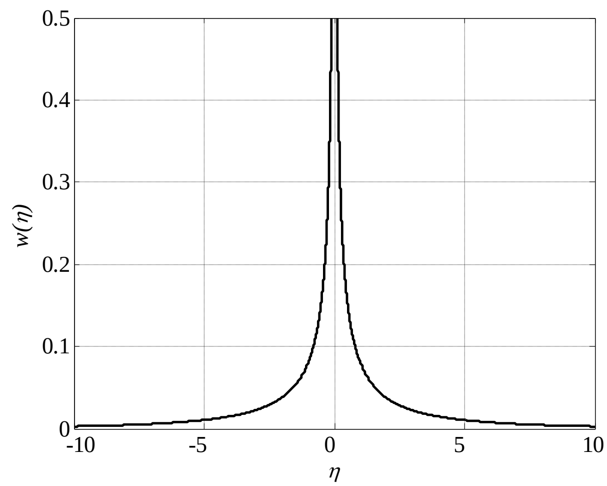

Fig. 11 Pdf of the process on the output of the polynomial converter $\eta(t) = -\xi^3(t)$

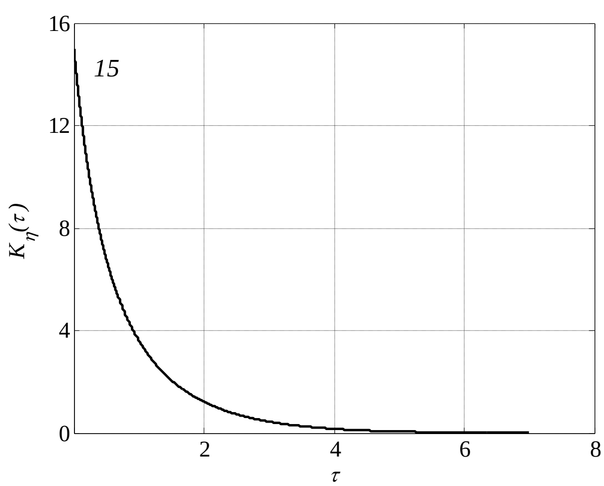

Fig. 12 Covariance function of the process on the output of the polynomial converter $\eta(t) = -\xi^3(t)$

Fig. 13 shows the reconstruction functions for both algorithms. In this graph is more noticeable the difference between the optimal algorithm (continuous lines) and the non optimal algorithm (dotted lines). The curves in the optimal algorithm have a non linear behavior according to their transfer function.

The curves forms of the reconstruction error functions are illustrated in Fig. 14. There is one curve for each pair of samples in both algorithms. The curves have an inclination toward the side where the sample is higher. The inclination grows if the sample is bigger. This effect is more appreciable in the optimal curves. It is possible that some curves in the optimal methodology have a lower error than the curves in the non optimal calculation in some parts of time. It depends on the value of the samples and their separation.

It is important to mention that in the non optimal algorithm we can obtain two error curves. The first curve is the first part of the reconstruction error $\varepsilon_d^2(t)$. This curve is symmetrical like in Gaussian case, and it does not depend on the samples. For that reason there is one curve in each process only. The form of this curve is strange, taking into account that we are reconstructing non Gaussian processes. The second curve represents the total approximate reconstruction error function $\varepsilon_t^2(t)$. One can see that the total error for all examples is bigger. This is a natural effect.

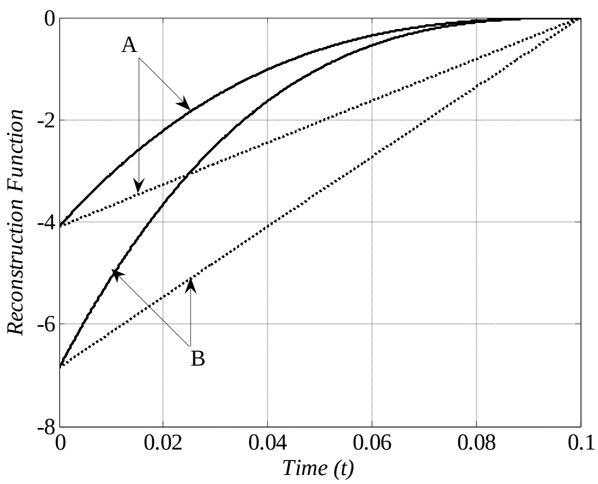

Fig. 13 Reconstruction functions for $\eta(t) = -\xi^3(t)$

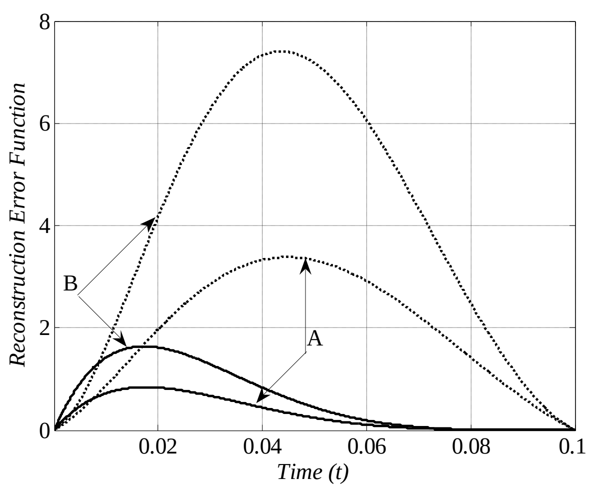

Fig. 14 Reconstruction error functions for $\eta(t) = -\xi^3(t)$

In order to have an unique total approximate reconstruction error function curve, it is necessary to use the statistical operation with respect of two dimensional pdf of both samples $w(\eta(T_1), \eta(T_2))$. It means that we have to find the average of the conditional variance on the output $\left\langle \varepsilon_t^2(t) \right\rangle$.

Analysis of curves in Fig. 14 shows the great difference between two algorithms under consideration. The precise calculations of the reconstruction error functions of the optimal algorithm give the family of various dependences. We need to take into account that there is a big difference in the physical interpretation of both curves: one curve describes the approximate error of the non optimal algorithm, and the second curve illustrates the result of the statistical description of the optimal algorithm.

The real situation is: the optimal reconstruction functions of non Gaussian processes are generally non linear function of samples.

VII. Conclusion

Two different reconstruction algorithms for some non Gaussian processes are analyzed. They describe the SRP for Rayleigh process realizations and for realizations of processes on the output of two non linear converters (exponential and polynomial) driven by a Gaussian Markov process. The principal characteristics, reconstruction function and reconstruction error function are obtained. The results of the investigation demonstrate that the reconstruction error function of non Gaussian processes depends on the samples and must be calculated by the average operation. Also we obtain the total approximate reconstruction error function. This curve has a highest error. So, the optimal algorithm throws the lowest error for a correct reconstruction. Comparison of these algorithms shows that it is necessary to take into account the pdf of the sampled process and more statistical characteristics of the process for an optimal reconstruction.

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