The Hyperbolic Region for Restricted Isometry Constants in Compressed Sensing

Shiqing Wang, Yan Shi, and Limin Su

***Abstract*—**The restricted isometry constants (RIC) play an important role in compressed sensing since if RIC satisfy some bounds then sparse signals can be recovered exactly in the noiseless case and estimated stably in the noisy case. During the last few years, some bounds of RIC have obtained. The bounds of RIC δ_{2k} among them were introduced by Candes (2008), Foucart and Lai (2009), Foucart (2010), Cai et al (2010), Mo and Li (2011). In the paper, we obtain a hyperbolic region on δ_{2k} and δ_k. It completely includes the regions of the bounds on δ_{2k} obtained by the authors above, and if δ_{2k} and δ_k belong to the hyperbolic region then sparse signals can be recovered exactly in the noiseless case.

***Keywords*—**Compressed sensing, L_1 minimization, restricted isometry property, sparse signal recovery.

I. Introduction

Compressed sensing aims to recover high-dimensional sparse signals based on considerably fewer linear measurements. We consider $y = \Phi\beta$, where the matrix $\Phi \in \mathbb{R}^{n\times p}$ with $n \ll p$, the unknown signal $\beta \in \mathbb{R}^p$. Let $\|\beta\|_0$ be the number of nonzero elements of β and $\|\beta\|_1 \triangleq \sum_{i=1}^{p}|\beta_i|$. The signal β is called k sparse if $\|\beta\|_0 \le k$. Our goal is to reconstruct β based on y and Φ.

A naive approach for solving this problem is to consider L_0 minimization where the goal is to find the sparsest solution in the feasible set of possible solutions. However, this is NP hard and thus is computationally infeasible. It is then natural to consider the method of L_1 minimization which can be viewed as a convex relaxation of L_0 minimization. The L_1 minimization method in this context is

$$\hat{\beta} = \arg\min_{\gamma\in\mathbb{R}^p}\left\{\|\gamma\|_1 \text{ subject to } y = \Phi\gamma\right\}. \quad (1)$$

This method has been successfully used as an effective way for reconstructing a sparse signal in many settings. See, e. g., Donoho and Huo [14], Donoho [13], Candes et al [8-11] and Cai et al [2, 3].

Recovery of high dimensional sparse signals is closely connected with Lasso and Dantzig selectors, e. g., see, Candes et al [11], Bickel et al [1], Wang and Su [19-22]. One of the most commonly used frameworks for sparse recovery via L_1 minimization is the restricted isometry property with a RIC introduced by Candes and Tao [9]. For an $n\times p$ matrix Φ and an integer k, $1\le k\le p$, the k restricted isometry constant $\delta_k(\Phi)$ is the smallest constant such that

$$\sqrt{1-\delta_k(\Phi)}\,\|u\|_2 \le \|\Phi u\|_2 \le \sqrt{1+\delta_k(\Phi)}\,\|u\|_2$$

for every k sparse vector u. If $k+k' \le p$, the k, k' restricted orthogonality constant $\theta_{k,k'}(\Phi)$ is the smallest number that satisfies

$$\left|\langle \Phi u, \Phi u'\rangle\right| \le \theta_{k,k'}(\Phi)\|u\|_2\,\|u'\|_2$$

for all u and u' such that u and u' are k sparse and k' sparse respectively, and have disjoint supports. For notational simplicity, we shall write δ_k for $\delta_k(\Phi)$ and $\theta_{k,k'}$ for $\theta_{k,k'}(\Phi)$ hereafter.

It has been shown that L_1 minimization can recover a sparse signal with a small or zero error under various conditions on δ_k and $\theta_{k,k'}$. So, a great deal of attention has been focused here during the last few year, for example, the conditions involving δ_{ak} and $\theta_{k,bk}$, where $a = 1,2,3,4,1.25,1.5$ and $b = 1,1.25,1.5,2$, see Candes et al [8-10] and Cai et al [2, 5]; the conditions involving only δ_{2k}, see Candes [7], Foucart and Lai [16], Foucart [15], Cai et al [2], Mo and Li [18], and only δ_k, see Cai et al [4], Ji and Peng [17], Cai and zhang [6].

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It is obvious that δ_{2k} and δ_k are two of the most important and basic parameters. In this paper, we obtain the sufficient conditions involving only δ_{2k} and δ_k. It is a hyperbolic region. This hyperbolic region completely includes the regions of the bounds on δ_{2k} in the literature [2, 7, 15, 16, 18], and if δ_{2k} and δ_k belong to the hyperbolic region then sparse signals can be recovered exactly in the noiseless case.

The rest of the paper is organized as follows. In Section 2, some basic notations are introduced and the functions on δ_{2k} and δ_k are given. Our hyperbolic region on δ_{2k} and δ_k is presented in Section 3. In Section 4, we discuss the problem that the hyperbolic region completely includes the regions of the bounds on δ_{2k} in the literature [2, 7, 15, 16, 18]. Other meaningful problems are also discussed.

II. The Functions of the Restricted Isometry Constants

We consider the simple setting where no noise is present. In this case the goal is to recover the signal β exactly when it is sparse. This case is of significant interest in its own right as it is also closely connected to the problem of decoding of linear codes. See, for example, Candes and Tao [9]. The ideas used in treating this special case can be easily extended to treat the general case where noise is present.

Let $\hat{\beta}$ be the minimizer to the problem (1). Let $h=\hat{\beta}-\beta$. For any subset $Q\subset\{1,2,\cdots,p\}$, we define $h_Q=hI_Q$, where I_Q denotes the indicator function of the set Q, i.e., $I_Q(j)=1$ if $j\in Q$ and 0 if $j\notin Q$. Let S_0 be the index set of the k largest elements (in absolute value). Rearrange the indices of S_0^c if necessary according to the descending order of $|h_i|$, $i\in S_0^c$. Partition S_0^c in order into $S_0^c=\sum_{i=1}^{l}S_i$, where $|S_i|=k$, the last S_i satisfies $|S_i|\le k$. For simplicity, when there is no ambiguity we write $h_i=h_{S_i}$, $i=1,2,\cdots,l$.

Let

$$\|h_1\|_1=t\sum_{i\ge1}\|h_i\|_1, \quad (2)$$

then there must be $t\in[1/l,1]$. In fact by the definition of S_i, we have

$$\sum_{i\ge1}\|h_i\|_1\le l\|h_1\|_1=tl\sum_{i\ge1}\|h_i\|_1. \quad (3)$$

If $\sum_{i\ge1}\|h_i\|_1\ne0$ then $t\ge1/l$, if $\sum_{i\ge1}\|h_i\|_1=0$, then the elements of S_0^c are all zero. The following we suppose that $l\ne1$ since $t=1$ when $l=1$.

From (2) we have

$$\sum_{i\ge2}\|h_i\|_1=(1-t)\sum_{i\ge1}\|h_i\|_1. \quad (4)$$

It is obvious from (2) and (4) that

$$\sum_{i\ge2}\|h_i\|_2^2\le\|h_2\|_\infty\sum_{i\ge2}\|h_i\|_1$$
$$\le\frac{\|h_1\|_1}{k}\sum_{i\ge2}\|h_i\|_1\le\frac{t}{k}(1-t)\left(\sum_{i\ge1}\|h_i\|_1\right)^2. \quad (5)$$

Further

$$\sum_{i\ge2}\|h_i\|_2\le\frac{1-3t/4}{\sqrt{k}}\sum_{i\ge1}\|h_i\|_1. \quad (6)$$

In fact from Cai et al [4], (2) and (4)

$$\sum_{i\ge2}\|h_i\|_2\le\frac{1}{\sqrt{k}}\sum_{i\ge2}\|h_i\|_1+\frac{\sqrt{k}}{4}\|h_2\|_\infty$$
$$\le\frac{1}{\sqrt{k}}\sum_{i\ge2}\|h_i\|_1+\frac{\sqrt{k}}{4}\frac{\|h_1\|_1}{k}=\frac{1-3t/4}{\sqrt{k}}\sum_{i\ge1}\|h_i\|_1.$$

Let $\Phi h=0$, then

$$\Phi(h_0+h_1)=-\Phi\left(\sum_{i\ge2}h_i\right).$$

Thus

$$\|\Phi(h_0+h_1)\|_2^2=\left\|\Phi\left(\sum_{i\ge2}h_i\right)\right\|_2^2.$$

The following need to use some basic facts:

$$0<\delta_k\le\delta_{2k}<1,\ 0<\theta_{k,k}\le\delta_{2k}<1,$$

see Candes et al [7, 9]. It is easy to see by Candes and Tao [9] that

$$\|\Phi(h_0+h_1)\|_2^2\ge(1-\delta_{2k})\|h_0+h_1\|_2^2$$
$$=(1-\delta_{2k})\left(\|h_0\|_2^2+\|h_1\|_2^2\right)\ge(1-\delta_{2k})\left(\|h_0\|_1^2+\|h_1\|_1^2\right)/k \quad (7)$$

By the definition of δ_k and δ_{2k}, (5) and (6), we have

$$\left\|\Phi\left(\sum_{i\ge2}h_i\right)\right\|_2^2=\sum_{i,j\ge2}\langle\Phi h_i,\Phi h_j\rangle$$

$$= \sum_{i\geq 2} \|\Phi h_i\|_2^2 + 2\sum_{j>i\geq 2} \langle \Phi h_i, \Phi h_j \rangle$$

$$\leq \sum_{i\geq 2} (1+\delta_k) \|h_i\|_2^2 + 2\delta_{2k} \sum_{j>i\geq 2} \|h_i\|_2 \|h_j\|_2$$

$$\leq \frac{1}{k}\left[(1+\delta_k)t(1-t) + \delta_{2k}(1-5t/4)^2\right]\left(\sum_{i\geq 1}\|h_i\|_1\right)^2.$$

From (7),

$$(1-\delta_{2k})\left(\|h_0\|_1^2 + \|h_1\|_1^2\right) \leq \left[(1+\delta_k)t(1-t) + \delta_{2k}(1-5t/4)^2\right]\left(\sum_{i\geq 1}\|h_i\|_1\right)^2.$$

From (2),

$$\|h_0\|_1^2 \leq \frac{\delta_k t(1-t) + \delta_{2k}\left(1 - 5t/2 + 41t^2/16\right) + t - 2t^2}{1-\delta_{2k}}\left(\sum_{i\geq 1}\|h_i\|_1\right)^2 \quad (8)$$

$$= \frac{16\delta_{2k} + 8(2+2\delta_k - 5\delta_{2k})t - (32+16\delta_k - 41\delta_{2k})t^2}{16(1-\delta_{2k})}\left(\sum_{i\geq 1}\|h_i\|_1\right)^2$$

.

Theorem 1. If $5\delta_{2k} - 2\delta_k < 2$, then

$$\|h_0\|_1 \leq \xi_k \sum_{i\geq 1}\|h_i\|_1,$$

where

$$\xi_k \triangleq \sqrt{\frac{4\left(1+2\delta_k + 3\delta_{2k} + \delta_k^2 - \delta_k\delta_{2k} - 4\delta_{2k}^2\right)}{(1-\delta_{2k})(32+16\delta_k - 41\delta_{2k})}}.$$

Proof Consider the function

$$f(t) \triangleq 16\delta_{2k} + 8(2+2\delta_k - 5\delta_{2k})t - (32+16\delta_k - 41\delta_{2k})t^2,$$

where $t \in (-\infty, +\infty)$. We firstly show that

$$32 + 16\delta_k - 41\delta_{2k} > 0.$$

Davies and Gribonval [10] constructed examples which showed that if $\delta_{2k} \geq 1/\sqrt{2}$, exact recovery of certain k-sparse signal can fail in the noiseless case. This implies that there must be $\delta_{2k} < 1/\sqrt{2}$ in order to guarantee stable recovery of k-sparse signals. So,

$$41\delta_{2k} < 41/\sqrt{2} < 32 < 32 + 16\delta_k.$$

Deriving $f(t)$ with respect to t , then have

$$f'(t) = 8(2+2\delta_k - 5\delta_{2k}) - 2(32+16\delta_k - 41\delta_{2k})t.$$

The stationary point is

$$t_0 = 4\frac{2+2\delta_k - 5\delta_{2k}}{32+16\delta_k - 41\delta_{2k}}.$$

If $5\delta_{2k} - 2\delta_k < 2$, then $t_0 > 0$. The function $f(t)$ is monotone increasing when $t \leq t_0$ and monotone decreasing when $t \geq t_0$. So $f(t)$ reaches the maximum when $t = t_0$, and the maximum is

$$f(t_0) = 64\frac{1+2\delta_k + 3\delta_{2k} + \delta_k^2 - \delta_k\delta_{2k} - 4\delta_{2k}^2}{32+16\delta_k - 41\delta_{2k}}.$$

Hence

$$\|h_0\|_1^2 \leq \frac{f(t_0)}{16(1-\delta_{2k})}\left(\sum_{i\geq 1}\|h_i\|_1\right)^2.$$

That is

$$\|h_0\|_1 \leq \xi_k \sum_{i\geq 1}\|h_i\|_1. \square$$

Remark 1. It is obvious that $0 < t_0 < 1$ since

$$4(2+2\delta_k - 5\delta_{2k}) < 32 + 16\delta_k - 41\delta_{2k}.$$

The inequality in Theorem 1 is equality when $t = t_0$ if $1/l \leq t_0 < 1$, but the strict inequality in Theorem 1 holds if $0 < t_0 < 1/l$.

Similar to the proof of Theorem 1, it follows that Theorem 3.1 by Mo and Li [18] is a special case of Theorem 1.

Corollary 1. If $\delta_{2k} < 2/3$, then

$$\|h_0\|_1 \leq \eta_k \sum_{i\geq 1}\|h_i\|_1,$$

where

$$\eta_k \triangleq \sqrt{\frac{4\left(1+5\delta_{2k} - 4\delta_{2k}^2\right)}{(1-\delta_{2k})(32-25\delta_{2k})}}. \square$$

III. The hyperbolic region of the restricted isometry constants

Lemma 1. If $\delta_k < 1/4$ and $\delta_{2k} < 1/2$, then

$$\|h_0\|_1 < \sum_{i\geq 1}\|h_i\|_1 .$$

Proof From $\delta_k < 1/4$ and (8), have

$$\|h_0\|_1^2 \leq \frac{t(1-t)/4+\delta_{2k}\left(1-5t/2+41t^2/16\right)+t-2t^2}{1-\delta_{2k}}\left(\sum_{i\geq 1}\|h_i\|_1\right)^2$$
$$=\frac{16\delta_{2k}+20\left(1-2\delta_{2k}\right)t-\left(36-41\delta_{2k}\right)t^2}{16\left(1-\delta_{2k}\right)}\left(\sum_{i\geq 1}\|h_i\|_1\right)^2.$$

Let

$$g(t)\triangleq 16\delta_{2k}+20\left(1-2\delta_{2k}\right)t-\left(36-41\delta_{2k}\right)t^2,$$

where $t\in(-\infty,+\infty)$. Similar to the proof of Theorem 1, the stationary point is

$$t_0 = 10\frac{1-2\delta_{2k}}{36-41\delta_{2k}}.$$

If $\delta_{2k} < 1/2$, then $t_0 > 0$. The function $g(t)$ is monotone increasing when $t \leq t_0$ and monotone decreasing when $t \geq t_0$. So $g(t)$ reaches the maximum when $t = t_0$, and the maximum is

$$g(t_0) = 4\frac{25+44\delta_{2k}-64\delta_{2k}^2}{36-41\delta_{2k}}.$$

Hence

$$\|h_0\|_1^2 \leq \frac{g(t_0)}{16\left(1-\delta_{2k}\right)}\left(\sum_{i\geq 1}\|h_i\|_1\right)^2.$$

That is

$$\|h_0\|_1 \leq \zeta_k\sum_{i\geq 1}\|h_i\|_1 ,$$

where

$$\zeta_k = \sqrt{\frac{25+44\delta_{2k}-64\delta_{2k}^2}{4\left(1-\delta_{2k}\right)\left(36-41\delta_{2k}\right)}}.$$

It is obvious that $\zeta_k < 1$ if and only if $\delta_{2k} < 1/2$. □

Lemma 2. If $5\delta_{2k}-2\delta_k < 2$ and

$$4\delta_k^2-57\delta_{2k}^2+12\delta_k\delta_{2k}-8\delta_k+85\delta_{2k} < 28,$$

then

$$\|h_0\|_1 < \sum_{i\geq 1}\|h_i\|_1 ,$$

where

$$4\delta_k^2-57\delta_{2k}^2+12\delta_k\delta_{2k}-8\delta_k+85\delta_{2k} < 28$$

is a hyperbolic region

$$\frac{\left(v-v_0\right)^2}{c_1}-\frac{\left(u+u_0\right)^2}{c_2} > 1$$

of the origin at $\left(-u_0, v_0\right)$. u_0, v_0, c_1, c_2 see Appendix.

Proof. Rewrite $x=\delta_k$ and $y=\delta_{2k}$ for the sake of convenience. From Theorem 1 we only need to prove $\xi_k < 1$. It is obvious that $\xi_k < 1$ if and only if

$$4x^2-57y^2+12xy-8x+85y < 28. \quad (9)$$

That is

$$\left(x,y\right)\begin{pmatrix}4 & 6\\ 6 & -57\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\left(-8,85\right)\begin{pmatrix}x\\ y\end{pmatrix} < 28.$$

Write

$$A\triangleq\begin{pmatrix}4 & 6\\ 6 & -57\end{pmatrix}, \cot 2\theta\triangleq\frac{a_{11}-a_{22}}{2a_{12}},$$

where $0<\theta<\pi/2$. Then

$$\cos\theta=\frac{1}{\sqrt{a^2+1}}, \sin\theta=\frac{a}{\sqrt{a^2+1}},$$

where $a=\tan\theta$. By trigonometric identity

$$6a^2+61a-6=0, \quad (10)$$

we have

$$a=\frac{\sqrt{3865}-61}{12}. \quad (11)$$

Let

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}. \quad (12)$$

Then

$$4x^2-57y^2+12xy-8x+85y<28$$

if and only if

$$\frac{(v-v_0)^2}{c_1}-\frac{(u+u_0)^2}{c_2}>1.$$

It is a hyperbolic region of the origin at $(-u_0,v_0)$. The specific calculations see Appendix. □

Write sets

$$U_1\triangleq\left\{(\delta_k,\delta_{2k}):0<\delta_k<\frac{1}{4},\delta_{2k}<\frac{1}{2},\delta_k\le\delta_{2k}\right\}.$$

$$U_2\triangleq\left\{(\delta_k,\delta_{2k}):\frac{1}{4}\le\delta_k<\frac{1}{2},g(\delta_k,\delta_{2k})<28,\delta_k\le\delta_{2k}\right\}.$$

where

$$g(\delta_k,\delta_{2k})\triangleq 4\delta_k^2-57\delta_{2k}^2+12\delta_k\delta_{2k}-8\delta_k+85\delta_{2k}.$$

$$U_3\triangleq\left\{(\delta_k,\delta_{2k}):5\delta_{2k}-2\delta_k<2,g(\delta_k,\delta_{2k})<28,\delta_k\le\delta_{2k}\right\}.$$

$$U_4\triangleq\left\{(\delta_k,\delta_{2k}):0<\delta_k<\frac{1}{4},5\delta_{2k}-2\delta_k<2,\delta_k\le\delta_{2k}\right\}.$$

Theorem 2. If $(\delta_k,\delta_{2k})\in U_1+U_2$, then

$$\|h_0\|_1<\sum_{i\ge1}\|h_i\|_1 .$$

Proof. From Lemma 1 and Lemma 2, if $(\delta_k,\delta_{2k})\in U_1+U_3$ then $\|h_0\|_1<\sum_{i\ge1}\|h_i\|_1$. Note that sets

$$\{5\delta_{2k}-2\delta_k<2\}\subset\{g(\delta_k,\delta_{2k})<28\}\subset\{\delta_{2k}<1/2\}$$

when $0<\delta_k<1/4$ and

$$\{5\delta_{2k}-2\delta_k<2\}\supset\{g(\delta_k,\delta_{2k})<28\}$$

when $1/4\le\delta_k<1/2$. So

$$U_3=U_4+U_2\subset U_1+U_2 .$$

This implies that

$$U_1+U_3=U_1+U_4+U_2=U_1+U_2 .□$$

Remark 2. Theorem 2 is intuitive. U_1+U_2 is the open region enclosed by straight lines $x=0$, $x=y$, $y=1/2$ and hyperbola $4x^2-57y^2+12xy-8x+85y=28$.

IV. Discussion and conclusion

Candes [7], Foucart and Lai [16], Foucart [15], Cai et al [2], Mo and Li [18] gave the conditions involving only δ_{2k}. Mo and Li [18] showed that if $\delta_{2k}<0.4931$, then $\|h_0\|_1<\sum_{i\ge1}\|h_i\|_1$. This is the best result on δ_{2k} so far. We illustrate that Theorem 2 completely improves the result by Mo and Li [18] below.

$\delta_{2k}<0.4931$ in fact corresponds to set $U_5=\{(\delta_k,\delta_{2k}):\delta_{2k}<0.4931,0<\delta_k\le\delta_{2k}\}$. We only need to show $U_5\subset U_1+U_2$. In order to precision, we take the exact value $\delta_{2k}<\left(77-\sqrt{1337}\right)/82$ instead of approximate value $\delta_{2k}<0.4931$. We will prove the intersection between straight line $y=\left(77-\sqrt{1337}\right)/82$ and hyperbola

$$4x^2-57y^2+12xy-8x+85y=28$$

is

$$(x_0,y_0)=\left(\frac{77-\sqrt{1337}}{82},\frac{77-\sqrt{1337}}{82}\right).$$

If use

$$x=\frac{82-123y-\sqrt{(123y-82)^2+161(226y-112)}}{82}=\frac{82-123y-\sqrt{41(1105y-536)}}{82},$$

can't get exact value. We use another method. Note that $y=\left(77-\sqrt{1337}\right)/82$ if and only if $41y^2-77y+28=0$ since $y<1$. So if $y=\left(77-\sqrt{1337}\right)/82$, then

$$4x^2-57y^2+12xy-8x+85y-28=4(x-y)(x+4y-2)=0 .$$

That is $x=y=\left(77-\sqrt{1337}\right)/82$ since $x\ge1/4$.

Note that U_5 is proper subset of U_1+U_2. In fact $U_1+U_2-U_5$ is the region enclosed by straight lines $x=0$, $y=\left(77-\sqrt{1337}\right)/82$, $y=1/2$ and hyperbola

$$4x^2-57y^2+12xy-8x+85y=28 .$$

Davies and Gribonval [12] constructed examples which showed that if $\delta_{2k}\ge1/\sqrt{2}$, exact recovery of certain k-sparse signal can fail in the noiseless case. Cai et al [4] constructed an example which showed that if $\delta_k \ge 1/2$, it is impossible to recover certain k-sparse signals. These imply that there must be $\delta_{2k} < 1/\sqrt{2}$ and $\delta_k < 1/2$ in order to guarantee stable recovery of k-sparse signals. In addition, Cai and Zhang [6] showed that if $\delta_k < 1/3$ then k-sparse signals can be recovered exactly in the noiseless case. Therefore, the remaining work are to research recovery of k-sparse signals in region

$$U_6 \triangleq \left\{ (\delta_k, \delta_{2k}) : 1/3 \le \delta_k < 1/2, \delta_{2k} < 1/\sqrt{2}, g(\delta_k, \delta_{2k}) \ge 28 \right\}.$$

By the way, the result "if $p \le 4k$ and $\delta_k < 0.6569$, then $\|h_0\|_1 < \sum_{i\ge1} \|h_i\|_1$" (see Corollary 3.4 in Mo and Li [18]) is incorrect. In fact, the right side of the equation (3) in Mo and Li [18] is equal to zero when $l = 1$ since $t \in [1/l, 1]$. In the proof of Theorem 3.3 in Mo and Li [18], $t(1+\delta_{2k} - 2t) \le (1+\delta_{2k})^2/8$ if and only if $(1+\delta_{2k})/4 \ge 1/l$ since $t \in [1/l, 1]$. This implies that $\delta_{2k} \ge 1$ when $l = 2$ and $\delta_{2k} \ge 1/3$ when $l = 3$.

We discuss second problem. Why the demarcation point is $\delta_k < 1/4$ in Lemma 1? As we will see below, this discussion is meaningful. If $\delta_k < d$, from (8), similar to the proofs of Theorem 1 and Lemma 1, then have

$$\begin{aligned}
&\|h_0\|_1^2 \le \\
&\frac{16\delta_{2k} + 8(2+2d-5\delta_{2k})t - (32+16d-41\delta_{2k})t^2}{16(1-\delta_{2k})} \left(\sum_{i\ge1} \|h_i\|_1 \right)^2 \\
&\le 4 \frac{1+2d+d^2+3\delta_{2k} - d\delta_{2k} - 4\delta_{2k}^2}{(1-\delta_{2k})(32+16d-41\delta_{2k})} \left(\sum_{i\ge1} \|h_i\|_1 \right)^2 \\
&\triangleq l(d, \delta_{2k}) \left(\sum_{i\ge1} \|h_i\|_1 \right)^2.
\end{aligned}$$

Our goal is δ_{2k} to achieve maximum when $l(d, \delta_{2k}) < 1$. By specific calculating, $l(d, \delta_{2k}) < 1$ if and only if $\delta_{2k} < 1/2$ since $\delta_{2k} < 1$, and $\delta_{2k} < 1/2$ when $\delta_k < 1/4$. So the demarcation point is $\delta_k < 1/4$ in Lemma 1. This imply that using the method of Theorem 1 and Lemma 1 cannot have $\delta_{2k} \ge 1/2$. Therefore, to research recovery of k-sparse signals in region U_6 must use new methods.

V. Appendix: Completion of the Proof of Lemma 2

We give specific calculation in order to get the precise hyperbolic equation as follows. By (10) and (12), we have

$$\begin{aligned}
&4x^2 - 57y^2 + 12xy - 8x + 85y \\
&= \frac{4+12a-57a^2}{a^2+1} \left(u + \frac{(85a-8)\sqrt{a^2+1}}{2(4+12a-57a^2)} \right)^2 \\
&+ \frac{4a^2-12a-57}{a^2+1} \left(v + \frac{(8a+85)\sqrt{a^2+1}}{2(4a^2-12a-57)} \right)^2 \\
&- \frac{(85a-8)^2}{4(4+12a-57a^2)} - \frac{(8a+85)^2}{4(4a^2-12a-57)} \\
&= 3\frac{1183a-106}{12-61a} \left(u + \frac{(85a-8)\sqrt{a^2+1}}{(1183a-106)} \right)^2 \\
&-2\frac{158a+159}{12-61a} \left(v - \frac{3(8a+85)\sqrt{a^2+1}}{2(158a+159)} \right)^2 \\
&- \frac{(85a-8)^2}{2(1183a-106)} + \frac{3(8a+85)^2}{4(158a+159)} \\
&\triangleq 3\frac{A}{C} \left(u + \frac{D\sqrt{a^2+1}}{A} \right)^2 - 2\frac{B}{C} \left(v - \frac{3E\sqrt{a^2+1}}{2B} \right)^2 \\
&- \frac{D^2}{2A} + \frac{3E^2}{4B}.
\end{aligned}$$

Thus

$$4x^2 - 57y^2 + 12xy - 8x + 85y < 28$$

if and only if

$$\frac{\left(v - \frac{3E\sqrt{a^2+1}}{2B} \right)^2}{\left(\frac{3E^2}{4B} - \frac{D^2}{2A} - 28 \right) \frac{C}{2B}} - \frac{\left(u + \frac{D\sqrt{a^2+1}}{A} \right)^2}{\left(\frac{3E^2}{4B} - \frac{D^2}{2A} - 28 \right) \frac{C}{3A}} > 1.$$

By (10) and (11), then have

$$\frac{3A}{C} = \frac{\sqrt{b}-53}{2},$$

$$\frac{2B}{C} = \frac{\sqrt{b}+53}{2},$$

$$\frac{1}{C} = \frac{\sqrt{b}+61}{12\sqrt{b}},$$

$$E^2 = \frac{(2\sqrt{b}+133)^2}{3^2},$$

$$D^2 = \frac{(85\sqrt{b}-5281)^2}{12^2},$$

where $b = 3865$. Thus

$$\frac{3E^2}{4B}\frac{C}{2B}=\frac{3E^2}{2}\frac{1}{C}\left(\frac{C}{2B}\right)^2$$
$$=\frac{\left(\sqrt{b}-53\right)^2\left(2\sqrt{b}+133\right)^2\left(\sqrt{b}+61\right)}{2^{11}\cdot 3^4\cdot 11^2\sqrt{b}},$$
$$\frac{D^2}{2A}\frac{C}{2B}=\frac{D^2}{4^2\cdot 11C}$$
$$=\frac{\left(85\sqrt{b}-5281\right)^2\left(\sqrt{b}+61\right)}{2^{10}\cdot 3^3\cdot 11\sqrt{b}},$$
$$\frac{C}{2B}=\frac{\sqrt{b}-53}{2^4\cdot 3\cdot 11}.$$

Based on the above results, by direct calculation, we have

$$c_1\triangleq\left(\frac{3E^2}{4B}-\frac{D^2}{2A}-28\right)\frac{C}{2B}$$
$$=\frac{743123290b-45231681366\sqrt{b}}{2^{13}\cdot 3^5\cdot 11^2 b}.$$

Similarly, have

$$c_2\triangleq\left(\frac{3E^2}{4B}-\frac{D^2}{2A}-28\right)\frac{C}{3A}=c_1\frac{2B}{3A}$$
$$=\frac{20630826583b+321742405427\sqrt{b}}{2^{15}\cdot 3^6\cdot 11^3 b},$$
$$v_0^2\triangleq\left(\frac{3E\sqrt{a^2+1}}{2B}\right)^2=\frac{3E^2}{2}\frac{1}{C}\left(\frac{C}{2B}\right)^2$$
$$=\frac{38365b+2377039\sqrt{b}}{2^7\cdot 3^2\cdot 11^2 b},$$
$$u_0^2\triangleq\left(\frac{D\sqrt{a^2+1}}{A}\right)^2=3D^2\left(\frac{C}{3A}\right)^2\frac{1}{2C}$$
$$=\frac{345285b-21393351\sqrt{b}}{2^7\cdot 3^4\cdot 11^2 b}.$$

The approximate values are $c_1=0.0646$, $c_2=0.7872$, $v_0=0.7413$ and $u_0=0.0305$, respectively.

REFERENCES

[1] P. J. Bickel, Y. Ritov, and A. B. Tsybakov, “Simultaneous analysis of Lasso and Dantzig selector,” *Ann. Stat.*, vol. 37, pp. 1705-1732, 2009.

[2] T. Cai, L. Wang, and G. Xu, “Shifting inequality and recovery of sparse signals,” *IEEE Trans. Signal Process.*, vol. 58, pp. 1300-1308, 2010.

[3] T. Cai, L. Wang, and G. Xu, “Stable recovery of sparse signals and an oracle inequality,” *IEEE Trans. Inf. Theory*, vol. 56, pp. 3516-3522, 2010.

[4] T. Cai, L. Wang, and G. Xu, “New bounds for restricted isometry constants,” *IEEE Trans. Inf. Theory*, vol. 56, pp. 4388-4394, 2010.

[5] T. Cai, G. Xu, and J. Zhang, “On recovery of sparse signals via L_1 minimization,” *IEEE Trans. Inf. Theory*, vol. 55, pp. 3388-3397, 2009.

[6] T. Cai, A. Zhang, “Sharp RIP bound for sparse signal and low-rank matrix recovery,” *Appl. Comput. Harmon. Anal.*, vol. 35, pp. 74-93, 2012.

[7] E. J. Candes, “The restricted isometry property and its implications for compressed sensing,” *Comptes Rendus Mathematique*, vol. 346, pp. 589-592, 2008.

[8] E. J. Candes, J. Romberg, and T. Tao, “Stable signal recovery from incomplete and inaccurate measurements,” *Comm. Pure Appl. Math.*, vol. 59, pp. 1207-1223, 2006.

[9] E. J. Candes, T. Tao, “Decoding by linear programming,” *IEEE Trans. Inf. Theory*, vol. 51, pp. 4203-4215, 2005.

[10] E. J. Candes, T. Tao, “Near-optimal signal recovery from random projections: Universal encoding strategies,” *IEEE Trans. Inf. Theory*, vol. 52, pp. 5406-5425, 2006.

[11] E. J. Candes, T. Tao, “The Dantzig selector: Statistical estimation when p is much larger than n (with discussion),” *Ann. Stat.*, vol. 35, pp. 2313-2351, 2007.

[12] M. E. Davies, R. Gribonval, “Restricted isometry constants where L_p sparse recovery can fail for $0<p\le 1$,” *IEEE Trans. Inf. Theory*, vol. 55, pp. 2203-2214, 2009.

[13] D. L. Donoho, “Compressed sensing,” *IEEE Trans. Inf. Theory*, vol. 52, pp. 1289-1306, 2006.

[14] D. L. Donoho, X. Huo, “Uncertainty principles and ideal atomic decomposition,” *IEEE Trans. Inf. Theory*, vol. 47, pp. 2845-2862, 2001.

[15] S. Foucart, “A note on guaranteed sparse recovery via L_1 minimization,” *Appl. Comput. Harmon. Anal.*, vol. 29, pp. 97-103, 2010.

[16] S. Foucart, M. Lai, “Sparsest solutions of underdetermined linear systems via L_q minimization for $0<q\le 1$,” *Appl. Comput. Harmon. Anal.*, vol. 26, pp. 395-407, 2009.

[17] J. Ji, J. Peng, “Improved bounds for restricted isometry constants,” *Discrete Dyn. Nat. Soc.*, vol. 2012. Available: http://www.hindawi.com/journals/ddns/2012/841261/abs/.

[18] Q. Mo, S. Li, “New bounds on the restricted isometry constant δ_{2k},” *Appl. Comput. Harmon. Anal.*, vol. 31, pp. 460-468, 2011.

[19] S. Q. Wang, L. M. Su, “The oracle inequalities on simultaneous Lasso and Dantzig selector in high dimensional nonparametric regression,” *Math. Probl. Eng.*, vol. 2013. Available: http://www.hindawi.com/journals/mpe/2013/571361/abs/

[20] S. Q. Wang, L. M. Su, “Recovery of high-dimensional sparse signals via L_1 -minimization,” *J. Appl. Math.*, vol. 2013. Available: http://www.hindawi.com/journals/jam/2013/636094/.

[21] S. Q. Wang, L. M. Su, “Simultaneous Lasso and Dantzig selector in high dimensional nonparametric regression,” *Int. J. Appl. Math. Stat.*, vol. 42, pp. 103-118.

[22] S. Q. Wang, L. M. Su, “New bounds of mutual incoherence property on sparse signals recovery,” *Int. J. Appl. Math. Stat.*, vol. 47, pp. 462-477.

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