An Efficient Method for Representing the Lightning Base Current in the Frequency-Domain by means of Orthogonal Polynomials

Dario Assante and Clemente Cesarano

Abstract**—** An efficient procedure to express the lightning base current in the frequency domain is presented. The method is based on a decomposition of the time-domain lightning base current expression in a part that can be analytically transformed and in another part that is efficiently approximated by a series of orthogonal polynomial. The method allows to easily compute the coefficients of the series so to have a semi-analytical expression of the lightning base current in the frequency domain. The presented procedure is fast and general, since it can be used with different kinds of current waveshapes.

Keywords**—**Lightning Base Current, Heidler model, Legendre polynomials.

I. Introduction

THE efficient modeling of the lightning current is essential for the analysis of all the effects due to the lighting phenomenon, such us the lightning electromagnetic field propagation or the effects induced on power lines and on electric and electronic devices. All these aspects are strongly related to the current distribution along the lightning channel.

The lightning channel modeling has been widely discussed in literature [1] and several models have been proposed. Nowadays, the most accredited one is the so-called *engineering model*, defined by the expression:

$$i(z,t)=i(0,t-z/v)\,P(z) \quad , \tag{1}$$

where the current along the channel is represented as the lightning base current i(0,t), propagating along the channel at a return stroke velocity v (~ 2c/3) and attenuating along the channel according to an height-dependent attenuation function P(z) [2]. The model is shown in Fig. 1.

The attenuation function has been widely discussed in literature as well, and several models have been proposed and validated with measurements showing strengths and weaknesses [3], however the discussion is still open. More recently, an alternative procedure has been discussed in literature, the idea is to find an efficient representation of the attenuation function by means of an inverse identification starting from the measured electromagnetic field [4-6].

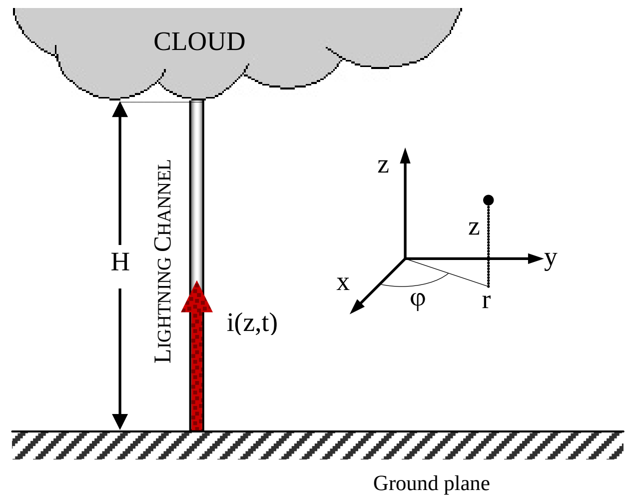

Fig. 1 Lightning model.

In this paper we focus our attention on the lightning base current i(0,t). This parameter, being the current at the hit point, has been measured by using both natural [7] and artificial lightning [8] at the hit points, enabling the acquisition of a good amount of experimental data. This has led to the definition of accurate models able to describe the lightning base current. The first model proposed in literature has been the so called double-exponential model, and for a first lightning stroke is:

$$i(t) = I_0\left(e^{-\alpha t} - e^{-\beta t}\right), \tag{2}$$

being where I_0 is the maximum value and α and β are two time constants. In case of subsequent lightning, a more complex expression considering two couples of double exponentials is preferred:

$$i(t) = I_{01}\left(e^{-\alpha t} - e^{-\beta t}\right) + I_{02}\left(e^{-\gamma t} - e^{-\delta t}\right), \tag{3}$$

Nowadays the most used model, being an evolution of (2), is the one proposed by Heidler [9] according to the expression

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$$i(t) = \frac{I_0}{\eta} \frac{(t/\tau_1)^n}{1+(t/\tau_1)^n} e^{-t/\tau_2}, \tag{4}$$

where I_0 is the amplitude of the channel base current, τ_1 and τ_2 are respectively the front and decay time constants, n is an integer number between 2 and 10, and η is the amplitude correction factor, given by

$$\eta = e^{-(\tau_1/\tau_2)(n\tau_2/\tau_1)^{1/n}}. \tag{5}$$

The expression (4) is used for first lightning strokes, while for subsequent lightning strokes it is preferable to use

$$i(t) = \frac{I_{01}}{\eta_1} \frac{(t/\tau_{11})^{n_1}}{1+(t/\tau_{11})^{n_1}} e^{-t/\tau_{12}} + \frac{I_{02}}{\eta_2} \frac{(t/\tau_{21})^{n_2}}{1+(t/\tau_{21})^{n_2}} e^{-t/\tau_{22}}. \tag{6}$$

The parameters appearing in (4) and (6) can be chosen in order to properly fit the specific lightning base current waveshape.

This model allows to represent very well a large class of lighting base currents in time domain. However, it exhibits a weakness in the frequency domain, since the Fourier transform of the (4) can't be expressed in terms of elementary functions [10]. This limitation is unfair, since several electromagnetic field problems can be formulated more easily in the frequency domain [11]. Also, in the frequency domain it is easier to take into account finite conductivity ground [12-13] or even multilayered structures.

In order to still adopt the Heidler model of the lightning base current in the frequency domain, usually the FFT is adopted. However, this solution too has some disadvantages as well. First of all, the lightning base current and the attenuation function are the starting point for any further calculation (electromagnetic fields, induced overvoltage, etc.) [14-18]. The use of the FFT for the computation of the lightning base current in the frequency domain produces a numerical result, that indeed limits the possibility to develop analytical calculations wherever it is possible. Then, the lightning base current essentially has two dynamics very different in time: an initial rising time of few μs and a descending part of hundreds of μs. Since the FFT requires a linear sampling of the signal, a very high number of samplings is required to achieve a good numerical representation in the frequency domain, with a consequent high numerical effort.

A possibility to overcome this problem is to adopt lightning base current models different from (4), adopting representations that can be elementary transformed; this possibility have been discussed in literature [19-20], however these models have not gained a good success and the Heidler model is currently the most used and accepted one.

In this paper we show an alternative approach for expressing the lightning base current in the frequency domain. The proposed procedure allows to find a simple expression, with minimal computational effort, using anyway the Heidler model. In addition, being a semi-analytical expression, it can be efficiently used in further calculations.

The paper is divided in two parts: at first the procedure is formalized in time and discussed with some numerical results, that the problem is solved in the frequency domain too.

II. Representation of the Lightning Base Current

A. *Procedure for the representation in time domain*

Let us consider again the lighting base current (4): it can be decomposed in two terms:

$$i(t) = \frac{I_0}{\eta} e^{-t/\tau_2} - \frac{I_0}{\eta} \frac{1}{1+(t/\tau_1)^n} e^{-t/\tau_2}. \tag{7}$$

The Fourier transform of the first term is trivial and expressed by a rational function, while the second term hasn't an analytical Fourier transform. Regarding the second term, it is possible to observe that the function

$$\frac{1}{1+(t/\tau_1)^n} \tag{8}$$

is equal to 1 for t = 0, quickly decreases and can be considered almost zero for $t > 2\tau_1$ at most. Due to this properties, it is possible to represent the function (8) in terms of orthogonal polynomials [21-25], that can be efficiently applied for solution of several classes of electromagnetic problems [26-27]. Considering the behavior of the expressions (7) and (8), it convenient to adopt the following representation

$$\frac{1}{1+(t/\tau_1)^n} = \sum_{p=0}^{+\infty} a_p L_p(t/\tau_1) e^{-t/2\tau_1}, \tag{9}$$

where $L_p(x)$ is the Laguerre polynomial of order p [28-29].

By projecting the (9) on the basis functions of the series and considering the orthogonality relation of the Laguerre polynomials, it is trivial to find out that

$$a_p = \int_0^{+\infty} \frac{L_p(x) e^{-x}}{1+(x)^n} dx. \tag{10}$$

This result is interesting because it is found that the coefficients a_p only depends by the parameter n and it is not related to lightning current intensity or to the time constants. Considering that, in the Heidler model, the parameter n is and integer between 2 and 10, once the coefficients a_p have been computed once and stored, they can be used for representing every kind of lightning.

In Table 1 the first 11 coefficients have been computed and shown. In is found that for low values of n, the coefficients quickly go to zero, while more coefficients are required to achieve a good representation for higher values of n.

TABLE 1. FIRST ELEVEN VALUES OF THE COEFFICIENTS a_p

p \ n	2	3	4	5	6	7	8	9	10
0	0.8605	0.8293	0.8135	0.8049	0.7998	0.7966	0.7944	0.7929	0.7918
1	0.1878	0.3100	0.3602	0.3842	0.3973	0.4051	0.4101	0.4135	0.4160
2	0.0849	0.0972	0.1208	0.1379	0.1492	0.1568	0.1620	0.1657	0.1683
3	-0.0029	-0.0043	-0.0029	0.0012	0.0051	0.0084	0.0109	0.0129	0.0143
4	-0.0108	-0.0445	-0.0598	-0.0669	-0.0702	-0.0717	-0.0723	-0.0727	-0.0728
5	-0.0274	-0.0568	-0.0792	-0.0929	-0.1012	-0.1064	-0.1097	-0.1120	-0.1136
6	-0.0204	-0.0542	-0.0780	-0.0942	-0.1050	-0.1123	-0.1173	-0.1208	-0.1234
7	-0.0225	-0.0458	-0.0667	-0.0822	-0.0933	-0.1011	-0.1067	-0.1107	-0.1138
8	-0.0149	-0.0352	-0.0514	-0.0641	-0.0737	-0.0809	-0.0862	-0.0902	-0.0932
9	-0.0143	-0.0250	-0.0355	-0.0444	-0.0516	-0.0572	-0.0616	-0.0649	-0.0675
10	-0.0086	-0.0159	-0.0210	-0.0258	-0.0301	-0.0337	-0.0366	-0.0390	-0.0409

In Fig. 1 we show the behavior of the coefficients a_p for n equal to 2, 6 and 10, in order to confirm the consideration of the coefficients behavior as function of the order p.

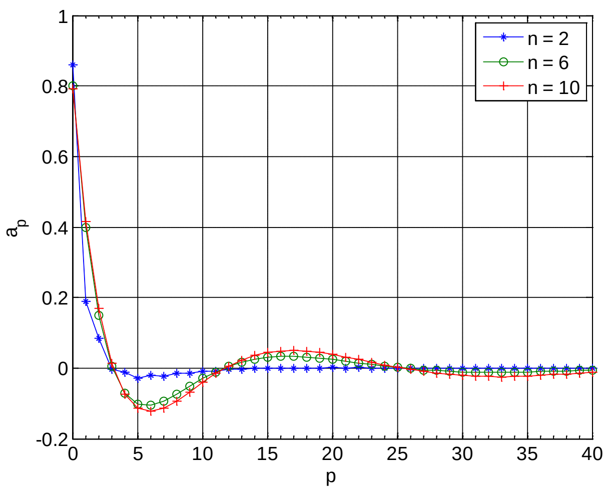

Fig. 1 Coefficients a_p.

B. Numerical results in time domain

In order to show the efficiency of the representation proposed in the previous paragraph, first of all we consider a typical lightning first and subsequent return stroke [30]. The adopted parameters are shown in Table 2.

TABLE 2. LIGHTNING CURRENT PARAMETERS FOR FIRST AND SUBSEQUENT STROKES

	I_{01} (kA)	n_1	τ_{11} (μs)	T_{21} (μs)	I_{02} (kA)	n_2	τ_{12} (μs)	τ_{22} (μs)
First	28	2	1.8	95	-	-	-	-
Subseq.	10.7	2	0.25	2.5	6.5	2	2.1	230

We consider the lightning base current computed in interval of 1 ms and we compare the analytical expressions (4) and (6) with the ones obtained by means of the representation (9). In Fig. 2 and 3 we show the comparison, by using the first 11 coefficients as in Table 1.

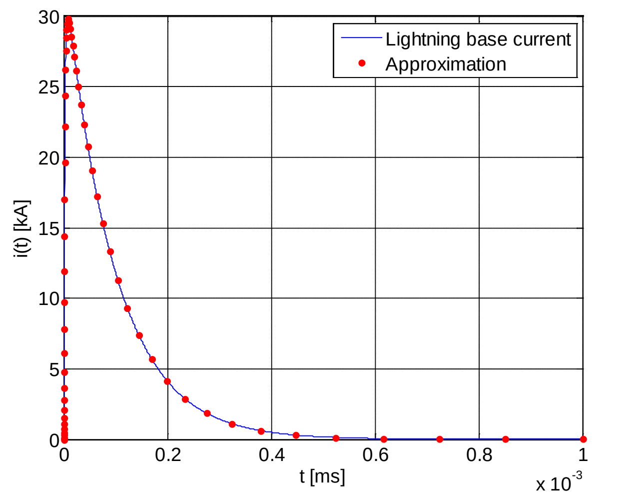

Fig. 2 First stroke lightning base current in time domain: comparison between analytical expression (4) and approximation (p = 0-10).

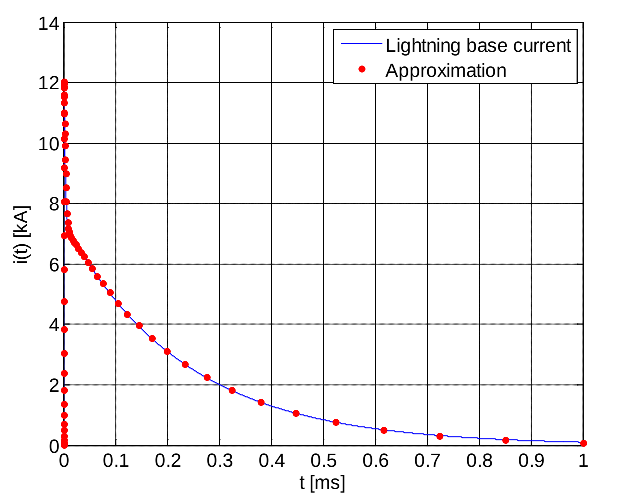

Fig. 3 Subsequent stroke lightning base current in time domain: comparison between analytical expression (4) and approximation (p = 0-10).

Then, in order to better estimate the approximation as function of the number of coefficients, we also define an error function, namely

$$err_p\,\% = \frac{\left| i(t) - i_p(t) \right|}{\left| i(t) \right|} \cdot 100\,, \tag{11}$$

where i(t) is the lightning base current as in (4) or (6) and $i_p(t)$ is the lightning base current approximated with p terms of the series (9). In Fig. 4 and 5 we show the error as function of p, evaluated in an interval between 0 and 1 ms.

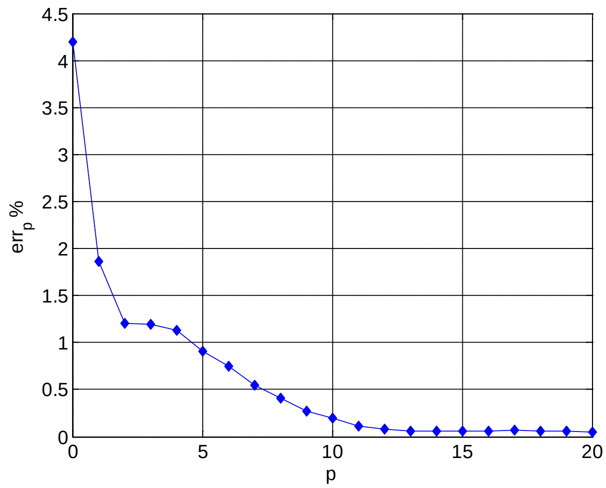

Fig. 4 First stroke lightning base current in time domain: error as function of the number of coefficients.

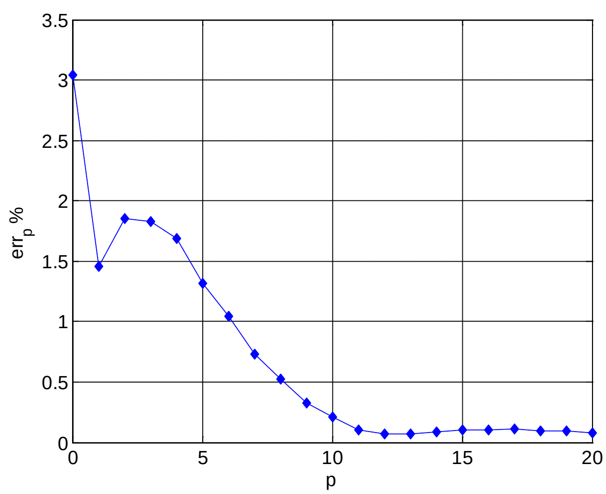

Fig. 5 Subsequent stroke lightning base current in time domain: error as function of the number of coefficients.

From this numerical results we can conclude that the proposed method is very efficient in order to approximate lightning base current in real cases.

III. Lightning Base Current in the Frequency-Domain

Once the lightning base current has been decomposed as in (7) and then the representation (9) is applied, an expression of the lightning base current in the frequency domain can be found in terms of elementary functions. Thanks to the useful integral

$$\int_0^{+\infty} e^{-bx} L_n(x)\,dx = (b-1)^n\, b^{-n-1}\,, \tag{12}$$

in case of first stroke it is possible to express the lightning base current I(ω) in the frequency domain as

$$I(\omega) = \\ = \frac{I_{01}}{\eta_1}\left(\frac{\tau_{12}}{1 + j\omega\tau_{12}} - \tau_{11} \sum_{p=0}^{+\infty} a_p \frac{\left(\tau_{11}/\tau_{12} - 1/2 + j\omega\tau_{11}\right)^p}{\left(\tau_{11}/\tau_{12} + 1/2 + j\omega\tau_{11}\right)^{p+1}} \right). \tag{13}$$

In case of subsequent lightning stroke, according to (6) the expression of the lightning base current I(ω) in the frequency domain will be the sum of two terms like (13).

Then, in Fig. 6 and 7 we show the lightning base current in the frequency domain obtained implementing the (13), computed with the first and subsequent stroke parameters adopted in Table 2, comparing the result with the frequency behavior obtained with the FFT. A very good agreement is found by using just 10 coefficients of the series.

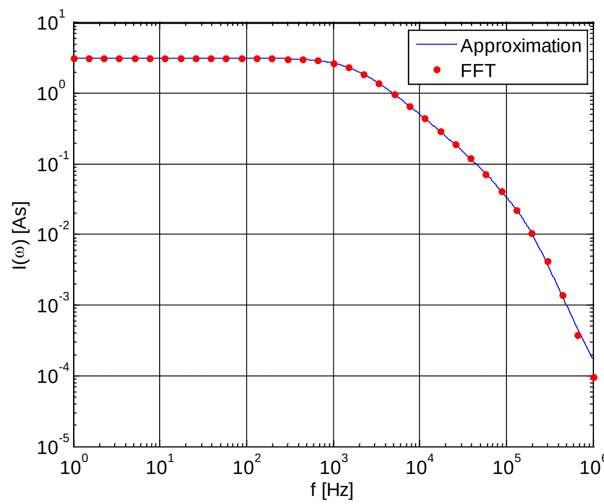

Fig. 6 First stroke lightning base current in the frequency domain: comparison between FFT and approximation (p = 0-10).

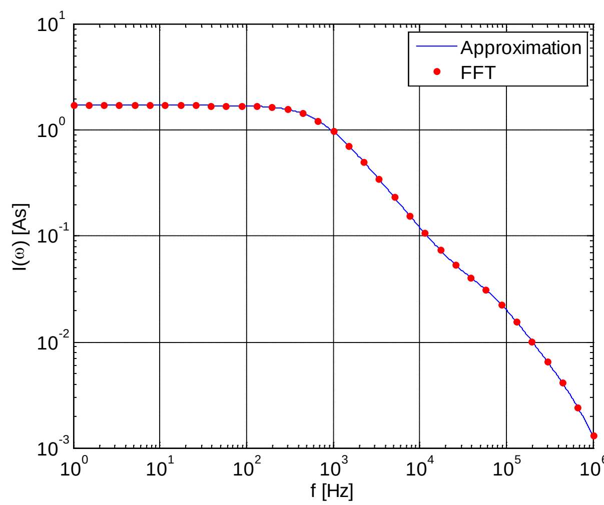

Fig. 7 Subsequent stroke lightning base current in the frequency domain: comparison between FFT and approximation (p = 0-10).

Finally, we also compute the error as function of the number of coefficients, using the same definition of (11) with the currents in the frequency domain instead of the time domain. The error is computed on an interval between 0 and 1 MHz. The results are shown in Fig. 8 and 9.

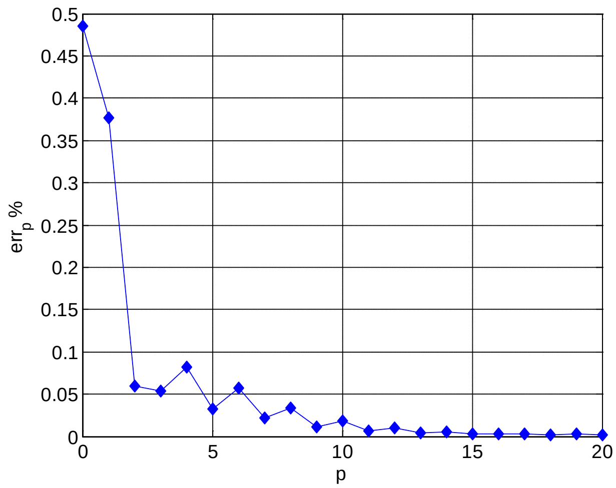

Fig. 8 First stroke lightning base current in the frequency domain: error as function of the number of coefficients.

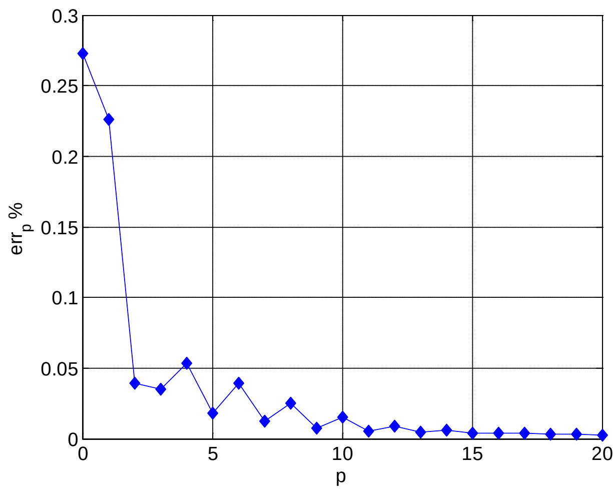

Fig. 9 Subsequent stroke lightning base current in the frequency domain: error as function of the number of coefficients.

In both cases it is clear that a very good accuracy can be achieved with a very small number of coefficients. This confirms the efficiency of the proposed method.

IV. Conclusions

A simple procedure has been shown to represent the lightning base current in the frequency domain, by using an efficient representation in terms of Laguerre polynomials. The method has been successfully tested on a typical Heidler base current. However, it can be applied to any other current models with similar performance results.

The proposed method is fast and allows to choose the desired accuracy. Then, differently from the FFT, the method doesn't impose a relationship between the time and frequency samplings. This is an advantage for the computational time.

Considering the results obtained with this method, it would be interesting to investigate the performances obtained with other kinds of orthogonal polynomial [31-33] and special functions [34-35]. Then, another interesting perspective for a future investigation on this topic is the use of the truncated exponential polynomials [36-38], that can be successfully adopted for the solution of several classes of numerical methods.

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