Sufficient Conditions for Controllability and Observability of Serial and Parallel Concatenated Linear Systems

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Abstract—This paper deals with the sufficient conditions for controllability and observability characters of finite-dimensional linear continuous-time-invariant systems of serial and parallel concatenated systems. The obtained conditions depend on the controllability and observability of the systems and in some cases, the functional output-controllability of the first one.

Keywords—Linear systems, serial composite systems,parallel composite systems, controllability, observability, functional-output controllability.

I. Introduction

It is well known that many physical problems as for example electrical networks, multibody systems, chemical engineering, convolutional codes among others (see [4] [15] [8] for example), the most popular and the most frequently used mathematical model for its description is the state space representation in the form

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}\right\} \tag{1}$$

where $A \in M_n(\mathbb{R})$, $B \in M_{n\times m}(\mathbb{R})$, $C \in M_{p\times n}(\mathbb{R})$ and $D \in M_{p\times m}(\mathbb{R})$

These linear systems can be described with an input-output relation called transfer function obtained by applying Laplace transformation at $x = 0$, to equation 1

$$\left.\begin{aligned} sX &= AX + BU \\ Y &= CX + DU \end{aligned}\right\},$$

obtaining the following relation

$$H(s) = C(sI_n - A)^{-1}B + D, \tag{2}$$

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In engineering problems, a system is sometimes built by interconnecting some other systems, this kind of systems are called concatenated (or composite) systems.

Let $\dot{x}_i = A_i x_i + B_i u_i$, $y_i = C_i x_i + D_i u_i$ for $i = 1, 2$, be two systems that can be connected in different ways. The most common are the following: a) Serialized one after the other, so that the input information $u_2 = y_1(t)$. Consequently

$$\left.\begin{aligned} \dot{x} &= \begin{pmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{pmatrix} \begin{pmatrix} cx_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2D_1 \end{pmatrix} u \\ y &= \begin{pmatrix} D_2C_1 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D_2D_1u. \end{aligned}\right\} \tag{3}$$

Another model of serial concatenation is the systematic serial concatenation. In this case we consider the same kind of systems than the case of serial concatenation systems.

$$\left.\begin{aligned} \dot{x} &= \begin{pmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2D_1 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & 0 \\ D_2C_1 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2D_1 \end{pmatrix} u. \end{aligned}\right\} \tag{4}$$

b) The second concatenated model that we will study is the parallel concatenation. So that the input information is $u_2(t) = u_1(t) = u(t)$ and the output is $y(t) = y_1(t) + y_2(t)$. Consequently

$$\left.\begin{aligned} \dot{x} &= \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (D_1 + D_2)u. \end{aligned}\right\} \tag{5}$$

Obviously, both in the case of serial concatenation as parallel the sizes of the systems must be adequate to make sense the concatenation.

The transfer functions for concatenated systems are

i) serial concatenated case:

$$H(s) = H_2(s) \cdot H_1(s),$$

ii) systematic serial concatenated case:

$$H(s) = \begin{pmatrix} H_1(s) \\ H_2(s) \cdot H_1(s) \end{pmatrix},$$

iii) parallel concatenated case:

$$H(s) = H_1(s) + H_2(s).$$

Controllability and observability are of important and fundamental properties of control systems.

The controllability concept of a dynamical standard system is largely studied by several authors and under many different points of view (see [2], [4], [7], [15], [16] for example). Nevertheless, controllability for the output vector of a system has been less treated (see [5], [8], [13] for example).

The functional output-controllability generally means, that the system can steer output of dynamical system along the arbitrary given curve over any interval of time, independently of its state vector. A similar but least essentially restrictive condition is the pointwise output-controllability ([8]).

It is well known the importance of observable systems because it is possible to use observers for to track and reconstruct states of a model ([3]).

Roughly speaking observability means the possibility of identifying the internal state of a system from measurements of the outputs.

In this paper controllability and observability for serial and parallel composite systems are analyzed and a sufficient conditions are presented.

II. Preliminaries

In this paper, it is considered the state space system introduced in equation 1

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}\right\},$$

where x is the state vector, y is the output vector, u is the input (or control) vector, $A \in M_n(\mathbb{R})$ is the state matrix, $B \in M_{n\times m}(\mathbb{R})$ is the input matrix, $C \in M_{p\times n}(\mathbb{R})$ is the output matrix and $D \in M_{p\times m}(\mathbb{R})$.

For simplicity we will write the systems by a triple of matrices (A, B, C, D).

A. Controllability

The most frequently used fundamental definition of controllability for linear control systems with constant coefficients is the following.

Definition 2.1: Dynamical system 1 is said to be controllable if for every initial condition $x(0)$ and every vector $x_1 \in \mathbb{R}^n$, there exist a finite time t_1 and control $u(t) \in \mathbb{R}^m$, $t \in [0, t_1]$, such that $x(t_1) = x_1$. This definition requires only that any initial state $x(0)$ can be steered to any final state x_1 at time t_1. However, the trajectory of the dynamical system between 0 and t_1 is not specified. Furthermore, there is no constraints posed on the control vector $u(t)$ and the state vector $x(t)$.

In order to formulate easily computable algebraic controllability criteria let us introduce the so-called controllability matrix $\mathcal{C}$, which is known as controllability matrix and defined as follows.

$$\mathcal{C} = \begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{pmatrix}. \tag{6}$$

Theorem 2.1: Dynamical system 1 is controllable if and only if $\operatorname{rank} \mathcal{C} = n$.

Or equivalently (test de Hautus [14]), if and only if

$$\operatorname{rank} \begin{pmatrix} sI_n - A & B \end{pmatrix} = n, \text{ for all } s \in \mathbb{C}, \tag{7}$$

(Observe that $s \in \mathbb{C}$ the algebraic closure of $\mathbb{R}$).

B. Observability

The observability character deals with the question of if it is possible to determine the state from the input and output, does not knowing the initial state.

Definition 2.2: Dynamical system 1 is said to be observable on $[0, t_1]$ if for all inputs, $u(t)$, and outputs $y(t)$, $t \in [0, t_1]$, the state $x(0) = x_0$ can be uniquely determined.

There is a duality between controllability and observability.

Definition 2.3: Given the system (A, B, C, D), we call adjoint system to the system (A^t, C^t, B^t, D^t)

Theorem 2.2 (Duality):

i) The system (A, B, C, D) is controllable if and only if (A^t, C^t, B^t, D^t) is observable,

ii) The system (A, B, C, D) is observable if and only if (A^t, C^t, B^t, D^t) is controllable

As a consequence we can formulate an easily computable algebraic observability criteria introducing the so-called observability matrix $\mathcal{O}$, which is known as observability matrix and defined as follows.

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{8}$$

Theorem 2.3: Dynamical system 1 is observable if and only if $\operatorname{rank} \mathcal{O} = n$.
or equivalently (test de Hautus [14]), if and only if

$$\operatorname{rank} \begin{pmatrix} sI - A \\ C \end{pmatrix} = n, \text{ for all } \in \mathbb{C}, \tag{9}$$

C. Functional output-controllability

The functional output-controllability generally means, that the system can steer output of dynamical system independently of its state vector. Concretely

Definition 2.4: A system is functional output-controllable if and only if its output can be steered along the arbitrary given curve over any interval of time. It means that if it is given any output $y_d(t)$, $t \geq 0$, there exists t_1 and a control u_t, $t \geq 0$, such that for any $t \geq t_1$, $y(t) = y_d(t)$.

Proposition 2.1 ([4]): A system is functional output-controllable if and only

$$\operatorname{rank} C(sI_n - A)^{-1}B + D = p$$

in the field of rational functions.
A necessary and sufficient condition for functional output-controllability is

Proposition 2.2 ([4], [6]):

$$\operatorname{rank} \begin{pmatrix} sI - A & -B \\ -C & -D \end{pmatrix} = n + p,$$

1) Test for functional output-controllability: The functional output-controllability can be computed by means of the rank of a constant matrix in the following manner

Theorem 2.4 ([8]): The system (A, B, C, D) is functional output-controllable if and only if

$$\operatorname{rank} oC_{\mathrm{f}}(A, B, C, D) =$$

$$\operatorname{rank} \begin{pmatrix} C & D & & & & \\ CA & CB & D & & & \\ CA^2 & CAB & CB & D & & \\ \vdots & & \ddots & & & \\ CA^n & CA^{n-1}B & \dots & CAB & CB & D \end{pmatrix}$$
$$= (n + 1)p.$$

The null terms are not written in the matrix.

Remark 2.1: We call $oC_i(A, B, C, D)$ or simply oC_i if confusion is not possible, the following matrix

$$oC_i = \begin{pmatrix} C & D & & & & \\ CA & CB & D & & & \\ CA^2 & CAB & CB & & & \\ \vdots & & \ddots & & & \\ CA^i & CA^{i-1}B & \dots & CAB & CB & D \end{pmatrix},$$

$\forall i \geq 1$.

i) If the system (A, B, C, D) is functional output-controllable, then the matrices oC_i have full row rank for all $0 \leq i \leq n$.
ii) If the matrix oC_{n-1} has full row rank, it is not necessary that the matrix oC_n has full row rank.

Example 2.1: Let (A, B, C, D) a system with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $D = 0$.

$$\operatorname{rank} \begin{pmatrix} C & D & \\ CA & CB & D \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 2,$$

$$\operatorname{rank} \begin{pmatrix} C & D & & \\ CA & CB & D & \\ CA^2 & CAB & CB & D \end{pmatrix} =$$
$$\operatorname{rank} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 2.$$

III. Controllability of composite systems

A. Controllability of serial composite systems

We will try to characterize the controllability of serial concatenated systems in terms of the original systems.

Let (A, B, C, D) the serial concatenated system of the systems (A_i, B_i, C_i, D_i), $i = 1, 2$, defined in 3.

The controllability character for serial concatenated systems can be described in terms of the both systems using the Hautus test in the following manner.

Theorem 3.1: A serial concatenated system is controllable if and only if

$$\operatorname{rank} \begin{pmatrix} sI_{n_1} - A_1 & 0 & B_1 \\ -B_2C_1 & sI_{n_2} - A_2 & B_2D_1 \end{pmatrix} = n_1 + n_2$$

$\forall s \in \mathbb{C}$.

Corollary 3.1: A necessary condition for controllability of serial concatenated system is that the pair (A_1, B_1) be controllable.

Corollary 3.2: A necessary condition for controllability of serial concatenated system is that the pair $(A_2, \overline{B})$ where $\overline{B} = \begin{pmatrix} -B_2C_1 & B_2D_1 \end{pmatrix}$ be controllable.

Corollary 3.3: A necessary condition for controllability of serial concatenated system is that the pair (A_2, B_2) be controllable.

Proof:

$$\begin{aligned}
&\text{rank}\begin{pmatrix} sI_{n_2} - A_2 & B_2C_1 & B_2D_1 \end{pmatrix} = \\
&\text{rank}\begin{pmatrix} sI_{n_2} - A_2 & B_2 & B_2 \end{pmatrix}\begin{pmatrix} I_2 & & \\ & C_1 & \\ & & D_1 \end{pmatrix} \le \\
&\min\left(\text{rank}\begin{pmatrix} sI_{n_2} - A_2 & B_2 \end{pmatrix}, \text{rank}\begin{pmatrix} I_{n_2} & & \\ & C_1 & \\ & & D_1 \end{pmatrix}\right).
\end{aligned}$$

So,

$$\begin{aligned}
&\text{rank}\begin{pmatrix} sI_{n_2} - A_2 & B_2C_1 & B_2D_1 \end{pmatrix} \le \\
&\text{rank}\begin{pmatrix} sI_{n_2} - A_2 & B_2 \end{pmatrix} \le n_2.
\end{aligned}$$

■

Example 3.1: Let (A, B, C, D) be a serial concatenated system of (A_1, B_1, C_1, D_1), and (A_2, B_2, C_2, D_2), where

$$A_1 = (0),\ B_1 = \begin{pmatrix} 1 & -3 \end{pmatrix},$$

$$C_1 = (4),\ D_1 = \begin{pmatrix} 1 & -2 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 1 & 0 \end{pmatrix},\ D_2 = (1),$$

the concatenated serial system (A, B, C, D) of both is:

$$A = \begin{pmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} B_1 \\ B_2D_1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 1 & -2 \\ 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} D_2C_1 & C_2 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} D_2D_1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \end{pmatrix}$$

In this case is it easy to observe that all systems are controllable, this can be easily observed from the Hautus representation of our serial controllability matrix,

$$\begin{aligned}
&\text{rank}\begin{pmatrix} sI_{n_1} - A_1 & B_1 \end{pmatrix} = \\
&\quad \text{rank}\begin{pmatrix} s & 1 & -3 \end{pmatrix} = 1,\ \forall s \in \mathbb{C} \\
&\text{rank}\begin{pmatrix} sI_{n_2} - A_2 & B_2 \end{pmatrix} = \\
&\quad \text{rank}\begin{pmatrix} s & 1 & 1 \\ -1 & s & 0 \end{pmatrix} = 2,\ \forall s \in \mathbb{C} \\
&\text{rank}\begin{pmatrix} sI_{n_1} - A_1 & 0 & B_1 \\ -B_2C_1 & sI_{n_2} - A_2 & B_2D_1 \end{pmatrix} = \\
&\quad \text{rank}\begin{pmatrix} s & 0 & 0 & 1 & -3 \\ -4 & s & 1 & 1 & -2 \\ 0 & -1 & s & 0 & 0 \end{pmatrix} = 3,\ \forall s \in \mathbb{C}
\end{aligned}$$

Nevertheless the corollary only gives us a necessary condition, but not sufficient, as we can see in the following example.

Example 3.2: Let (A, B, C, D) a serial concatenated system of (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) where $A_1 = (1)$, $B_1 = (1)$, $C_1 = (1)$, $D_1 = (1)$, and $A_2 = (0)$, $B_2 = (1)$, $C_2 = (1)$, $D_2 = (1)$.

Both systems are controllable because of

$$\begin{aligned}
&\text{rank}\begin{pmatrix} s-1 & 1 \end{pmatrix} = 1\ \forall s \in \mathbb{C} \\
&\text{rank}\begin{pmatrix} s & 1 \end{pmatrix} = 1\ \forall s \in \mathbb{C},
\end{aligned}$$

but the serial concatenated system (A, B, C, D) where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 \end{pmatrix},\ D = (1).$$

is not controllable because of

$$\begin{aligned}
&\text{rank}\begin{pmatrix} sI_2 - A & B \end{pmatrix} = \begin{pmatrix} s-1 & 0 & 1 \\ -1 & s & 1 \end{pmatrix} = \\
&\begin{cases} 2 & \text{for all } s \neq 0, \\ 1 & \text{for } s = 0. \end{cases}
\end{aligned}$$

Remark 3.1: Obviously, if the matrix

$$\begin{pmatrix} B_1 \\ B_2D_1 \end{pmatrix}$$

has full row rank, then the concatenated serial system (A, B, C, D) is controllable.

A sufficient condition is obtained in the case where $Spec(A_1) \cap Spec(A_2) = \emptyset$. (Spectrum of A_i, $Spec(A_i)$, is the set of eigenvalues of matrix A_i for $i = 1, 2$).

Proposition 3.1: Let (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) be the systems with $Spec(A_1) \cap Spec(A_2) = \emptyset$. If the pairs (A_1, B_1)

and (A_2, B_2) are controllable and the system (A_1, B_1, C_1, D_1) is functional output controllable, then the serial concatenated system is controllable.

Proof: For all $s \notin Spec(A_1) \cup Spec(A_2)$, we have that

$$\text{rank}\begin{pmatrix} sI_{n_1} - A_1 & \\ -B_2C_1 & sI_{n_2} - A_2 \end{pmatrix} = n_1 + n_2,$$

so

$$\text{rank}\begin{pmatrix} sI_{n_1} - A_1 & & B_1 \\ -B_2C_1 & sI_{n_2} - A_2 & B_2D_1 \end{pmatrix} = n_1 + n_2.$$

If $s_0 \in Spec(A_1)$, taking into account that $Spec(A_1) \cap Spec(A_2) = \emptyset$ we have that rank $(s_0 I_{n_2} - A_2) = n_2$, then

$$\text{rank}\begin{pmatrix} s_0I_{n_1} - A_1 & & B_1 \\ -B_2C_1 & s_0I_{n_2} - A_2 & B_2D_1 \end{pmatrix} = \text{rank}\begin{pmatrix} s_0I_{n_1} - A_1 & B_1 & \\ -B_2C_1 & B_2D_1 & s_0I_{n_2} - A_2 \end{pmatrix} = \text{rank}\begin{pmatrix} s_0I_{n_1} - A_1 & B_1 \end{pmatrix} + n_2.$$

But, taking into account that (A_1, B_1) is controllable we have rank $\begin{pmatrix} s_0I_{n_1} - A_1 & B_1 \end{pmatrix} = n_1$. Then,

$$\text{rank}\begin{pmatrix} s_0I_{n_1} - A_1 & & B_1 \\ -B_2C_1 & s_0I_{n_2} - A_2 & B_2D_1 \end{pmatrix} = n_1 + n_2.$$

Finally, if $s_0 \in Spec(A_2)$, we have that rank $(s_0I_{n_1} - A_1) = n_1$, then

$$\text{rank}\begin{pmatrix} s_0I_{n_1} - A_1 & & B_1 \\ -B_2C_1 & s_0I_{n_2} - A_2 & B_2D_1 \end{pmatrix} =$$

$$n_1 + \text{rank}\begin{pmatrix} s_0I - A_2 & B_2 \end{pmatrix} \cdot \cdot \begin{pmatrix} I_{n_2} & 0 \\ & C_1(s_0I - A_1)^{-1}B_1 + D_1 \end{pmatrix} = (a),$$

for all $s_0 \in Spec(A_2)$, and knowing that (A_2, B_2) is controllable and (A_1, B_1, C_1, D_1) is functional output-controllable we have that $(a) = n_1 + n_2$. ■

Remark 3.2: Example 3 shows that if $C_1(s_0I_{n_1} - A_1)^{-1}B_1 + D_1$ has not full rank for all $s_0 \in Spec(A_2)$ the result 3.1 is not true.

For systematic serial concatenated systems the controllability character can be described in terms of the both systems using the Hautus test in the following manner.

Theorem 3.2: A systematic serial controllability concatenated system of (A_i, B_i, C_i, D_i), $i = 1, 2$ is controllable, if and only if.

$$\text{rank}\begin{pmatrix} sI_{\delta_1} - A_1 & 0 & B_1 \\ -B_2C_1 & sI_{\delta_2} - A_2 & B_2D_1 \end{pmatrix} = n_1 + n_2,$$

$\forall s \in \mathbb{C}$

We observe that this result coincides with the case of serial concatenation. Then all results about controllability of serial concatenation are valid for systematic serial concatenation and vice-versa.

B. Controllability of parallel composite systems

As in the serial concatenated case, we will try to characterize the observability of serial concatenated systems in terms of the original systems.

Theorem 3.3: The parallel concatenated system (A, B, C, D) is controllable, if and only if the following matrix

$$\text{rank}\begin{pmatrix} sI_{n_1} - A_1 & & B_1 \\ & nI_{n_2} - A_2 & B_2 \end{pmatrix} = n_1 + n_2,$$

$\forall s \in \mathbb{C}$.

Proposition 3.2: A necessary condition for controllability of parallel concatenated system is that the pairs (A_1, B_1) and (A_2, B_2) are controllable

Unfortunately, this condition is necessary, but not sufficient. As we can see in this particular case:

Example 3.3: Let (A_i, B_i, C_i, D_i), $i = 1, 2$ be two systems with

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$C_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, D_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$C_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, D_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The pair (A_1, B_1) is controllable because

$$\text{rank}\begin{pmatrix} B_1 & A_1B_1 \end{pmatrix} = \text{rank}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 2,$$

and (A_2, B_2) is also controllable because

$$\text{rank}\begin{pmatrix} B_2 & A_2B_2 \end{pmatrix} = \text{rank}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = 2$$

However, the parallel concatenated model

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},\ D = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is not controllable because

$$\text{rank} \begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix} = \text{rank} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 2 < 4$$

A sufficient condition is obtained in the case where $Spec(A_1) \cap Spec(A_2) = \emptyset$.

Proposition 3.3: Let (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) be two systems such that $Spec(A_1) \cap Spec(A_2) = \emptyset$. If the pairs (A_1, B_1) and (A_2, B_2) are controllable, then the parallel concatenated system is controllable.

Proof: For all $s \notin Spec(A_1) \cup Spec(A_2)$, we have that

$$\text{rank} \begin{pmatrix} sI_{n_1} - A_1 & \\ & sI_{n_2} - A_2 \end{pmatrix} = n_1 + n_2,$$

so

$$\text{rank} \begin{pmatrix} sI_{n_1} - A_1 & & B_1 \\ & sI_{n_2} - A_2 & B_2 \end{pmatrix} = n_1 + n_2.$$

If $s_0 \in Spec(A_1)$, taking into account that $Spec(A_1) \cap Spec(A_2) = \emptyset$ we have that rank $(s_0 I_{n_2} - A_2) = n_2$, then

$$\text{rank} \begin{pmatrix} s_0 I_{n_1} - A_1 & & B_1 \\ & s_0 I_{n_2} - A_2 & B_2 \end{pmatrix} = \text{rank} \begin{pmatrix} s_0 I_{n_1} - A_1 & B_1 & \\ & B_2 & s_0 I_{n_2} - A_2 \end{pmatrix} = \text{rank} \begin{pmatrix} s_0 I_{n_1} - A_1 & B_1 \end{pmatrix} + n_2.$$

But, taking into account that (A_1, B_1) is controllable we have rank $\begin{pmatrix} s_0 I_{n_1} - A_1 & B_1 \end{pmatrix} = n_1$. Then,

$$\text{rank} \begin{pmatrix} s_0 I_{n_1} - A_1 & & B_1 \\ & s_0 I_{n_2} - A_2 & B_2 \end{pmatrix} = n_1 + n_2.$$

Analogously, if $s_0 \in Spec(A_2)$, we have that rank $(s_0 I_{n_1} - A_1) = n_1$, then

$$\text{rank} \begin{pmatrix} s_0 I_{n_1} - A_1 & & B_1 \\ & s_0 I_{n_2} - A_2 & B_2 \end{pmatrix} = n_1 + n_2,$$

for all $s_0 \in Spec(A_2)$. ■

Remark 3.3: Example 3.3 shows that if $Spec(A_1) \cap Spec(A_2) \neq \emptyset$ the result 3.3 is not true.

IV. Observability of composite systems

A. Observability of serial composite systems

The serial observability concatenated character is obtained from the Hautus test:

Theorem 4.1:

$$\text{rank} \begin{pmatrix} sI_{n_1} - A_1 & 0 \\ -B_2 C_1 & sI_{n_2} - A_2 \\ D_2 C_1 & C_2 \end{pmatrix} = n_1 + n_2, \forall s \in \mathbb{C}$$

Corollary 4.1: A necessary condition for observability of concatenated system is that the pair (A_2, C_2) be observable.

Corollary 4.2: A necessary condition for observability of concatenated system is that the pair $(A_1, \bar{C}_1)$, with $\bar{C}_1 = \begin{pmatrix} -B_2 C_1 \\ D_2 C_1 \end{pmatrix}$ be observable.

Corollary 4.3: A necessary condition for observability of concatenated system is that the pair (A_1, C_1), be observable.

Proof:

$$\text{rank} \begin{pmatrix} sI_{n_1} - A_1 \\ -B_2 C_1 \\ D_2 C_1 \end{pmatrix} = \text{rank} \begin{pmatrix} I_{n_1} & & \\ & -B_2 & \\ & & D_2 \end{pmatrix} \begin{pmatrix} sI_{n_1} - A_1 \\ C_1 \\ C_1 \end{pmatrix} \leq \min \left(\text{rank} \begin{pmatrix} I_{n_1} & & \\ & -B_2 & \\ & & D_2 \end{pmatrix}, \text{rank} \begin{pmatrix} sI_{n_1} - A_1 \\ C_1 \\ C_1 \end{pmatrix} \right)$$

Then

$$\text{rank} \begin{pmatrix} sI_{n_1} - A_1 \\ -B_2 C_1 \\ D_2 C_1 \end{pmatrix} \leq \text{rank} \begin{pmatrix} sI_{n_1} - A_1 \\ C_1 \end{pmatrix} \leq n_1.$$

■

In this specific case, a sufficient condition is obtained after observation of the spectrum of both matrices A_1 and A_2.

Proposition 4.1: Let (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) be two systems with $Spec(A_1) \cap Spec(A_2) = \emptyset$. If the pairs (A_1, C_1) and (A_2, C_2) are observable, with B_2 having full column rank, then the serial concatenated system is observable.

Proof: Let us suppose that we have: $s \notin Spec(A_1) \cup Spec(A_2)$; then,

$$\text{rank} \begin{pmatrix} sI_{n_1} - A_1 & 0 \\ -B_2 C_1 & sI_{n_2} - A_2 \end{pmatrix} = n_1 + n_2, \forall s \in \mathbb{C}$$

which means that

$$\text{rank}\begin{pmatrix} sI_{n_1}-A_1 & 0 \\ -B_2C_1 & sI_{n_2}-A_2 \\ D_2C_1 & C_2 \end{pmatrix} = n_1+n_2, \forall s \in \mathbb{C}$$

Let us consider $s_0 \in Spec(A_2)$; knowing that $Spec(A_1) \cap Spec(A_2) = \emptyset$, then $\text{rank}\left(s_0 I_{n_1} - A_1\right) = n_1$.

Then,

$$\text{rank}\begin{pmatrix} s_0I_{n_1}-A_1 & 0 \\ -B_2C_1 & s_0I_{n_2}-A_2 \\ D_2C_1 & C_2 \end{pmatrix} = n_1 + \text{rank}\begin{pmatrix} s_0I_{n_2}-A_2 \\ C_2 \end{pmatrix} = n_1+n_2$$

since (A_2, C_2) is observable.

Let us consider $s_0 \in Spec(A_1)$; knowing that $Spec(A_1) \cap Spec(A_2) = \emptyset$, then $\text{rank}\left(s_0 I_{n_2} - A_2\right) = n_2$.

Then,

$$\text{rank}\begin{pmatrix} s_0I_{n_1}-A_1 & 0 \\ -B_2C_1 & s_0I_{n_2}-A_2 \\ D_2C_1 & C_2 \end{pmatrix} = n_2 + \text{rank}\begin{pmatrix} s_0I_{n_1}-A_1 \\ -B_2C_1 \\ D_2C_1 \end{pmatrix}$$

If we suppose that B_2 has full column rank, we can see that:

$$\text{rank}\begin{pmatrix} s_0I_{n_1}-A_1 \\ -B_2C_1 \\ D_2C_1 \end{pmatrix} = \text{rank}\begin{pmatrix} I_{n_1} & & \\ & -B_2 & \\ & & D_2 \end{pmatrix}\begin{pmatrix} s_0I_{n_1}-A_1 \\ C_1 \\ C_1 \end{pmatrix} = \text{rank}\begin{pmatrix} s_0I_{n_1}-A_1 \\ C_1 \\ 0 \\ D_2C_1 \end{pmatrix} = n_1$$

since (A_1, C_1) is observable

Which means that:

$$\text{rank}\begin{pmatrix} sI_{n_1}-A_1 & 0 \\ -B_2C_1 & sI_{n_2}-A_2 \\ D_2C_1 & C_2 \end{pmatrix} = n_1+n_2,$$

$\forall s \in \mathbb{C}$ when $Spec(A_1) \cap Spec(A_2) = \emptyset$. ■

In the systematic serial concatenated system (A, B, C, D) obtained from the systems (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2), the observability concatenated character is obtained from the Hautus test:

Theorem 4.2: The systematic serial concatenated system (A, B, C, D) is observable if and only if the following relation holds.

$$\text{rank}\begin{pmatrix} sI_{n_1}-A_1 & 0 \\ -B_2C_1 & sI_{n_2}-A_2 \\ C_1 & 0 \\ D_2C_1 & C_2 \end{pmatrix} = n_1+n_2,\ \forall s \in \mathbb{C}$$

After this theorem it is obvious the following sufficient condition for observability of systematic serial concatenated system.

Corollary 4.4: A sufficient condition for observability of systematic serial concatenated system is that the serial concatenated system is.

Proof:

$$n_1+n_2 = \text{rank}\begin{pmatrix} sI_{n_1}-A_1 & 0 \\ -B_2C_1 & zI_{n_2}-A_2 \\ D_2C_1 & C_2 \end{pmatrix} \le \text{rank}\begin{pmatrix} sI_{n_1}-A_1 & 0 \\ -B_2C_1 & sI_{n_2}-A_2 \\ C_1 & 0 \\ D_2C_1 & C_2 \end{pmatrix} \le n_1+n_2.$$

■

We have the following result.

Theorem 4.3: A necessary and sufficient condition for observability of systematic serial concatenated system is that both systems are observable.

Proof:

$$\text{rank}\begin{pmatrix} sI_{n_1}-A_1 & 0 \\ -B_2C1 & sI_{n_2}-A_2 \\ C_1 & 0 \\ D_2C_1 & C_2 \end{pmatrix} = \text{rank}\begin{pmatrix} sI_{n_1}-A_1 & 0 \\ C_1 & 0 \\ -B_2C1 & sI_{n_2}-A_2 \\ D_2C_1 & C_2 \end{pmatrix}.$$

■

Example 4.1: Let us consider the systems (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2) two systems to be concatenated in a serial systematic form, and $(A_1, B_1, C_1, D_1$ and (A_2, B_2, C_2, D_2) as follows:

$$A_1 = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 2 \end{pmatrix},\ B_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix},\ D_1 = \begin{pmatrix} 3 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & 4 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 0 & 4 \end{pmatrix},\ D_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Then, both systems are observable;

$$\operatorname{rank} \begin{pmatrix} C_1 \\ C_1A_1 \\ C_1A_1^2 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & 0 & 2 \\ 7 & 3 & 5 \\ 22 & 27 & 29 \end{pmatrix} = 3$$

$$\operatorname{rank} \begin{pmatrix} C_2 \\ C_2A_2 \\ C_2A_2^2 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 1 & 4 & 2 \\ 1 & 0 & 4 \\ 11 & 7 & 16 \\ 5 & 3 & 16 \\ 41 & 40 & 78 \\ 27 & 18 & 70 \end{pmatrix} = 3.$$

The concatenated system A, B, C, D is observable, because:

$$A = \begin{pmatrix} 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 2 & 0 & 4 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 4 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 6 \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 3 & 0 & 6 & 1 & 4 & 2 \\ 1 & 0 & 2 & 1 & 0 & 4 \end{pmatrix},\ D = \begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix}$$

and $\operatorname{rank} \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{pmatrix} =$

$$\operatorname{rank} \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 3 & 0 & 6 & 1 & 4 & 2 \\ 1 & 0 & 2 & 1 & 0 & 4 \\ 7 & 3 & 5 & 0 & 0 & 0 \\ 29 & 9 & 31 & 11 & 7 & 16 \\ 7 & 3 & 5 & 5 & 3 & 16 \\ 22 & 27 & 29 & 0 & 0 & 0 \\ 136 & 105 & 155 & 41 & 40 & 78 \\ 28 & 27 & 41 & 27 & 18 & 70 \\ 109 & 120 & 188 & 0 & 0 & 0 \\ 681 & 618 & 1026 & 199 & 163 & 392 \\ 187 & 138 & 290 & 133 & 99 & 316 \\ 673 & 567 & 965 & 0 & 0 & 0 \\ 4085 & 3279 & 5857 & 917 & 760 & 1894 \\ 1255 & 837 & 1715 & 647 & 498 & 1462 \\ 3568 & 3153 & 4871 & 0 & 0 & 0 \\ 23176 & 18813 & 31955 & 4331 & 3511 & 9096 \\ 7396 & 5439 & 10025 & 3105 & 2439 & 6844 \end{pmatrix} = 6.$$

Then, the systematic serial concatenated matrix is observable.

(The example has been computed using Matlab).

B. *Observability of parallel composite systems*

In parallel concatenated model (A, B, C, D) obtained from the systems (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2), the observability matrix is:

$$\begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n_1+n_2-1} \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ C_1A_1 & C_2A_2 \\ C_1A_1^2 & C_2A_2^2 \\ \vdots & \\ C_1A_1^{n_1+n_2-1} & C_2A_2^{n_1+n_2-1} \end{pmatrix} \tag{10}$$

So, we have the following theorem.

Theorem 4.4: The parallel concatenated system $\mathcal{C}(A, B, C, D)$ is observable, if and only if, the matrix (10), has full rank.

Using the Hautus test we have

Theorem 4.5: The parallel concatenated system (A, B, C, D) is observable, if and only if the following matrix

$$\operatorname{rank} \begin{pmatrix} sI_{n_1} - A_1 & 0 \\ 0 & sI_{n_2} - A_2 \\ C_1 & C_2 \end{pmatrix} = n_1 + n_2,\ \forall s \in \mathbb{C}$$

which means that:

Proposition 4.2: A necessary condition for observability of parallel concatenated system is that the pairs (A_1, C_1) and (A_2, C_2) are observable.

Nevertheless, this condition is not sufficient as we can see in the following example.

Example 4.2: Let (A, B, C, D) be the parallel concatenated system of the following realizations. The first system is (A_1, B_1, C_1, D_1) with $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $C_1 = \begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$ and $D_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}$,
and the second one is the system (A_2, B_2, C_2, D_2) with

$$A_2 = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 2 & 0 & 1 \end{pmatrix},\ D_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Both systems are observable:

$$\operatorname{rank} \begin{pmatrix} C_1 \\ C_1A_1 \\ C_1A_1^2 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} = 3,$$

$$\operatorname{rank}\begin{pmatrix} C_2 \\ C_2A_2 \\ C_2A_2^2 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 2 & 0 & 1 \\ -1 & -2 & 0 \\ 3 & 1 & 2 \end{pmatrix} = 3.$$

However, the parallel concatenated model is not observable:

$$\operatorname{rank}\begin{pmatrix} 2 & 1 & 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & -1 & -2 & 0 \\ 1 & 0 & 2 & 3 & 1 & 2 \\ 2 & 1 & 0 & -2 & -3 & -1 \\ 0 & 2 & 1 & 4 & 2 & 3 \\ 1 & 0 & 2 & -3 & -4 & -2 \end{pmatrix} = 5 < 6.$$

Proposition 4.3: Let (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) be two systems where $Spec(A_1) \cap Spec(A_2) = \emptyset$. If the pairs (A_1, C_1) and (A_2, C_2) are observable, then the parallel concatenated system is observable.

Proof: Analogous to 3.3. ■

V. Conclusion

In this paper a sufficient condition for controllability and observability characters of serial and parallel concatenated finite-dimensional linear continuous-time-invariant systems are obtained.

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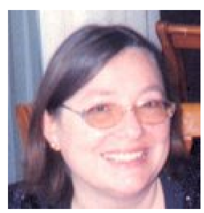

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