Eigenvalue intervals for infinite-point fractional boundary value problem and application in Systems Theory

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Abstract– In this paper, we solve an important problem in Systems Theory: We obtain the eigenvalue intervals of the infinite-point fractional boundary value problem. Fractional Eivenvalue Problems are important in Decentralized Systems, Decentralized Control, Robotics, Distributed Systems, Electromagnetic Fields, Eleasticity Theory, 1-D and 2-D Systems etc. We prove its existence of at least one or two positive solutions for the fractional equations arising in control. The results can describe the corresponding control system accurately.

KeyWords– Systems Theory; Control; Eigenvalue intervals; Infinite-point fractional boundary value problem; Positive solutions; Fixed point theorem.

1 Introduction

System and control theory has long been a rich source of problems for the numerical linear algebra community. In some problems, conditions on analytic functions of a complex variable are usually evaluated by solving a special generalized eigenvalue problem. Our principal contribution in this paper is to demonstrate the eigenvalue problem of some fractional equations. In last few decades, researchers found that fractional order differential equations could model various materials more adequately than integer order ones and provide an excellent tool for describing dynamic process [1][2][3]. The fractional order models need fractional order controllers for more effective control of dynamic systems [4]. This necessity motivated renewed interest in various applications of fractional order control. And with the rapid development of computer performances, modeling and realization of fractional order control systems also became possible and much easier than before.

Fractional differentiation's applications in automatic control is now an important issue for the international scientific community. The first Symposium on Fractional Derivatives and Their Applications of the 19th Biennial Conference on Mechanical Vibration and Noise was held from September 2 to September 6, 2003. 29 papers concerning Fractional Derivatives and Their Applications in Automatic Control, Automatic Control and System, Robotics and Dynamic Systems, Analysis Tools and Numerical Methods, Modeling and Thermal Systems were presented in the symposium.

In the research of fractional order low pass filter, in order to achieve a proper controller, which is neither conservative nor aggressive, a fractional order low-pass filter $\frac{1}{(Ts+1)^\alpha}$ is introduced. By choosing proper fractional order α, the tradeoff between stability margin loss and vibration suppression strength can be adjusted in a clear-cut way.

We propose a generalization of the PID-controller, which can be called the $PI^\lambda D^\mu$-controller because it involves an integrator of order λ and differentiator of order μ. The transfer function of such a controller has the form:

$$G_c(s) = \frac{U(s)}{E(s)} = K_P + K_I s^{-\lambda} + K_D s^\mu.$$

The equation for the $PI^\lambda D^\mu$-controller's output in the time domain is:

$$u(t) = K_P e(t) + K_I D^{-\lambda} e(t) + K_D D^\mu e(t).$$

Taking $\lambda = 1$ and $\mu = 1$, we obtain a classical PID-controller, $\lambda = 1$ and $\mu = 0$ give a PI-controller, $\lambda = 0$ and $\mu = 1$ give a PD-controller, $\lambda = 0$ and $\mu = 0$ give a gain.

All these classical types of PID-controllers are the particular cases of the fractional $PI^\lambda D^\mu$-controller. However, the $PI^\lambda D^\mu$-controller is more flexible and gives a11 opportunity to better adjust the dynamical properties of a fractional-order control system. We can also see that the use of the fractional-order controller leads to the improvement of the control of the fractional-order system. The use of fractional-order derivatives and integrals in control theory leads to better results than integral-order approaches, in addition,

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it provides strong motivation for further development of control theory in generalizing classical methods of study and the interpretation of results.

In this paper, we solve an important problem in control systems theory:

$$(\phi_p(D_{0+}^{\alpha}u(t)))' + \lambda f(u(t)) = 0, \quad 0 < t < 1, \quad (1)$$

$$u(0) = 0, \ \ u'(0) = 0, \ \ u(1) = \sum_{i=1}^{\infty} \alpha_i u(\xi_i), \quad (2)$$

where $\phi_p(s) = |s|^{p-2}s, p > 1, \phi_q = (\phi_p)^{-1}, \frac{1}{p} + \frac{1}{q} = 1, 2 < \alpha \leq 3$, D_{0+}^{α} is the standard Riemann-Liouville differentiation and $\alpha_i \geq 0, \ 0 < \xi_1 < \xi_2 < \cdots < \xi_{i-1} < \xi_i < \cdots < 1, (i = 1, 2, \cdots)$, with $\sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1} < 1, \lambda > 0, f(u) \in C([0, +\infty), [0, +\infty))$.

In section 4, we consider the following problem in control systems theory:

$$(\phi_p(D_{0+}^{\alpha}u(t)))' + q(t)f(t, u(t)) = 0, \quad 0 < t < 1, \quad (3)$$

$$u(0) = 0, \ \ u'(0) = 0, \ \ u(1) = \sum_{i=1}^{\infty} \alpha_i u(\xi_i), \quad (4)$$

where $\phi_p(s) = |s|^{p-2}s, p > 1, \phi_q = (\phi_p)^{-1}, \frac{1}{p} + \frac{1}{q} = 1, 2 < \alpha \leq 3$, D_{0+}^{α} is the standard Riemann-Liouville differentiation and $\alpha_i \geq 0, \ 0 < \xi_1 < \xi_2 < \cdots < \xi_{i-1} < \xi_i < \cdots < 1, (i = 1, 2, \cdots)$, with $\sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1} < 1, q(t) \in C([0, 1], [0, +\infty)), f$ may be singular about both the time and space variables.

2 Preliminaries and Lemmas

Definition 1 *[22] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to R$ is given by*

$$I_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided the right side integral is pointwise defined on $(0, +\infty)$.

Definition 2 *[22] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0, +\infty) \to R$ is given by*

$$D_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds,$$

where $n = [\alpha] + 1$, provided the right side integral is pointwise defined on $(0, \infty)$.

Lemma 3 [22] *Let $\alpha > 0$. If we assume $u \in C(0, 1) \cap L(0, 1)$, then the fractional differential equation*

$$D_{0+}^{\alpha}u(t) = 0$$

has a unique solution

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N},$$

where $c_i \in R, \ \ i = 1, 2, \cdots, N, N = [\alpha] + 1$.

Lemma 4 [22] *Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L(0, 1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha}u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}$$

for some $c_i \in R, \ \ i = 1, 2, \cdots, N$.

Lemma 5 *Let $y \in C[0, 1]$, and $2 < \alpha \leq 3$, the unique solution of*

$$(\phi_p(D_{0+}^{\alpha}u(t)))' + y(t) = 0, \quad 0 < t < 1, \quad (5)$$

$$u(0) = 0, \ \ u'(0) = 0, \ \ u(1) = \sum_{i=1}^{\infty} \alpha_i u(\xi_i), \quad (6)$$

is given by

$$u(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds + \frac{\sum\limits_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds}{1 - \sum\limits_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}} t^{\alpha-1}, \quad (7)$$

where

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (8)$$

Proof: Integrating both sides of the equation (5), we can get

$$\phi_p(D_{0+}^{\alpha}u(t)) = -\int_0^t y(s)ds,$$

hence

$$D_{0+}^{\alpha}u(t) = -\phi_q\Big(\int_0^t y(s)ds\Big).$$

From Lemma 4, it follows that

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3},$$

condition (6) imply that $C_2 = 0, C_3 = 0$.

$$C_1 = \frac{1}{\Gamma(\alpha)(1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1}\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds - \frac{\sum_{i=1}^{\infty}\alpha_i}{\Gamma(\alpha)(1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1})}\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds.$$

Therefore, the unique solution of problem (5), (6) is

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds + \frac{1}{\Gamma(\alpha)(1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1}t^{\alpha-1}\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds \\
&\quad - \frac{\sum_{i=1}^{\infty}\alpha_i}{\Gamma(\alpha)(1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1})}\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}t^{\alpha-1}\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds \\
&= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}t^{\alpha-1}\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds \\
&\quad + \frac{\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1}t^{\alpha-1}\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds \\
&\quad - \frac{\sum_{i=1}^{\infty}\alpha_i}{\Gamma(\alpha)(1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1})}\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}t^{\alpha-1}\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t [(1-s)^{\alpha-1}t^{\alpha-1}-(t-s)^{\alpha-1}]\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds + \frac{1}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}t^{\alpha-1}\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds \\
&\quad + \frac{\sum_{i=1}^{\infty}\alpha_i t^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1})}\Big\{\int_0^{\xi_i}[(1-s)^{\alpha-1}\xi_i^{\alpha-1}-(\xi_i-s)^{\alpha-1}]\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds + \int_{\xi_i}^1 (1-s)^{\alpha-1}\xi_i^{\alpha-1}\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds\Big\} \\
&= \int_0^1 G(t,s)\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds + \frac{\sum_{i=1}^{\infty}\alpha_i t^{\alpha-1}\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s y(\tau)d\tau\Big)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}.
\end{aligned}$$

This completes the proof.

Lemma 6 *[40] Let $2 < \alpha \le 3$. The function $G(t,s)$ defined by (8) has the following properties.*
(i) For any $(t,s) \in [0,1]\times[0,1]$, $G(t,s) \ge 0$;
(ii) Fix $s \in [0,1]$, then for any $t \in [0,1]$,

$$G(t,s) \le G(t_0,s) = \frac{s^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)[1-(1-s)^{\frac{\alpha-1}{\alpha-2}}]^{\alpha-2}}, \tag{9}$$

where $t_0 = \dfrac{s}{1-(1-s)^{\frac{\alpha-1}{\alpha-2}}} \in [s,1)$.
(iii) Fix $s \in [0,1]$, then for any $t \in [0,1]$, $G(t,s) \ge \rho(t)G(t_0,s) \ge \rho(t)t^{\alpha-1}G(t_0,s)$, where

$$\rho(t) = \begin{cases} t(1-t), & \frac{1}{2} \le t \le 1, \\ t^2, & 0 \le t \le \frac{1}{2}. \end{cases} \tag{10}$$

(iv) Fix $s \in [0,1]$, then for any $t \in [\frac{1}{4},\frac{3}{4}]$,

$$G(t,s) \ge \frac{1}{16}G(t_0,s).$$

Lemma 7 *[40] Let $y \in C([0,1],[0,+\infty))$, then the solution $u(t)$ of the boundary value problem (5),(6) satisfies:*

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t) \ge \frac{1}{16}\|u\|. \tag{11}$$

Lemma 8 *[26] Let P be a cone in real Banach space E, and $\Omega_1,\ \Omega_2$ two bounded open sets of E centered at the origin with $\overline{\Omega_1} \subset \Omega_2$. Assume that $T : P \cap (\overline{\Omega_2}\backslash\Omega_1) \to P$ is a completely continuous operator such that, either*
(i) $\|Tx\| \le \|x\|,\quad x \in P\cap\partial\Omega_1,$ and $\|Tx\| \ge \|x\|,\quad x \in P\cap\partial\Omega_2$ or
(ii) $\|Tx\| \ge \|x\|,\quad x \in P\cap\partial\Omega_1,$ and $\|Tx\| \le \|x\|,\quad x \in P\cap\partial\Omega_2,$
holds. Then T has at least one fixed point in $P \cap (\overline{\Omega_2}\backslash\Omega_1)$.

3 Eigenvalue intervals for problem in control systems theory (1), (2)

The following assumptions will be used in this section.
(H_1) $f \in C([0,+\infty],[0,+\infty))$, and there exists $t_n \to 0$ such that $f(t_n) > 0, n = 1,2,\cdots$;
(H_2) $\alpha_i \geq 0,\ 0 < \xi_1 < \xi_2 < \cdots < \xi_{i-1} < \xi_i < \cdots < 1, (i = 1,2,\cdots)$, with $\sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1} < 1$;
(H_3) $\sup_{b>0} \min_{\frac{b}{16} \leq t \leq b} f(t) > 0$;
(A_1) $\lim\limits_{t\to 0} \frac{f(t)}{\phi_p(t)} = \infty$;
(A_2) $\lim\limits_{t\to \infty} \frac{f(t)}{\phi_p(t)} = \infty$;
(A_3) $\lim\limits_{t\to 0} \frac{f(t)}{\phi_p(t)} = 0$;
(A_4) $\lim\limits_{t\to \infty} \frac{f(t)}{\phi_p(t)} = 0$.
Set

$$A = \int_0^1 G(t_0,s)ds + \frac{\sum\limits_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i,s)ds}{1 - \sum\limits_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}},$$

$$B = \frac{\sum\limits_{i=1}^{\infty} \alpha_i \int_{\frac{1}{4}}^{\frac{3}{4}} G(\xi_i,s)(s-\frac{1}{4})^{q-1}ds}{1 - \sum\limits_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}} (\frac{1}{4})^{\alpha-1},$$

$$\lambda^* = \frac{1}{\phi_p(A)} \sup_{r>0} \frac{\phi_p(r)}{\max\limits_{0\leq t\leq r} f(t)},$$

$$\lambda^{**} = \frac{1}{\phi_p(B)} \inf_{r>0} \frac{\phi_p(r)}{\min\limits_{\frac{b}{16}\leq t\leq b} f(t)}.$$

In this section, let $E = C[0,1]$ be endowed with the maximum norm

$$\|u\| = \max_{0\leq t\leq 1} |u(t)|,$$

then E is a Banach space. Let $P \subset E$ be defined as

$$P = \{u \in E \mid u(t) \geq 0,\ 0 \leq t \leq 1,\ \min_{\frac{1}{4}\leq t\leq \frac{3}{4}} u(t) \geq \frac{1}{16}\|u\|\}, \tag{12}$$

then P is a cone in E.

If $u \in E,\ u(t) \geq 0,\ t \in [0,1]$ and satisfies the boundary value problem (1), (2), we call u is a nonnegative solution of the problem (1), (2).

If u is a nonnegative solution of boundary value problem (1), (2) with $\|u\| > 0$, then we call u is a positive solution of the problem (1), (2).

Define an operator $T : P \to C[0,1]$ as

$$(Tu)(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^s \lambda f(u(\tau))d\tau\Big)ds + \frac{\sum\limits_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s \lambda f(u(\tau))d\tau\Big)ds}{1 - \sum\limits_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}} t^{\alpha-1}, \tag{13}$$

where $G(t,s)$ is given by (8).

For $r > 0$, let

$$\Omega_r = \{u \in P \,|\, \|u\| < r\},$$

$$\partial\Omega_r = \{u \in P \,|\, \|u\| = r\}.$$

Lemma 9 *The operator $T : P \to P$ is completely continuous.*

Proof: The proof is similar to Lemma 3.1 in [26], so we omit it.

Lemma 10 *Assume that $(H_1)-(H_3)$ hold, and there exist two positive constants a, b such that*

$$\max_{0\leq t\leq a} f(t) \leq \frac{1}{\lambda}\phi_p(\frac{a}{A}),\ \min_{\frac{b}{16}\leq t\leq b} f(t) \geq \frac{1}{\lambda}\phi_p(\frac{b}{B}). \tag{14}$$

Then problem (1), (2) has at least one positive solution $u^* \in P$ such that

$$\min\{a,b\} \leq \|u^*\| \leq \max\{a,b\}.$$

Proof: Without loss of generality, we assume that $a < b$. For $u \in \partial\Omega_a,\ 0 \leq t \leq 1$, one has

$$f(u(t)) \leq \frac{1}{\lambda}\phi_p(\frac{a}{A}),$$

then

$$\begin{aligned}(Tu)(t) &\leq \int_0^1 G(t_0,s)\phi_q\left(\int_0^1 \lambda \tfrac{1}{\lambda}\phi_p(\tfrac{a}{A})d\tau\right)ds \\ &+ \frac{\sum\limits_{i=1}^{\infty} \alpha_i}{1 - \sum\limits_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}} \int_0^1 G(\xi_i,s)\phi_q\left(\int_0^1 \lambda \frac{1}{\lambda}\phi_p(\frac{a}{A})d\tau\right)ds \\ &= \int_0^1 G(t_0,s)\tfrac{a}{A}ds \\ &+ \frac{\sum\limits_{i=1}^{\infty} \alpha_i}{1 - \sum\limits_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}} \int_0^1 G(\xi_i,s)\frac{a}{A}ds \\ &= \frac{a}{A}\left[\int_0^1 G(t_0,s)ds + \frac{\sum\limits_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i,s)ds}{1 - \sum\limits_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}}\right] \\ &= \frac{a}{A}A = a,\end{aligned}$$

this implies $\|Tu\| \le \|u\|$, for $u \in \partial\Omega_a$.

For $u \in \partial\Omega_b$, $\frac{1}{4} \le t \le \frac{3}{4}$, there is

$$f(u(t)) \ge \frac{1}{\lambda}\phi_p(\frac{b}{B}),$$

so

$$\begin{aligned}(Tu)(t) &\ge \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)\phi_q\left(\int_{\frac{1}{4}}^{s} \lambda f(u(\tau))d\tau\right)ds \\ &+ \frac{\sum\limits_{i=1}^{\infty}\alpha_i}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \\ &\int_{\frac{1}{4}}^{\frac{3}{4}} G(\xi_i,s)\phi_q\left(\int_{\frac{1}{4}}^{s} \lambda f(u(\tau))d\tau\right)ds(\tfrac{1}{4})^{\alpha-1} \\ &\ge \frac{\sum\limits_{i=1}^{\infty}\alpha_i}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}\end{aligned}$$

$$\begin{aligned}&\int_{\frac{1}{4}}^{\frac{3}{4}} G(\xi_i,s)\phi_q\left(\int_{\frac{1}{4}}^{s} \phi_p(\tfrac{b}{B})d\tau\right)ds(\tfrac{1}{4})^{\alpha-1} \\ &= \frac{b}{B}\left(\frac{\sum\limits_{i=1}^{\infty}\alpha_i\int_{\frac{1}{4}}^{\frac{3}{4}} G(\xi_i,s)(s-\frac{1}{4})^{q-1}ds}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}(\frac{1}{4})^{\alpha-1}\right) \\ &= \frac{b}{B}B = b,\end{aligned}$$

this implies $\|Tu\| \ge \|u\|$, for $u \in \partial\Omega_b$.

As a consequence of Lemma 7, there exists $\omega^* \in \overline{\Omega_b}\backslash\Omega_a$, such that $T\omega^* = \omega^*$. This means ω^* is a solution of problem (1), (2) and $a \le \|\omega^*\| \le b$. $\|\omega^*\| \ge a > 0$ implies that $\omega^*(t) > 0$ for $t \in [\frac{1}{4}, \frac{3}{4}]$. This combines with (H_1) and $\omega^* = T\omega^*$, we can get $\omega^*(t) > 0, 0 < t < 1$.

Theorem 11 *Assume that* $(H_1)-(H_3),(A_1),(A_2)$ *hold. Then for every* $0 < \lambda < \lambda^*$, *problem (1), (2) has at least two positive solutions.*

Proof: Let

$$q(r) = \phi_p\left(\frac{r}{A}\right)\frac{1}{\max\limits_{0\le t\le r} f(t)},$$

condition (H_1) implies that $q : (0,+\infty) \to (0,+\infty)$ is continuous. So for $0 < \lambda < \lambda^*$, there exists $0 < r_0 < +\infty$ such that

$$f(t) \le \frac{1}{\lambda}\phi_p\left(\frac{r_0}{A}\right), \quad t \in [0, r_0].$$

On the other hand, since (A_1) and (A_2) hold, there exist $0 < b_1 < r_0 < b_2 < +\infty$ such that

$$\frac{f(t)}{\phi_p(t)} \ge \frac{1}{\lambda}\phi_p\left(\frac{4}{B}\right), \quad t \in [0, b_1] \cup [\frac{b_2}{4}, +\infty).$$

Therefore,

$$f(t) \ge \frac{1}{\lambda}\phi_p\left(\frac{b_1}{B}\right), \quad t \in [\frac{b_1}{4}, b_1],$$

$$f(t) \ge \frac{1}{\lambda}\phi_p\left(\frac{b_2}{B}\right), \quad t \in [\frac{b_2}{4}, b_2].$$

By the application of Lemma 10, the proof is complete.

Theorem 12 *Assume that* $(H_1)-(H_3),(A_3),(A_4)$ *hold.* $f(t) > 0$ *for* $t > 0$. *Then for every* $\lambda^{**} < \lambda < +\infty$, *problem (1), (2) has at least two positive solutions.*

Proof: Denote function

$$p(r) = \frac{\phi_p(r)}{\phi_p(B)\min\limits_{\frac{r}{16}\le t\le r} f(t)}.$$

It is obvious that the function $p : (0,+\infty) \to (0,+\infty)$ is continuous. For $\lambda^{**} < \lambda < +\infty$, there exists $0 < r_1 < +\infty$ such that

$$f(t) \ge \frac{1}{\lambda}\phi_p\left(\frac{r_1}{B}\right), \quad t \in [\frac{r_1}{4}, r_1].$$

On the other hand, since condition (A_3) holds, there exists $0 < a_1 < r_0$ such that

$$\frac{f(t)}{\phi_p(t)} \le \frac{1}{\lambda\phi_p(A)}, \quad t \in (0, a_1].$$

Then

$$f(t) \le \frac{\phi_p(t)}{\lambda\phi_p(A)} \le \frac{1}{\lambda}\phi_p\left(\frac{a_1}{A}\right).$$

By condition (A_4), there exists $r_1 < a < +\infty$, such that

$$\frac{f(t)}{\phi_p(t)} \le \frac{1}{\lambda\phi_p(A)}, \quad t \in [a, +\infty).$$

Define $M = \max\limits_{0\le t\le a} f(t)$. Let $a_2 > a$ such that $a_2 \ge \phi_q(M\lambda)A$. Then

$$f(t) \le \frac{1}{\lambda}\phi_p\left(\frac{a_2}{A}\right), \ t \in [0, a_2].$$

As an application of Lemma 10, the proof is complete.

4 Positive solutions of singular fractional problem in control systems theory (3), (4)

In this section, we consider the following singular fractional differential equation with infinite-point boundary value conditions

$$(\phi_p(D_{0+}^{\alpha}u(t)))' + q(t)f(t,u(t)) = 0, \quad 0 < t < 1,$$

$$u(0)=0,\ \ u'(0)=0,\ \ u(1)=\sum_{i=1}^{\infty}\alpha_i u(\xi_i),$$

where $\phi_p(s)=|s|^{p-2}s, p>1, \phi_q=(\phi_p)^{-1}, \frac{1}{p}+\frac{1}{q}=1, 2<\alpha\le 3$, D_{0+}^{α} is the standard Riemann-Liouville differentiation and $\alpha_i\ge 0,\ 0<\xi_1<\xi_2<\cdots<\xi_{i-1}<\xi_i<\cdots<1, (i=1,2,\cdots)$, with $\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}<1, q(t)\in C([0,1],[0,+\infty)), f$ may be singular about both the time and space variables.

We make the following conditions.
(L_1) $f\in C((0,1)\times(0,+\infty),[0,+\infty))$;
(L_2) $q(t)\in C((0,1),[0,+\infty))$, and not identically zero on any subinterval of $(0,1)$;
(L_3) for any positive constants $r_1<r_2$, there exists a continuous function $\phi_{r_1,r_2}:(0,1)\to[0,+\infty)$ such that

$$\int_0^1 q(t)\phi_{r_1,r_2}(t)dt<+\infty$$

and $f(t,u)\le\phi_{r_1,r_2}(t),\ \ 0<t<1$.

Lemma 13 *$u(t)$ is a solution of the boundary value problem (3), (4) if and only if $u(t)$ is a solution of the following integral equation*

$$u(t)=\int_0^1 G(t,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds+\frac{\sum\limits_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}t^{\alpha-1}, \tag{15}$$

where $G(t,s)$ is given by (8).

Proof: As an immediate result of Lemma 5, we can easily complete the proof, so we omit it.

Lemma 14 *The solution $u(t)$ of boundary value problem (3), (4) satisfies*

$$\min_{0\le t\le 1}u(t)\ge\rho(t)t^{\alpha-1}\|u\|. \tag{16}$$

Proof: From Lemma 13, it follows that

$$\|u\|\le\int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds+\frac{\sum\limits_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}.$$

On the other hand,

$$\begin{aligned}u(t)&\ge\int_0^1\rho(t)t^{\alpha-1}G(t_0,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds\\&\quad+\rho(t)t^{\alpha-1}\frac{\sum\limits_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}\\&=\rho(t)t^{\alpha-1}\Big(\int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds\\&\quad+\frac{\sum\limits_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}\Big)\\&=\rho(t)t^{\alpha-1}\|u\|,\end{aligned}$$

which means that

$$\min_{0\le t\le 1}u(t)\ge\rho(t)t^{\alpha-1}\|u\|.$$

Let $E=C[0,1]$, then E is a Banach space equipped with the norm

$$\|u\|=\max_{0\le t\le 1}|u(t)|.$$

Denote

$$K=\{u\in C[0,1]|u(t)\ge 0,\ \min_{0\le t\le 1}u(t)\ge\rho(t)t^{\alpha-1}\|u\|\}. \tag{17}$$

It is obvious that K is a cone.
Define an operator T as follows:

$$(Tu)(t)=\int_0^1 G(t,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds+\frac{\sum\limits_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}t^{\alpha-1}. \tag{18}$$

Let

$$\Omega(r)=\{u\in K:\ \|u\|<r\},$$
$$\partial\Omega(r)=\{u\in K:\ \|u\|=r\}.$$

Lemma 15 *Fix $2<\alpha\le 3$, then*

$$\max_{0\le t\le 1}\rho(t)t^{\alpha-1}=\max\left\{(\frac{1}{2})^{\alpha+1},\ \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\right\},$$

where $\rho(t)$ is given by (10).

Proof: For $\frac{1}{2}\le t\le 1$,

$$\rho(t)t^{\alpha-1}=t(1-t)t^{\alpha-1}=t^{\alpha}-t^{\alpha+1},$$

$$\frac{d(\rho(t)t^{\alpha-1})}{dt}=\alpha t^{\alpha-1}-(\alpha+1)t^{\alpha}=t^{\alpha-1}(\alpha-(\alpha+1)t).$$

If $t = \frac{\alpha}{\alpha+1}$, then $\dfrac{d(\rho(t)t^{\alpha-1})}{dt} = 0$,
If $\frac{1}{2} \le t < \frac{\alpha}{\alpha+1}$, then $\dfrac{d(\rho(t)t^{\alpha-1})}{dt} > 0$,
If $\frac{\alpha}{\alpha+1} < t \le 1$, then $\dfrac{d(\rho(t)t^{\alpha-1})}{dt} < 0$.
So for $\frac{1}{2} \le t \le 1$, there is

$$\max_{\frac{1}{2}\le t\le 1} \rho(t)t^{\alpha-1} = \rho(\frac{\alpha}{\alpha+1})(\frac{\alpha}{\alpha+1})^{\alpha-1} = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}},$$

for $0 \le t \le \frac{1}{2}$, $\rho(t)t^{\alpha-1} = t^2 t^{\alpha-1} = t^{\alpha+1}$, so

$$\max_{0\le t\le\frac{1}{2}} \rho(t)t^{\alpha-1} = \max_{0\le t\le\frac{1}{2}} t^{\alpha+1} = (\frac{1}{2})^{\alpha+1}.$$

The proof is completed.

Lemma 16 *Suppose that $(L_1) - (L_3)$ hold. Then $T : K \to K$ is a completely continuous operator.*

We give the following height function to control the growth of nonlinearity.

$$\varphi(t,r) = \max\{f(t,u) :\ \rho(t)t^{\alpha-1}r \le u \le r\},$$

$$\psi(t,r) = \min\{f(t,u) :\ \rho(t)t^{\alpha-1}r \le u \le r\}.$$

Theorem 17 *Suppose that $(L_1) - (L_3)$ hold. Furthermore, there exist two positive real numbers $a < b$ such that one of the following conditions is satisfied:*

$$(a_1)\quad \int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\varphi(\tau,a)d\tau\Big)ds + \frac{\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)\varphi(\tau,a)d\tau\Big)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \le a$$

and

$$+\infty > \max\Big\{(\frac{1}{2})^{\alpha+1},\ \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\Big\} \int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,b)d\tau\Big)ds + \frac{\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,b)d\tau\Big)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \ge b;$$

$$(a_2)\quad +\infty > \max\Big\{(\frac{1}{2})^{\alpha+1},\ \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\Big\} \int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,a)d\tau\Big)ds + \frac{\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,a)d\tau\Big)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \ge a$$

and

$$\int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\varphi(\tau,b)d\tau\Big)ds + \frac{\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)\varphi(\tau,b)d\tau\Big)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \le b.$$

Then the problem (3),(4) has at least one positive solution $u^* \in K$ such that $a \le \|u^*\| \le b$.

Proof: We only prove the case (a_1), similarly, we can prove the case (a_2).
If $u \in \partial\Omega(a)$, then $\|u\| = a$ and $\rho(t)t^{\alpha-1}a \le u(t) \le a,\ \ 0 \le t \le 1$.
The definition $\varphi(t,a)$ implies that

$$f(t,u(t)) \le \varphi(t,a), \quad 0 < t < 1,$$

furthermore,

$$\|Tu\| \le \int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\varphi(\tau,a)d\tau\Big)ds + \frac{\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)\varphi(\tau,a)d\tau\Big)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \le a = \|u\|.$$

If $u \in \partial\Omega(b)$, then $\|u\| = b$ and $\rho(t)t^{\alpha-1}b \le u(t) \le b,\ \ 0 \le t \le 1$.
The definition $\psi(t,b)$ implies that

$$f(t,u(t)) \ge \psi(t,b), \quad 0 < t < 1,$$

furthermore,

$$\begin{aligned}\|Tu\| &= \max_{0\le t\le 1}\Big\{\int_0^1 G(t,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds + \frac{\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}t^{\alpha-1}\Big\}\\
&\ge \max_{0\le t\le 1}\int_0^1 \rho(t)t^{\alpha-1}G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,b)d\tau\Big)ds + \frac{\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)f(\tau,u(\tau))d\tau\Big)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}\\
&= \max\Big\{(\frac{1}{2})^{\alpha+1},\ \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\Big\}\int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,b)d\tau\Big)ds + \frac{\sum_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,b)d\tau\Big)ds}{1-\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}\\
&\ge b = \|u\|.\end{aligned}$$

From Lemma 8, it follows that the operator T has a fixed point $u^* \in \overline{\Omega(b)} \backslash \Omega(a)$. Thus $a \leq \|u^*\| \leq b$, since $u^* \geq \rho(t)t^{\alpha-1}\|u\| \geq a\rho(t)t^{\alpha-1} > 0, \ 0 < t < 1$, we deduce that u^* is a positive solution.

Theorem 18 *Suppose that* $(L_1)-(L_3)$ *hold. Furthermore, there exist three positive real constants* $a < b < c$ *such that one of the following conditions is satisfied:*

$$(b_1) \quad \frac{\int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\varphi(\tau,a)d\tau\Big)ds + \sum\limits_{i=1}^{\infty}\alpha_i \int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)\varphi(\tau,a)d\tau\Big)ds}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \leq a$$

$$+\infty > \max\left\{(\frac{1}{2})^{\alpha+1},\ \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\right\}\frac{\int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,b)d\tau\Big)ds + \sum\limits_{i=1}^{\infty}\alpha_i \int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,b)d\tau\Big)ds}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} > b$$

and

$$\frac{\int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\varphi(\tau,c)d\tau\Big)ds + \sum\limits_{i=1}^{\infty}\alpha_i \int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)\varphi(\tau,c)d\tau\Big)ds}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \leq c\ ;$$

$$(b_2) \quad +\infty > \max\left\{(\frac{1}{2})^{\alpha+1},\ \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\right\}\frac{\int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,a)d\tau\Big)ds + \sum\limits_{i=1}^{\infty}\alpha_i \int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,a)d\tau\Big)ds}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \geq a$$

$$\frac{\int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\varphi(\tau,b)d\tau\Big)ds + \sum\limits_{i=1}^{\infty}\alpha_i \int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)\varphi(\tau,b)d\tau\Big)ds}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} < b$$

and

$$+\infty > \max\left\{(\frac{1}{2})^{\alpha+1},\ \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\right\}\frac{\int_0^1 G(t_0,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,c)d\tau\Big)ds + \sum\limits_{i=1}^{\infty}\alpha_i \int_0^1 G(\xi_i,s)\phi_q\Big(\int_0^s q(\tau)\psi(\tau,c)d\tau\Big)ds}{1-\sum\limits_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \geq c\ .$$

Then the problem (3),(4) has at least two positive solution $u_1^*, u_2^* \in K$ *such that* $a \leq \|u_1^*\| < b < \|u_2^*\| \leq c$.

5 Conclusion

In this paper we have examined some well known problems in systems and control theory. Fractional eigenvalue problems are important Problems in Automatic Control, Electromagnetic Fields 1-D and 2-D Systems. We have demonstrated that many of these problems can be solved with resorting to generalized eigenvalue problems. We prove its existence of at least one or two positive solutions for the fractional eigenvalue problems arising in control. As far as we know, no work has been done to get existence and positive solutions of the infinite-point fractional eigenvalue problems with p-Laplacian. The aim of this paper is to fill the gap in the relevant literatures. Such investigations will provide an important platform for gaining a deeper understanding of our environment. Some relevant studies with Engineering Applications can be found in [41],[42]

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