A Discrete Model of Jitter for Coarsely Quantized Waveforms

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***Abstract*—**This work deals with the derivation and validation of a discrete model of timing jitter suited to coarsely quantized waveforms, i.e., for waveforms digitized by low-resolution high-speed analog-to-digital converters. In fact, in the paper it is shown that when a coarsely quantized waveform is considered, the classical continuous model for timing jitter is no longer valid since the discrete nature of the waveform must be taken into account. In particular, it is shown that the classical continuous model leads to significant underestimation of the variance of frequency-domain waveform parameters when repeated measurements are performed. Analytical derivations related to the statistical properties of Fourier coefficients of a jittered waveform are validated through numerical simulations.

***Keywords*—**Analog-to-digital conversion, discrete Fourier transform, frequency-domain analysis, jitter, noise, statistical analysis.

I. Introduction

ANALOG-to-digital (A/D) conversion of signals is nowadays a widespread technique allowing fast and effective measurements in most of engineering applications. For this reason, the whole A/D conversion process has received in the past literature a lot of attention in order to investigate all the sources of uncertainty resulting from the non-ideal behavior of real A/D converters (ADCs) (e.g., see [1]-[6]). Timing jitter consists in the uncertainty related to the actual sampling instants in A/D conversion of signals [7]-[12]. It is well known that this kind of uncertainty becomes more and more relevant as the speed of ADCs increases. Much work has been done in order to obtain experimental characterization of jitter effects. Theoretical treatment of such a phenomenon, however, has received much less attention, and some analytical results are available only in the case of the sinusoidal waveform [7]-[11].

In this work, a general model is proposed able to provide the probabilistic description of timing jitter effects in the frequency domain after the discrete Fourier transform (DFT) of a sampled and quantized signal [12]. Indeed, the main novelty of the proposed model consists in taking simultaneously into account quantization and jitter. Of course this is of special relevance when a high-speed low-resolution A/D conversion is performed. In such a case, as an essential enhancement of the results reported in [12], in this paper it is clearly shown that the well-known classical model provides an underestimate of jitter effects on the variance and on the probability density function (PDF) of the DFT coefficients. This happens because the classical model provides a mathematical description leading to an equivalent additive noise with standard deviation proportional to the signal amplitude. Thus, by decreasing the signal amplitude, the jitter effects predicted by the classical model decrease on a proportional base. However, when a low-amplitude signal is considered, quantization cannot be neglected since jitter refers to quantized samples. Thus, quantization tends to emphasize jitter effects because even a small sampling jitter can result in an amplitude error equal to one quantization step. It follows that a more accurate model must result in a discrete additive noise instead of a continuous one.

The derivations are carried out by assuming a Gaussian distribution of each sampling instant. However, the proposed approach can be readily extended to different distributions. In particular, the PDF, the mean value and the variance of both the real and the imaginary parts of the DFT are evaluated. The effects of some A/D conversion features, such as jitter standard deviation, ADC resolution, and number of acquired samples, are put into evidence. The predictions provided by the proposed probabilistic model are validated numerically by simulating the A/D conversion of sinusoidal and multisine waveforms.

II. Jitter Discrete Model

It is well known that, due to technological limits, high-speed ADCs are typically characterized by low resolution since the need for high sampling frequencies contrasts with the need for a fine quantization. As a consequence, low-amplitude waveforms are subject to coarse quantization in high-speed ADCs. It is worth noticing, however, that the general theory of quantization guarantees that the so-called noise model of quantization can be applied only when fine quantization is performed. On the contrary, in case of rough quantization such a process can be no longer treated as additive independent noise, but a deterministic approach in that case is more appropriate. In that case, indeed, repeated A/D conversions of a given waveform in triggered mode (i.e., by assuming no phase variation in different A/D conversions) provide results with low statistical dispersion provided that sampling is an ideal process and input additive noise is negligible. In case of jitter, however, sampling is not an ideal process and therefore repeated low-resolution A/D conversions of a given waveform lead to results with statistical dispersion. The jitter model

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derived in this Section provides the statistical properties of low-resolution A/D conversions of a given waveform in triggered mode.

A/D conversion of a signal can be physically considered as the ordered sequence of two processes, i.e., sampling and quantization. However, from a modeling viewpoint, the order of the two processes can be exchanged without changing the overall result. This choice leads to consider the problem statement depicted in Fig. 1 where the quantized version $q(t)$ of a short stretch of the underlying analog signal $x(t)$ is represented [12]. It is assumed that, as far as the time extension of one quantization output level is considered (i.e., the level $m\Delta$ in Fig. 1, where Δ is the quantization step), the slope s of the analog signal can be treated as a constant. Under the hypothesis of uniform quantization, the time extension of the considered quantization level is therefore $\Delta t = \Delta / s$.

Let us consider now the case of a signal sample taken at the ideal time instant kT_S (where T_S is the sampling period) falling within the above defined Δt. Its position is defined by the distance r with respect to the center of Δt . In the case of absence of jitter this would result in the quantized sample $m\Delta$. In the case of a jittered sample, however, the actual sampling instant is $kT_S + \delta$, where δ is a random variable (RV) described by the related PDF $f_\delta(\delta)$. Three different cases can happen in the proposed model. In the first case, if the actual sampling instant is included within the duration Δt of the quantization level $m\Delta$ then the output is still $m\Delta$, i.e., jitter has no effect. In the second case, if the actual sampling instant exceeds the right edge of the quantization level $m\Delta$ then the output is $(m+1)\Delta$, i.e., an error equal to $+\Delta$ is introduced. Finally, in the third case, if the actual sampling instant exceeds the left edge of the quantization level $m\Delta$ then the output is $(m-1)\Delta$, i.e., an error equal to $-\Delta$ is introduced. Thus, the quantized sample is affected by additive discrete noise d , whose possible values are $\{-\Delta, 0, +\Delta\}$. The occurrence of one of such values depends on $\sigma_\delta / \Delta t$ (where σ_δ is the standard deviation of the timing jitter δ), and on the relative location r of the ideal sampling instant kT_S and the center of the considered quantization level. By assuming a given value for r, and a zero-mean Gaussian PDF $f_\delta(\delta)$, the corresponding conditional probabilities can be evaluated as [13]-[15]

$$P(d = -\Delta \mid r) = \int_{-\infty}^{\delta_1} f_\delta(\delta) d\delta \tag{1}$$

$$P(d = 0 \mid r) = \int_{\delta_1}^{\delta_2} f_\delta(\delta) d\delta \tag{2}$$

$$P(d = +\Delta \mid r) = \int_{\delta_2}^{+\infty} f_\delta(\delta) d\delta \tag{3}$$

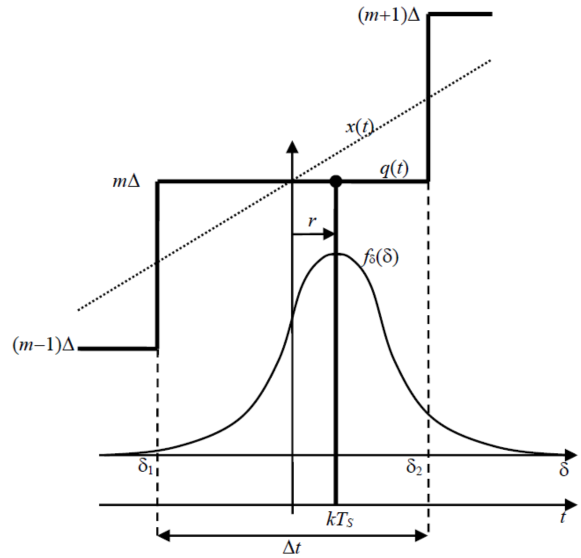

Fig. 1. Quantized version $q(t)$ of the input signal $x(t)$, and definition of the probabilistic quantities related to the sampling instant kT_S.

where $\delta_1 = -r - \Delta t / 2$ and $\delta_2 = -r + \Delta t / 2$. In order to include the randomness of the relative position r , by assuming a uniform PDF $f_r(r) = 1/\Delta t$ within the interval $(-\Delta t / 2, \Delta t / 2)$, from the total probability theorem one can obtain [13]-[15]:

$$\begin{aligned} P(d = +\Delta) &= \int_{-\Delta t/2}^{\Delta t/2} P(+\Delta \mid r) f_r(r) dr = \\ &= \frac{1}{2}\left[(1 - \operatorname{erf}(z)) + \frac{1}{z\sqrt{\pi}} (1 - \exp(-z^2)) \right] \end{aligned} \tag{4}$$

$$P(d = -\Delta) = P(d = +\Delta) \tag{5}$$

$$P(d = 0) = 1 - 2P(d = +\Delta) \tag{6}$$

where $z = \Delta / (s\sigma_\delta \sqrt{2})$. Thus, the discrete RV d has zero mean value, and variance given by:

$$\sigma_d^2 = 2\Delta^2 P(d = +\Delta) . \tag{7}$$

The above results hold also for the case of signal with negative slope provided that the magnitude of s is considered, i.e., $z = \Delta / (|s| \sigma_\delta \sqrt{2})$.

The proposed discrete model is a time-domain model resulting in additive noise with respect to the quantized waveform $q(t)$. In most of the applications, the digitized waveform is transformed into the frequency domain through the DFT which is effectively implemented by the well-known fast Fourier transform (FFT) [13]. In order to evaluate the frequency-domain effects of jitter, it must be considered that the amplitude of the n-th frequency component c_n is evaluated through the DFT as:

$$c_n = \frac{2}{N_S} \sum_{k=0}^{N_S-1} \left[q(kT_S) + d_k \right] \times \exp(-j2\pi nk / N_S) = \\ = a_n - jb_n \quad (8)$$

where N_S is the number of samples, and $\{d_k\}$ are independent RVs with zero mean and variance given by (7) where z is replaced by $z_k = \Delta / (|s_k| \sigma_\delta \sqrt{2})$:

$$\sigma_{d_k}^2 = \Delta^2 \left[\left(1 - \mathrm{erf}(z_k)\right) + \frac{1}{z_k \sqrt{\pi}} \left(1 - \exp(-z_k^2)\right) \right] \quad (9)$$

where $|s_k|$ is the local slope of the input signal which can be readily estimated from the reconstructed analog signal. By denoting as a_n and b_n the RVs corresponding to the real and the imaginary parts of c_n, respectively, from the Central Limit Theorem [13]-[15] it follows that they can be approximated as Gaussian RVs (see Section III), with mean values given by the deterministic components of the spectrum (i.e., the spectrum of the jitter-free quantized samples), and variances given by [4]-[6]:

$$\sigma_{a_n}^2 = \frac{4}{N_S^2} \sum_{k=0}^{N_S-1} \sigma_{d_k}^2 \cos^2\left(2\pi nk / N_S\right) \quad (10)$$

$$\sigma_{b_n}^2 = \frac{4}{N_S^2} \sum_{k=0}^{N_S-1} \sigma_{d_k}^2 \sin^2\left(2\pi nk / N_S\right). \quad (11)$$

Notice that (10) and (11) were obtained under the assumption of linear behavior of the underlying analog signal. Therefore, (10) and (11) provide exact results for linear waveforms such as the triangular waveform, whereas a correction factor lower than one is needed for a smooth waveform. Extensive numerical simulations have shown that for a sine wave the correction factor for (10) and (11) is $1/\sqrt{2}$. Moreover, notice that in the case of the use of a time window in (8) against spectral leakage, the variances (10)-(11) must be multiplied by the Equivalent Noise Bandwidth (ENBW) of the selected window [4]-[6].

Finally, notice that in the general case a_n and b_n are not uncorrelated RVs, therefore the magnitude $|c_n|$ cannot be treated as a RV with Rician distribution (or as a Rayleigh distribution in the special case of an only-noise frequency bin). In Section III it will be shown that the distribution of the magnitude $|c_n|$ can be approximated by a Gaussian distribution provided that its variance is estimated by taking into account the statistical correlation between a_n and b_n [13]-[15]:

$$\sigma_{|c_n|}^2 \cong \frac{\mu_{a_n}^2 \sigma_{a_n}^2 + \mu_{b_n}^2 \sigma_{b_n}^2 + 2\mu_{a_n}\mu_{b_n} cov(a_n, b_n)}{|c_n|^2} \quad (12)$$

where μ denotes the mean value, and *cov* is the covariance.

A. *Additive noise*

Additive noise can be readily included in the discrete model derived above. In fact, by considering Fig. 1, in the noise-free case time jitter can be provided with the following alternative interpretation. Instead of assigning to the actual sampling instant a random nature, i.e., an RV with standard deviation σ_δ, the sampling instant can be regarded as deterministic whereas the random nature can be assigned to the left and right edges of the relevant quantization level $m\Delta$. It means that an equivalent viewpoint for time jitter is treating the sampling instant as deterministic and the edges of each quantization level as RVs with standard deviation σ_δ. This equivalent viewpoint allows a straightforward extension of the discrete jitter model to the case of additive noise. In fact, in case of additive noise with standard deviation σ_n, each quantization level edge is characterized by a standard deviation $\sigma_n/|s|$ where s is the local slope of the waveform. It follows that in (9) the quantities z_k must be rewritten as

$$z_k = \Delta / (|s_k| \sigma \sqrt{2}) \quad (13)$$

where

$$\sigma = \sqrt{\sigma_\delta^2 + \left(\frac{\sigma_n}{s_k}\right)^2} \quad (14)$$

III. NUMERICAL VALIDATION

In this Section the main analytical results derived in Section II are validated through numerical simulations performed in MATLAB®.

Let us consider a sinusoidal waveform $x(t) = A \cdot \sin(2\pi f_0\, t + \varphi)$ with frequency $f_0 = 1$ GHz and arbitrarily selected phase $\varphi = \pi/3$. The amplitude A is normalized with respect to the quantization step Δ. Therefore, rough quantization corresponds to low amplitude A, whereas finer quantization is obtained by increasing A. Coherent sampling was performed by taking $N_S = 2^{12} = 4096$ samples within $N_p = 401$ periods of the sine wave. This choice corresponds to a sampling frequency $f_S = f_0 N_S / N_p = 10.214$ GS/s. Each sampling instant was corrupted by Gaussian zero-mean noise (i.e., random jitter) with standard deviation (STD) σ_δ. Two values were selected for σ_δ, i.e., 0.1 ps and 1 ps in order to represent typical jitter values for high-speed ADCs. Each jittered sample was quantized by rounding its value. The Hann window was used to weight the samples. The related ENBW=1.5 must be taken into account as a multiplicative factor for the variances (10) and (11).

In Fig. 2 the PDF of the imaginary part b of the DFT coefficient corresponding to the input sine wave is shown. The sine wave amplitude A was equal to 10, and the numerical PDF (dashed line) was obtained through 10^4 repeated runs by assuming jitter with STD equal to 1 ps. The continuous line corresponds to the discrete model proposed in the paper, i.e., a

Gaussian PDF with STD given by the square root of (11) (actually (11) is multiplied by ENBW of the weighting window and by the correction factor $1/\sqrt{2}$ to take into account the non-linear behavior of the sinusoidal waveform). The dotted line corresponds to the conventional continuous model. Clearly the conventional model provides an underestimate of the actual STD.

Fig. 3 shows the PDF of the magnitude $|c|$ of the DFT coefficient corresponding to the input sine wave. Numerical (dashed line) and analytical (solid line) results corresponding to (12) taking into account the covariance contribution show a good agreement. The analytical approximate result (dotted line) does not take into account the covariance.

Fig. 4 shows the behavior of the signal-to-noise ratio (SNR) as a function of the sine wave amplitude, assuming time jitter with STD equal to 1 ps. Here the SNR is defined as the ratio between the sine wave amplitude A and the STD of the magnitude $|c|$ of the DFT sine-wave coefficient, i.e., the square root of (12). The discrete model derived in the paper (solid line) is in good agreement with numerical results (dashed line). The dotted line represents the conventional continuous model which provides a significant overestimation of SNR for low-amplitude sine waves. Fig. 5 shows the same curves with time jitter with STD equal to 0.1 ps instead of 1 ps. Clearly, since the jitter STD is lower, the SNR takes higher values. Notice that the overestimate provided by the conventional model is significant even for sine wave amplitude around 60, i.e., around the full scale of an 8-bit ADC.

A second set of simulations was performed in order to validate (14) to take into account additive noise. Figures 6 and 7 show the behavior of SNR, as defined in Figures 4 and 5, as a function of the sine wave amplitude, for two different values of timing jitter (i.e., 1 ps and 0.1 ps, respectively), and for different levels of additive Gaussian noise, i.e., $\sigma_n = 0, 0.05, 0.1, 0.2$. The analytical model derived in the paper (solid lines) shows a good agreement with numerical results (dashed lines).

A third set of simulations was performed to provide further validation of (14) and to prove the capability of the proposed model to handle also waveform different from sine waves. To this aim a third harmonic has been added to the sine wave used in Figures 2 and 3 with amplitude $A_3 = A/2 = 5$. Figures 8 and 9 show the PDF of the fundamental imaginary part b_1 and magnitude $|c_1|$ in both the noise-free case and the case $\sigma_n = 0.2$. Moreover, Fig. 10 shows the behavior of the PDF of the third harmonic amplitude $|c_3|$ in both the noise-free case and the case $\sigma_n = 0.2$. The proposed model confirmed its capability to handle waveforms different from the sine wave, and to take into account the contribution of additive noise.

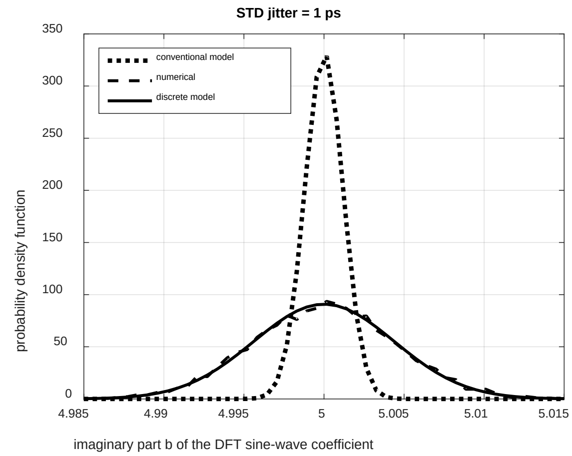

Fig. 2. Probability density function of the imaginary part b_1 of the DFT coefficient corresponding to the input sine wave having amplitude 10 and phase $\pi/3$.

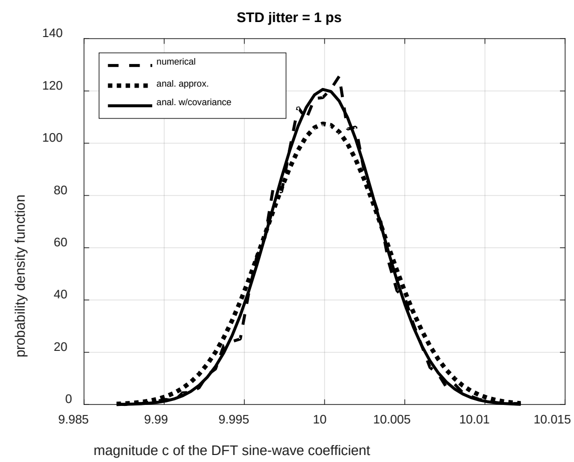

Fig. 3. Probability density function of the magnitude $|c_1|$ of the DFT coefficient corresponding to the input sine wave having amplitude 10 and phase $\pi/3$.

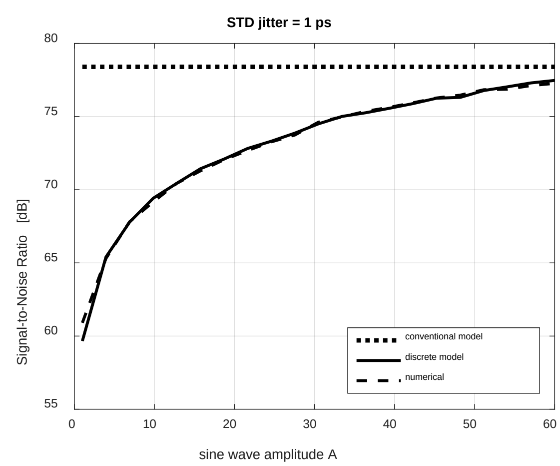

Fig. 4. Ratio between the sine wave amplitude A and the STD $\sigma_{|c_1|}$ of jitter noise within the sine-wave frequency bin as a function of A.

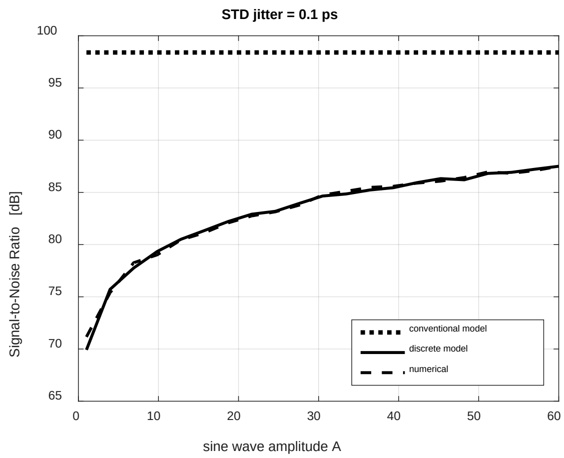

Fig. 5. Same as Fig. 4 but with jitter STD equal to 0.1 ps.

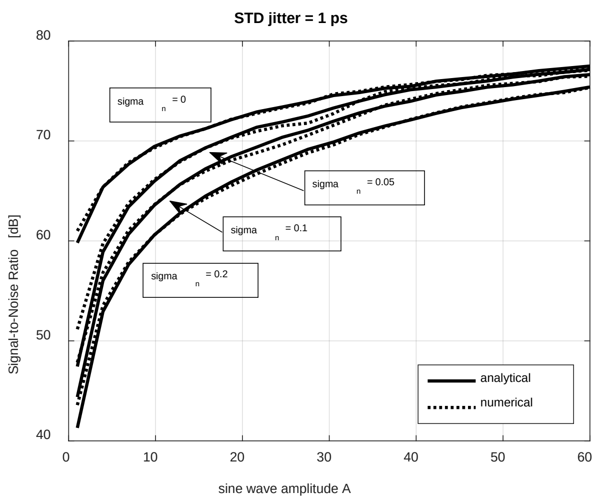

Fig. 6. Ratio between the sine wave amplitude A and the STD $\sigma_{|c_1|}$ of jitter noise within the sine-wave frequency bin as a function of A and for different levels of additive Gaussian noise.

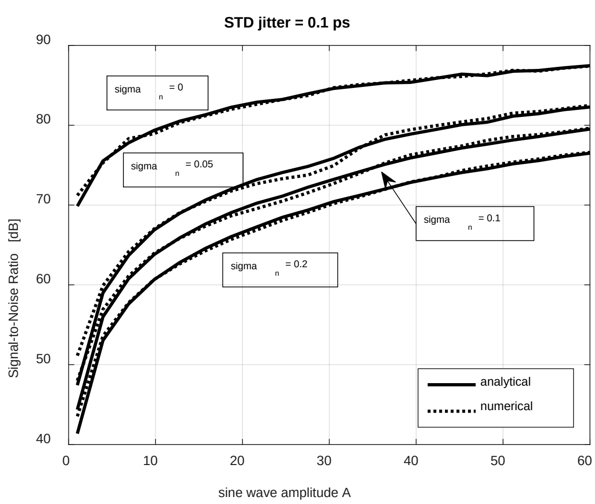

Fig. 7. Same as Fig. 6 but with jitter STD equal to 0.1 ps.

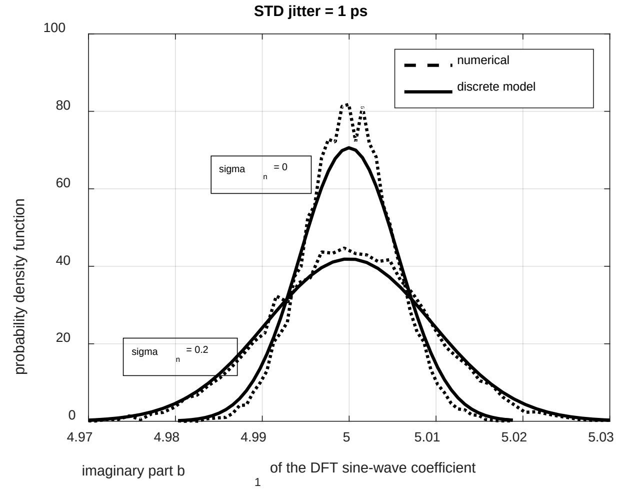

Fig. 8. Probability density function of the imaginary part b_1 of the DFT coefficient corresponding to the input waveform consisting in a sine wave and a third harmonic. The noise-free case and the case $\sigma_n = 0.2$ are reported in the figure.

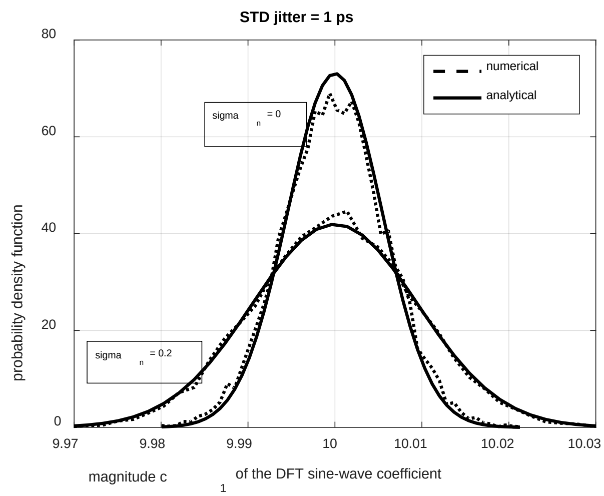

Fig. 9. Probability density function of the magnitude $|c_1|$ of the DFT coefficient corresponding to the input waveform consisting in a sine wave and a third harmonic. The noise-free case and the case $\sigma_n = 0.2$ are reported in the figure.

IV. Conclusion

The discrete model derived in the paper proved that repeated jittered measurements of a given waveform through a high-speed low-resolution ADC result in DFT coefficients whose statistical properties cannot be predicted by the classical continuous model. In particular, it was shown that the spread of the PDF of a DFT coefficient is much larger than that predicted by the classical model. The results presented in the paper are useful to predict the uncertainty due to timing jitter in A/D conversion of roughly quantized waveforms in triggered mode. In fact, it was proven that the proposed model can handle also waveforms different from sine waves. Moreover, a slight modification of the proposed discrete model allowed to extend its validity by including the contribution of additive noise.

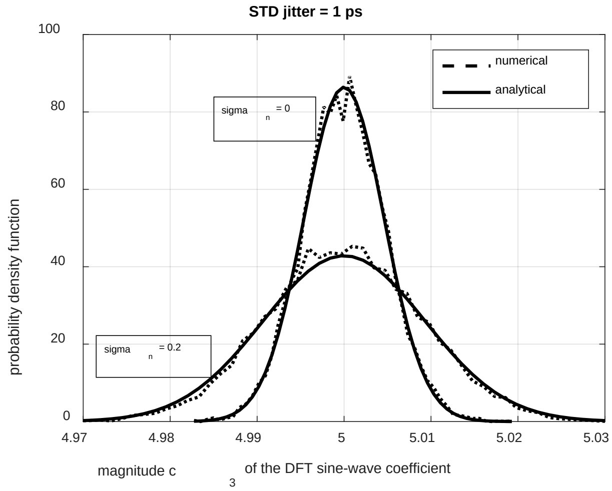

Fig. 10. Probability density function of the magnitude $|c_3|$ of the DFT coefficient corresponding to the input waveform consisting in a sine wave and a third harmonic. The noise-free case and the case $\sigma_n = 0.2$ are reported in the figure.

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