Extended critical directions for time-control constrained problems

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Abstract—We derive in this paper second order necessary conditions for certain classes of optimal control problems involving inequality and equality constraints in the time variable and the control functions. We study different normality conditions found in the literature which can be imposed on solutions to the problem and provide a new approach, partly based on the theory for constrained problems in finite dimensional spaces, which under mild assumptions allows us to enlarge the usual set of critical directions.

Keywords—Optimal control, second order necessary conditions, extremals, normality.

I. INTRODUCTION

This paper deals with the derivation of second order necessary conditions for certain classes of Lagrange optimal control problems posed over piecewise C^1 trajectories and piecewise continuous controls. For these problems, the integrand of the cost function is independent of the state variable and the constraints are expressed in terms of the dynamics, fixed endpoint conditions, and inequalities and equalities depending on the time variable and the control functions. The latter are of the form

$$\varphi_\alpha(t, u(t)) \leq 0 \quad \text{and} \quad \varphi_\beta(t, u(t)) = 0 \quad (t \in T)$$

with α, β in finite sets of indices and T a compact time interval.

To illustrate the kind of problems we shall deal with let us give a simple example which can be found in the literature of optimal control processes (see [19]). Suppose that a landing vehicle separates from a spacecraft with initial velocity v, at time t_0 and altitude h from some surface. Not taking into account gravitational forces and assuming the mass of the vehicle is constant, consider vertical motion only, with upwards regarded as the positive direction. If $x_1(t)$ denotes altitude, $x_2(t)$ velocity, and $u(t)$ the thrust exerted by the rocket motor subject to $|u(t) + e(t)| \leq 1$ with suitable scaling and $e(t)$ some error at time t, we have the equations of motion

$$(\dot{x}_1(t), \dot{x}_2(t)) = (x_2(t), u(t))$$

and the initial conditions $(x_1(t_0), x_2(t_0)) = (h, -v)$. For a soft landing at time t_1 we require also that $(x_1(t_1), x_2(t_1)) = (0, 0)$. We might then be interested in minimizing

$$I(x, u) = \int_{t_0}^{t_1} (|u(t)| + k)dt$$

which represents a sum of the total fuel consumption and time to landing, k being a factor which weights the relative importance of these two quantities.

First order conditions for such problems are well established in the literature (see, for example, [7, 9, 10, 16] and, more recently, [4] where mixed constraints with nonsmooth data are studied), providing a natural definition of extremals. This notion will be our starting point. Our aim is to derive in a simple way second order conditions which those extremals, if solving the problem, should satisfy. The main idea relies on linking up the properties that characterize the extremals with any solution to the underlying problem of minimizing the same functional over the set of equality and inequality constraints without dynamics or endpoint conditions.

Second order necessary conditions in terms of the accessory problem can be found in [9, 22–24] for the problem posed over controls u in $L^\infty(T, \mathbf{R}^m)$. However, for the problem we shall deal with, the control functions are piecewise continuous and so do not form a Banach space. Thus, the technique used for L^∞ controls where results from abstract optimization theory on Banach spaces are applied to the optimal control

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problem, does not work for our problem. Let us also point out that it provides a natural setting for results related to, in particular, "broken extremals" as defined and studied in [3, 18].

There is an extensive literature on second order conditions for optimal control problems and how the theory can be applied to practical problems (see, for example, [1–7, 9, 10, 12–18, 20–24] and references therein) but some fundamental questions remain unanswered. In particular, when dealing with such conditions, one usually faces two main features: (1) the normality assumptions imposed on the solution to the problem which imply, in particular, a positive cost multiplier, and (2) the set of critical directions where, under those normality assumptions, the second order conditions hold.

As we shall see, a weak notion of normality usually imposed for first order conditions is not enough to ensure the validity of the second order necessary conditions, and a stronger notion is needed. However, once this stronger assumption is imposed, the set of critical directions may be too restrictive. One is then, of course, interested in enlarging that set and, if possible, in weakening the normality assumptions. In this paper we show how, for the problem we shall deal with, these two aspects can be achieved. The results obtained here correspond to a generalization of second order conditions first derived in [21] (and with more detail in [5]) for certain classes of optimal control problems.

Let us point out that, in some of the references mentioned above, the approach followed to derive second order conditions is based precisely on that stronger notion of normality by taking into account only equality constraints for active indices and imposing normality assumptions with respect to the corresponding set of tangential constraints. The conditions one encounters in those cases will be said to be of a "weak type" since, as we shall see, the assumptions and the critical directions can be modified and, in the two senses mentioned above, improved.

This paper is organized as follows. In Section 2 we pose the problem we shall deal with together with a fundamental assumption on the full rank of the matrix of partial derivatives of the constraints with respect to the control functions. In Section 3 we state well-known first order necessary conditions and introduce the notions of extremal and normality relative to the set S, where the functional of the problem is to be minimized. Weak normality is defined in terms precisely of that set, while strong normality corresponds to applying that definition to the set S_0 defined by equality constraints for active indices. We then state second order necessary conditions found in the literature which hold, for strongly normal extremals, on the corresponding set of tangential constraints with respect to S_0.

We provide an example to illustrate that the nonnegativity of the quadratic form appearing in this result may not hold if the assumption of strong normality is replaced with that of weak normality. An important characterization of normality, applicable to any subset of S, is given in Section 4 and used in Section 5 where, taking into account the sign of the Lagrange multipliers, a new set of second order conditions is obtained by making use of the theory for constrained problems in finite dimensional spaces (see [8, 11]). A natural conjecture appearing in this result is then solved through an example.

II. Statement of the problem

Suppose we are given an interval $T := [t_0, t_1]$ in $\mathbf{R}$, two points ξ_0, ξ_1 in $\mathbf{R}^n$, and functions L and $\varphi = (\varphi_1, \dots, \varphi_q)$ mapping $T \times \mathbf{R}^m$ to $\mathbf{R}$ and $\mathbf{R}^q$ ($q \leq m$) respectively, and f mapping $T \times \mathbf{R}^n \times \mathbf{R}^m$ to $\mathbf{R}^n$.

Denote by X the space of piecewise C^1 functions mapping T to $\mathbf{R}^n$, by $\mathcal{U}_k$ the space of piecewise continuous functions mapping T to $\mathbf{R}^k$ ($k \in \mathbf{N}$), set $Z := X \times \mathcal{U}_m$,

$$D := \{(x,u) \in Z \mid \dot{x}(t) = f(t, x(t), u(t)) \; (t \in T),$$

$$x(t_0) = \xi_0, \; x(t_1) = \xi_1\},$$

$$S := \{(x,u) \in D \mid \varphi_\alpha(t, u(t)) \leq 0,$$

$$\varphi_\beta(t, u(t)) = 0 \; (\alpha \in R, \; \beta \in Q, \; t \in T)\}$$

where $R = \{1, \dots, r\}$, $Q = \{r+1, \dots, q\}$, and consider the functional $I \colon Z \to \mathbf{R}$ given by

$$I(x,u) := \int_{t_0}^{t_1} L(t, u(t))dt \quad ((x,u) \in Z).$$

The problem we shall deal with, which we label (P), is that of minimizing I over S. Note that the functional I is independent of x but we use the notation $I(x,u)$ to emphasize the fact that we shall be concerned with elements of $Z = X \times \mathcal{U}_m$ satisfying the dynamics and endpoint constraints defining membership of D. In other words, the problem is posed over those $u \in \mathcal{U}_m$ satisfying the inequality and equality constraints given in S and for which, if x is the unique solution of the differential equation $\dot{x}(t) = f(t, x(t), u(t))$ ($t \in T$) together with the initial condition $x(t_0) = \xi_0$, then $x(t_1) = \xi_1$.

A common and concise way of formulating this problem is as follows:

Minimize

$$I(x,u) = \int_{t_0}^{t_1} L(t,u(t))dt$$

subject to $(x,u) \in Z$ and

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t)) \ (t \in T); \\ x(t_0) = \xi_0, \ x(t_1) = \xi_1; \\ \varphi_\alpha(t,u(t)) \le 0 \ (\alpha \in R, \ t \in T), \\ \qquad \varphi_\beta(t,u(t)) = 0 \ (\beta \in Q, \ t \in T). \end{cases}$$

Elements of Z will be called *processes* and of S *admissible processes*. We shall say that a process (x,u) *solves* (P) if (x,u) is admissible and $I(x,u) \le I(y,v)$ for all admissible process (y,v).

Given $(x,u) \in Z$ we shall find convenient to use the notation $(\tilde{x}(t))$ to represent $(t,x(t),u(t))$, and '*' will be used to denote transpose.

We assume that L, f and φ are C^2 and the $q \times (m + r)$-dimensional matrix

$$\left(\frac{\partial \varphi_i}{\partial u^k} \ \ \delta_{i\alpha}\varphi_\alpha \right)$$

$(i = 1,\ldots,q; \ \alpha = 1,\ldots,r; \ k = 1,\ldots,m)$ has rank q on $\mathcal{A}$ (here $\delta_{\alpha\alpha} = 1$, $\delta_{\alpha\beta} = 0$ $(\alpha \ne \beta)$), where

$$\mathcal{A} := \{(t,u) \in T \times \mathbf{R}^m \mid \varphi_\alpha(t,u) \le 0 \ (\alpha \in R),$$

$$\varphi_\beta(t,u) = 0 \ (\beta \in Q)\}.$$

This condition (see [7, 10] for details) is equivalent to the condition that, at each point (t,u) in $\mathcal{A}$, the matrix

$$\left(\frac{\partial \varphi_i}{\partial u^k} \right)$$

$(i = i_1,\ldots,i_p; \ k = 1,\ldots,m)$ has rank p, where $i_1,\ldots,i_p$ are the indices $i \in \{1,\ldots,q\}$ such that $\varphi_i(t,u) = 0$.

III. Necessary conditions and normality

Usually first order conditions for this problem are established in terms of the Hamiltonian function (see, for example, [4, 7, 9, 10, 16]), and one version can be written as follows.

For all (t,x,u,p,μ,λ) in $T \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^q \times \mathbf{R}$ let

$$H(t,x,u,p,\mu,\lambda) := \langle p, f(t,x,u)\rangle -$$

$$\lambda L(t,u) - \langle \mu, \varphi(t,u)\rangle.$$

3.1 Theorem. *Suppose* (x_0,u_0) *solves* (P). *Then there exist* $\lambda_0 \ge 0$, $p \in X$, *and* $\mu \in \mathcal{U}_q$, *not vanishing simultaneously on* T, *such that*

a. $\mu_\alpha(t) \ge 0$ *and* $\mu_\alpha(t)\varphi_\alpha(t,u_0(t)) = 0$ *for all* $\alpha \in R$ *and* $t \in T$;

b. $\dot{p}(t) = -H_x^*(\tilde{x}_0(t),p(t),\mu(t),\lambda_0)$ *and*

$$H_u(\tilde{x}_0(t),p(t),\mu(t),\lambda_0) = 0$$

on every interval of continuity of u_0.

This result assures the existence of multipliers (p,μ,λ_0) associated to a solution to the problem. Let us denote by $M(x,u)$ the set of such multipliers and define a set $\mathcal{E}$ of "extremals" as the set of all (x,u,p,μ) which have associated a nonzero cost multiplier normalized to one.

3.2 Definition. For all $(x,u) \in Z$ let $M(x,u)$ be the set of all $(p,\mu,\lambda_0) \in X \times \mathcal{U}_q \times \mathbf{R}$ with $\lambda_0 + |p| \ne 0$ satisfying

a. $\mu_\alpha(t) \ge 0$ and $\mu_\alpha(t)\varphi_\alpha(t,u(t)) = 0$ for all $\alpha \in R$ and $t \in T$;

b. $\dot{p}(t) = -H_x^*(\tilde{x}(t),p(t),\mu(t),\lambda_0)$ and

$$H_u(\tilde{x}(t),p(t),\mu(t),\lambda_0) = 0 \ (t \in T).$$

Denote by $\mathcal{E}$ the set of all $(x,u,p,\mu) \in Z \times X \times \mathcal{U}_q$ such that $(p,\mu,1) \in M(x,u)$, that is,

a. $\mu_\alpha(t) \ge 0$ and $\mu_\alpha(t)\varphi_\alpha(t,u(t)) = 0$ for all $\alpha \in R$ and $t \in T$;

b. $\dot{p}(t) = -f_x^*(\tilde{x}(t))p(t)$ and $f_u^*(\tilde{x}(t))p(t) = L_u^*(\tilde{x}(t)) + \varphi_u^*(t,u(t))\mu(t)$ $(t \in T)$.

The notion of "normality" is introduced so that the non-vanishing of the cost multiplier can be assured. This is accomplished by having zero as the unique solution to the adjoint equation whenever $\lambda_0 = 0$.

3.3 Definition. A process $(x,u) \in S$ will be said to be *normal relative to* S if, given $p \in X$ and $\mu \in \mathcal{U}_q$ satisfying

i. $\mu_\alpha(t) \ge 0$ and $\mu_\alpha(t)\varphi_\alpha(t,u(t)) = 0$ for all $\alpha \in R$ and $t \in T$;

ii. $\dot{p}(t) = -f_x^*(\tilde{x}(t))p(t)$

$$[\, = -H_x^*(\tilde{x}(t),p(t),\mu(t),0)\,] \ (t \in T);$$

iii. $0 = f_u^*(\tilde{x}(t))p(t) - \varphi_u^*(t,u(t))\mu(t)$

$$[\, = H_u^*(\tilde{x}(t),p(t),\mu(t),0)\,] \ (t \in T),$$

then $p \equiv 0$. In this event, clearly, also $\mu \equiv 0$.

From Theorem 3.1 and the above definitions it follows that, if (x_0, u_0) solves (P) and is normal relative to S, then there exists $(p, \mu) \in X \times \mathcal{U}_q$ such that $(x_0, u_0, p, \mu) \in \mathcal{E}$. Note also that uniqueness of the pair (p, μ) cannot be assured and, for that purpose, a stronger notion of normality is required or, as we shall see below, the definition of normality given before but applied to a set S_0 involving only equality constraints.

Denote the set of *active indices at* $(t, u) \in T \times \mathbf{R}^m$ by

$$I_a(t, u) := \{\alpha \in R \mid \varphi_\alpha(t, u) = 0\}$$

and, given $(x_0, u_0) \in S$, consider the set

$$S_0 := \{(x, u) \in D \mid \varphi_\gamma(t, u(t)) = 0$$

$$(\gamma \in I_a(t, u_0(t)) \cup Q,\ t \in T)\}.$$

Note that (x_0, u_0) is normal relative to S_0 if, given $(p, \mu) \in X \times \mathcal{U}_q$ satisfying

i. $\mu_\alpha(t)\varphi_\alpha(t, u_0(t)) = 0$ $(\alpha \in R,\ t \in T)$;

ii. $\dot{p}(t) = -f_x^*(\tilde{x}_0(t))p(t)$ and $f_u^*(\tilde{x}_0(t))p(t) = \varphi_u^*(t, u_0(t))\mu(t)$ $(t \in T)$,

then $p \equiv 0$.

To be consistent with other references (see, for example, [5–7, 20, 21]), we shall refer to normality relative to S and S_0 as "weak" and "strong normality" respectively. The following proposition is crucial. It is a simple consequence of Theorem 3.1 and the definitions given above (see also [20]).

3.4 Proposition. *If* (x_0, u_0) *solves* (P) *then* $M(x_0, u_0) \neq \emptyset$. *If also* (x_0, u_0) *is strongly normal then there exists a unique* $(p, \mu) \in X \times \mathcal{U}_q$ *such that* $(x_0, u_0, p, \mu) \in \mathcal{E}$.

Proof: Suppose (x_0, u_0) solves (P). Theorem 3.1 states precisely that $M(x_0, u_0)$ is not empty. Let $(p, \mu, \lambda_0) \in M(x_0, u_0)$. If also (x_0, u_0) is strongly normal, clearly we have $\lambda_0 \neq 0$ and, if $(q, \nu, \lambda_0) \in M(x_0, u_0)$, then

i. For all $\alpha \in R$ and $t \in T$,

$$[\mu_\alpha(t) - \nu_\alpha(t)]\varphi_\alpha(t, u_0(t)) = 0;$$

ii. For all $t \in T$,

$$[\dot{p}(t) - \dot{q}(t)] = -f_x^*(\tilde{x}_0(t))[p(t) - q(t)];$$

iii. For all $t \in T$,

$$f_u^*(\tilde{x}_0(t))[p(t) - q(t)] -$$

$$\varphi_u^*(t, u_0(t))[\mu(t) - \nu(t)] = 0,$$

implying that $p \equiv q$ and $\mu \equiv \nu$. The result follows by choosing $\lambda_0 = 1$ since $(p/\lambda_0, \mu/\lambda_0, 1)$ belongs to $M(x_0, u_0)$. ∎

For second order conditions let us consider, for all (x, u, p, μ) in $Z \times X \times \mathcal{U}_q$ and (y, v) in Z, the quadratic form defined as

$$J((x, u, p, \mu); (y, v)) := \int_{t_0}^{t_1} 2\Omega(t, y(t), v(t))dt$$

where, for all $(t, y, v) \in T \times \mathbf{R}^n \times \mathbf{R}^m$,

$$2\Omega(t, y, v) := -[\langle y, H_{xx}(t)y\rangle +$$

$$2\langle y, H_{xu}(t)v\rangle + \langle v, H_{uu}(t)v\rangle]$$

and $H(t)$ denotes $H(\tilde{x}(t), p(t), \mu(t), 1)$.

As mentioned in the introduction, a set of weak second order conditions for problem (P) can be found in the literature. In particular, the following result was derived in [7] by reducing the original problem into a problem involving only equality constraints in the control. In what follows, the notation $\varphi_{\gamma u}(t, u_0(t))$ is short for $\dfrac{\partial \varphi_\gamma}{\partial u}(t, u_0(t))$.

3.5 Theorem. *Let* (x_0, u_0) *be an admissible process for which there exists* $(p, \mu) \in X \times \mathcal{U}_q$ *such that* $(x_0, u_0, p, \mu) \in \mathcal{E}$. *If* (x_0, u_0) *is a strongly normal solution to* (P) *then*

$$J((x_0, u_0, p, \mu); (y, v)) \geq 0$$

for all $(y, v) \in Z$ *satisfying*

i. $\dot{y}(t) = f_x(\tilde{x}_0(t))y(t) + f_u(\tilde{x}_0(t))v(t)$ $(t \in T)$, *and* $y(t_0) = y(t_1) = 0$;

ii. $\varphi_{\gamma u}(t, u_0(t))v(t) = 0$ $(\gamma \in I_a(t, u_0(t)) \cup Q,$ $t \in T)$.

It is of interest to see if, in Theorem 3.5, the assumption of strong normality can be weakened and the set of critical directions where the second order condition holds can be enlarged. Note however that, as the following example illustrates, the conclusion of Theorem 3.5 may not hold if we assume weak instead of strong normality.

3.6 Example. Consider the problem of minimizing

$$I(x, u) = \int_0^1 \{u_2(t) - u_1(t)\}dt$$

subject to $(x,u) \in Z$ and

$$\begin{cases} \dot{x}(t) = 2u_2(t) - (t+2)u_1(t) + tu_3^2(t) \ (t \in [0,1]); \\ x(0) = x(1) = 0; \\ u_2(t) \geq u_1(t), \ (t+1)u_1(t) \geq tu_2(t) \ (t \in [0,1]). \end{cases}$$

In this case $T = [0,1]$, $n = 1$, $m = 3$, $r = q = 2$, $\xi_0 = \xi_1 = 0$ and, for all $t \in T$, $x \in \mathbf{R}$, and $u = (u_1, u_2, u_3)$,

$$L(t,u) = u_2 - u_1, \quad f(t,x,u) = 2u_2 - (t+2)u_1 + tu_3^2,$$

$$\varphi_1(t,u) = u_1 - u_2, \quad \varphi_2(t,u) = tu_2 - (t+1)u_1.$$

Note that the matrix

$$\left(\frac{\partial \varphi_i}{\partial u^k}\right) = \begin{pmatrix} 1 & -1 & 0 \\ -(t+1) & t & 0 \end{pmatrix}$$

$(i = 1,2;\ k = 1,2,3)$ has rank 2 for all $t \in T$. We have

$$H(t,x,u,p,\mu,1) =$$
$$p(2u_2 - (t+2)u_1 + tu_3^2) + u_1 - u_2 -$$
$$\mu_1(u_1 - u_2) - \mu_2(tu_2 - (t+1)u_1)$$

and so

$$H_u(t,x,u,p,\mu,1) =$$
$$(-p(t+2) - \mu_1 + (t+1)\mu_2 + 1,$$
$$2p + \mu_1 - t\mu_2 - 1, 2tpu_3)$$

and

$$H_{uu}(t,x,u,p,\mu,1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2tp \end{pmatrix}$$

so that

$$J((x,u,p,\mu);(y,v)) = -\int_0^1 2tp(t)v_3^2(t)dt$$

for all $(x,u,p,\mu) \in Z \times X \times \mathcal{U}_2$ and $(y,v) \in Z$.

Clearly $(x_0,u_0) \equiv (0,0)$ is a solution to the problem. To test for normality, suppose $(p,\mu) \in X \times \mathcal{U}_2$ is such that

$$\dot{p}(t) = -f_x^*(\tilde{x}_0(t))p(t) = 0$$

and

$$f_u^*(\tilde{x}_0(t))p(t) = \varphi_u^*(t,u_0(t))\mu(t) \quad (t \in T),$$

that is, p is constant and, for all $t \in T$,

$$\begin{pmatrix} -(t+2) \\ 2 \\ 0 \end{pmatrix} p = \begin{pmatrix} \mu_1(t) - (t+1)\mu_2(t) \\ -\mu_1(t) + t\mu_2(t) \\ 0 \end{pmatrix}.$$

As one readily verifies, this implies that $tp = \mu_2(t)$ $(t \in T)$ and so

$$\mu_1(t) = t\mu_2(t) - 2p = (t^2 - 2)p.$$

If there are no restrictions on the sign of the multipliers, then $p \equiv 1$, $\mu_1(t) = t^2 - 2$ and $\mu_2(t) = t$ $(t \in T)$ solve the system, implying that (x_0,u_0) is not strongly normal. However, if we require that $\mu_\alpha(t) \geq 0$ $(\alpha = 1,2)$, then $\mu_1(1) = -p \geq 0$ and $\mu_2(1) = p \geq 0$ imply that $(p,\mu) \equiv (0,0)$ is the only solution and so (x_0,u_0) is weakly normal.

Now, note that (x_0,u_0,p,μ) is an extremal if $\mu_\alpha(t) \geq 0$ $(\alpha = 1,2)$, $\dot{p}(t) = 0$ and, for all $t \in T$,

$$f_u^*(\tilde{x}_0(t))p(t) = L_u^*(\tilde{x}_0(t)) + \varphi_u^*(t,u_0(t))\mu(t).$$

This last relation corresponds to

$$\begin{pmatrix} -(t+2) \\ 2 \\ 0 \end{pmatrix} p(t) =$$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu_1(t) - (t+1)\mu_2(t) \\ -\mu_1(t) + t\mu_2(t) \\ 0 \end{pmatrix}$$

and thus, if $p \equiv 1/2$, $\mu_1(t) = t^2/2$ and $\mu_2(t) = t/2$, then $(x_0,u_0,p,\mu) \in \mathcal{E}$. Let $v \equiv (0,0,1)$ and $y \equiv 0$. Then (y,v) solves (i) and (ii) of Theorem 3.5 since $y(0) = y(1) = 0$,

$$\dot{y}(t) = -(t+2)v_1(t) + 2v_2(t) \ (t \in T)$$

and $\varphi_{\gamma u}(t,u_0(t))v(t) = 0$ $(\gamma \in \{1,2\},\ t \in T)$. However,

$$J((x_0,u_0,p,\mu);(y,v)) = -\int_0^1 tdt = -1/2 < 0. \blacksquare$$

IV. A CHARACTERIZATION OF NORMALITY

The purpose of this section is to characterize the notion of normality given in Definition 3.3. The notion we now introduce is that of τ-regularity, based on a cone $\tau(t,u)$ in $\mathbf{R}^m$ defined for all (t,u) in $T \times \mathbf{R}^m$ and constructed according to how the set S of inequality and equality constraints is itself defined. This notion will then be applied to other subsets of D also defined by inequality and/or equality constraints. As we shall see this will allow us to verify, in a simple way, membership to different sets of normal processes.

Let us point out that a similar characterization was first established in [20] for problems where the constraints depend only on the control functions and are independent of the time variable. In that reference, each statement relating regularity with normality is proved as a single property while all those statements are unified (and generalized) in this section. For comparison reasons let us also mention that the following set (first introduced for constraints that depend only on the controls) corresponds to $\tau_2(u)$ in [20].

For any $(t,u) \in T \times \mathbf{R}^m$ define

$$\tau(t,u) := \{h \in \mathbf{R}^m \mid \varphi_{\alpha u}(t,u)h \leq 0 \ (\alpha \in I_a(t,u)),$$

$$\varphi_{\beta u}(t,u)h = 0 \ (\beta \in Q)\}.$$

4.1 Definition. Let $(x,u) \in Z$ and set

$$A(t) := f_x(\tilde{x}(t)),\ B(t) := f_u(\tilde{x}(t)) \quad (t \in T).$$

We shall say that (x,u) is τ-*regular* if there is no non-null solution $z \in X$ to the system

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h \leq 0 \quad \text{for all } h \in \tau(t,u(t)) \quad (t \in T).$$

Let us prove that the notions of weak normality (normality relative to S) and τ-regularity are equivalent.

4.2 Theorem. *For any $(x,u) \in S$ the following are equivalent:*

a. *(x,u) is τ-regular.*

b. *(x,u) is weakly normal.*

Proof:

(a) $\Rightarrow$ (b): Suppose $(p,\mu) \in X \times \mathcal{U}_q$ is such that

i. For all $\alpha \in R$ and $t \in T$, $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t)\varphi_\alpha(t,u(t)) = 0$;

ii. For all $t \in T$, we have $\dot{p}(t) = -A^*(t)p(t)$ and $B^*(t)p(t) = \varphi_u^*(t,u(t))\mu(t)$.

Let $t \in T$ and $h \in \tau(t,u(t))$. Then we have

$$\begin{aligned} p^*(t)B(t)h &= \mu^*(t)\varphi_u(t,u(t))h \\ &= \sum_{\gamma=1}^{q} \mu_\gamma(t)\varphi_{\gamma u}(t,u(t))h \leq 0 \end{aligned}$$

implying, by (a), that $p \equiv 0$.

(b) $\Rightarrow$ (a): Let $z \in X$ be such that

$$\dot{z}(t) = -A^*(t)z(t),$$

$$z^*(t)B(t)h \leq 0 \quad \text{for all } h \in \tau(t,u(t)) \quad (t \in T).$$

Fix $t \in T$ and let $i_1 < \cdots < i_p$ be the indices in $\{1,\ldots,q\}$ such that $\varphi_i(t,u(t)) = 0$. Define

$$\hat{\varphi} = (\varphi_{i_1},\ldots,\varphi_{i_p}) \text{ and } \hat{\mu}(t) = (\hat{\mu}_{i_1}(t),\ldots,\hat{\mu}_{i_p}(t))$$

where

$$\hat{\mu}(t) := \Lambda^{-1}(t)\hat{\varphi}_u(t,u(t))B^*(t)z(t) \quad (t \in T)$$

and $\Lambda(t) = \hat{\varphi}_u(t,u(t))\hat{\varphi}_u^*(t,u(t))$. Note that, since

$$\hat{\varphi}_u(t,u(t))\hat{\varphi}_u^*(t,u(t))\Lambda^{-1}(t) =$$

$$\Lambda^{-1}(t)^*\hat{\varphi}_u(t,u(t))\hat{\varphi}_u^*(t,u(t)) = I_{p\times p}$$

we have $\Lambda^{-1}(t) = \Lambda^{-1}(t)^*$.

Let $\mu(t) = (\mu_1(t),\ldots,\mu_q(t))$ where

$$\mu_\alpha(t) := \begin{cases} \hat{\mu}_{i_r}(t) & \text{if } \alpha = i_r,\ r = 1,\ldots,p \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

$$\mu_\alpha(t)\varphi_\alpha(t,u(t)) = 0 \quad (\alpha \in R,\ t \in T)$$

and, for all $t \in T$,

$$\hat{\mu}^*(t)\hat{\varphi}_u(t,u(t)) = \mu^*(t)\varphi_u(t,u(t)).$$

Now, let

$$G(t) := I_{m\times m} - \hat{\varphi}_u^*(t,u(t))\Lambda^{-1}(t)\hat{\varphi}_u(t,u(t))$$

and note that $\hat{\varphi}_u(t,u(t))G(t) = 0$. If $h_k(t)$ denotes the k-th column of $G(t)$ for $k = 1,\ldots,m$, we have

$$\frac{\partial \varphi_{i_j}}{\partial u}(t,u(t))h_k(t) = 0$$

($j = 1,\ldots,p,\ k = 1,\ldots,m$), that is, $h_k(t) \in \tau(t,u(t))$, and therefore

$$z^*(t)B(t)h_k(t) \leq 0 \quad (k = 1,\ldots,m).$$

Since also $-h_k(t) \in \tau(t,u(t))$, we have

$$z^*(t)B(t)h_k(t) = 0 \quad (k = 1,\ldots,m).$$

Thus we have

$$\begin{aligned} 0 &= z^*(t)B(t)G(t) \\ &= z^*(t)B(t) - \mu^*(t)\varphi_u(t,u(t)). \end{aligned}$$

The result will follow, by weak normality, if

$$\mu_\alpha(t) \geq 0 \quad (\alpha \in R,\ t \in T).$$

To prove it, let $C(t)$ be the $p \times m$ matrix

$$C(t) := \Lambda^{-1}(t)\hat{\varphi}_u(t, u(t))$$

and observe that

$$C(t)\hat{\varphi}_u^*(t, u(t)) = I_{p\times p} = \hat{\varphi}_u(t, u(t))C^*(t).$$

Therefore, if $c_j(t)$ denotes the j-th column of $C^*(t)$ and $\{e_j\}$ the canonical base in $\mathbf{R}^p$ for $j = 1, \ldots, p$, then

$$\left(\frac{\partial \varphi_{i_j}}{\partial u}(t, u(t))c_1(t), \ldots, \frac{\partial \varphi_{i_j}}{\partial u}(t, u(t))c_p(t)\right) = e_j^*$$

$(j = 1, \ldots, p)$. Thus, if $j \in \{1, \ldots, p\}$ is such that $i_j \in I_a(t, u(t))$, then

$$\frac{\partial \varphi_k}{\partial u}(t, u(t))(-c_j(t)) = \begin{cases} -1 & \text{if } k = i_j \\ 0 & \text{if } k \neq i_j \end{cases}$$

implying that $-c_j(t) \in \tau(t, u(t))$ and therefore

$$z^*(t)B(t)c_j(t) \geq 0 \quad (t \in T).$$

But $\hat{\mu}^*(t) = z^*(t)B(t)C^*(t)$ and so $\hat{\mu}_\alpha^*(t) \geq 0$ for all $\alpha \in I_a(t, u(t))$. This proves the claim. ▮

The set $\tau(t, u)$ defined above is a cone of "critical directions" associated with S. In a similar way, for any $(t, u) \in T \times \mathbf{R}^m$, let

$$\tau_0(t, u) := \{h \in \mathbf{R}^m \mid \varphi_{\gamma u}(t, u)h = 0$$
$$(\gamma \in I_a(t, u) \cup Q)\}.$$

By Definition 4.1, $(x, u) \in Z$ is τ_0-regular if there is no nonnull solution $z \in X$ to the system

$$\dot{z}(t) = -A^*(t)z(t),$$
$$z^*(t)B(t)h = 0 \quad \text{for all } h \in \tau_0(t, u(t)) \quad (t \in T).$$

Recall that the notion of strong normality, or normality relative to the set S_0 defined as

$$S_0 = \{(x, u) \in D \mid \varphi_\gamma(t, u(t)) = 0$$
$$(\gamma \in I_a(t, u_0(t)) \cup Q,\ t \in T)\}.$$

depends on a given admissible process (x_0, u_0). Note that, in view of Theorem 4.2, (x_0, u_0) is τ_0-regular $\Leftrightarrow$ (x_0, u_0) is strongly normal.

Suppose now that we are given $\mu \in \mathcal{U}_q$ with $\mu_\alpha(t) \geq 0$ $(\alpha \in R,\ t \in T)$, and we define

$$S_1 = \{(x, u) \in D \mid \ \varphi_\alpha(t, u(t)) \leq 0$$
$$(\alpha \in R,\ \mu_\alpha(t) = 0,\ t \in T),$$
$$\varphi_\beta(t, u(t)) = 0$$
$$(\beta \in R \text{ with } \mu_\beta(t) > 0, \text{ or } \beta \in Q,\ t \in T)\}.$$

Associate with this set, for all $(t, u) \in T \times \mathbf{R}^m$ and $\mu \in \mathbf{R}^q$, the cone of directions

$$\tau_1(t, u, \mu) := \{h \in \mathbf{R}^m \mid \ \varphi_{\alpha u}(t, u)h \leq 0$$
$$(\alpha \in I_a(t, u),\ \mu_\alpha = 0),$$
$$\varphi_{\beta u}(t, u)h = 0$$
$$(\beta \in R \text{ with } \mu_\beta > 0, \text{ or } \beta \in Q)\}.$$

Again following Definition 4.1, if there is no nonnull solution $z \in X$ to the system

$$\dot{z}(t) = -A^*(t)z(t),$$
$$z^*(t)B(t)h \leq 0 \quad \text{for all } h \in \tau_1(t, u(t), \mu(t))\ (t \in T)$$

then (x, u) is called τ_1-*regular*.

Note that, by Theorem 4.2, if $\mu \in \mathcal{U}_q$ is such that $\mu_\alpha(t) \geq 0$ $(\alpha \in R,\ t \in T)$ and $(x, u) \in S_1$, then (x, u) is τ_1-regular $\Leftrightarrow$ (x, u) is normal relative to S_1. Also, as one readily verifies, if (x_0, u_0) is normal relative to S_0 (strongly normal) then it is normal relative to S_1 (with μ as above) which in turn implies normality relative to S (weakly normal).

V. Second order conditions

In this section we shall derive second order necessary conditions for problem (P). Our main result will be proved with the help of an auxiliary result established in [5] and partially based on the theory presented in [11]. It is a consequence of the full rank assumption mentioned in Section 2.

Let us consider the problem, which we label (C), of minimizing $\int_{t_0}^{t_1} L(t, u(t))dt$ on the set

$$\mathcal{C} := \{u \in \mathcal{U}_m \mid (t, u(t)) \in \mathcal{A}\ (t \in T)\}.$$

5.1 Lemma. *Let $u_0 \in \mathcal{C}$. Then u_0 solves* (C) $\Leftrightarrow$ *$L(t, u) \geq L(t, u_0(t))$ $(t \in T)$ whenever $(t, u) \in \mathcal{A}$. In this event, there exists a unique $\mu \in \mathcal{U}_q$ such that $F_u(t, u_0(t), \mu(t)) = 0$ $(t \in T)$ where*

$$F(t, u, \mu) := L(t, u) + \langle \mu, \varphi(t, u) \rangle.$$

Moreover, $\mu_\alpha(t) \geq 0$ and $\mu_\alpha(t)\varphi_\alpha(t, u_0(t)) = 0$ $(\alpha \in R,\ t \in T)$, and

$$\langle h, F_{uu}(t, u_0(t), \mu(t))h \rangle \geq 0$$

for all $h \in \tau_1(t, u_0(t), \mu(t))$ $(t \in T)$.

Note that, if (x_0, u_0) is admissible for (P) and u_0 solves (C), then (x_0, u_0) solves (P). However, we may have a solution (x_0, u_0) to the problem (P) but u_0 does not solve (C). In fact, it may even afford a maximum to I on $\mathcal{C}$. A simple example illustrates this fact.

5.2 Example. Consider the problem of minimizing

$$I(x, u) = \int_0^1 u(t)dt$$

subject to $(x, u) \in Z$ and

$$\dot{x}(t) = u^2(t),\ x(0) = x(1) = 0,\ u(t) \leq 0.$$

Then $(x_0, u_0) \equiv (0, 0)$ is a solution to (P), being the only admissible process, but $u_0 \equiv 0$ does not solve (C), that is, it does not minimize $\int_0^1 u(t)dt$ over the set

$$\mathcal{C} = \{u \in \mathcal{U}_1 \mid u(t) \leq 0\ (t \in T)\}.$$

In this example, u_0 maximizes I on $\mathcal{C}$. Note also that (x_0, u_0) is not normal with respect to S since

$$B(t) = f_u(\tilde{x}_0(t)) = 0 \quad (t \in [0, 1]).$$

The following example illustrates an opposite situation where, though one can exhibit a solution to the problem (C), the original problem (P) may fail to have a solution.

5.3 Example. Consider the problem of minimizing

$$I(x, u) = \int_0^1 u_2^2(t)dt$$

subject to $(x, u) \in Z$ and

$$\begin{cases} \dot{x}(t) = u_3(t) - u_1(t) + u_1^2(t)u_2(t)\ (t \in [0, 1]); \\ x(0) = x(1) = 0; \\ u_3(t) - u_1(t) \leq -2,\ -u_3(t) \leq 1, \\ \qquad u_1^2(t)u_2(t) \leq 2\ (t \in [0, 1]). \end{cases}$$

Clearly $u_0 \equiv (1, 0, -1)$ is a solution to the problem (C) of minimizing $\int_0^1 u_2^2(t)dt$ on the set

$$\mathcal{C} = \{(u_1, u_2, u_3) \in \mathcal{U}_3 \mid u_3(t) - u_1(t) \leq -2,$$

$$-u_3(t) \leq 1,\ u_1^2(t)u_2(t) \leq 2\ (t \in [0, 1])\}.$$

However, if (x, u) is admissible, then necessarily

$$0 < u_2(t) \leq 2 \quad (t \in [0, 1]).$$

This can be easily seen since, by the constraints, we have $\dot{x}(t) \leq 0$ and $x(0) = x(1) = 0$ implying that $x \equiv 0$ and so $u_3 - u_1 + u_1^2 u_2 \equiv 0$. Therefore $u_1^2 u_2 \equiv 2$ and, since $-1 \leq u_3(t) \leq u_1(t) - 2$, we have $u_1(t) \geq 1$. This proves the claim.

5.4 Theorem. *Suppose* (x_0, u_0) *solves* (P) *and* $\exists(p, \mu) \in X \times \mathcal{U}_q$ *satisfying*

a. $\mu_\alpha(t) \geq 0$ *and* $\mu_\alpha(t)\varphi_\alpha(t, u_0(t)) = 0$ $(\alpha \in R,\ t \in T)$*;*

b. *For all* $t \in T$,

$$\dot{p}(t) = -f_x^*(\tilde{x}_0(t))p(t),$$

$$p^*(t)f_u(\tilde{x}_0(t)) = L_u(t, u_0(t)) + \mu^*(t)\varphi_u(t, u_0(t)).$$

Suppose also that u_0 *solves* (C)*. If* $p \equiv 0$ *then*

$$J((x_0, u_0, p, \mu); (y, v)) \geq 0$$

for all $(y, v) \in Z$ *with*

$$v(t) \in \tau_1(t, u_0(t), \mu(t)) \quad (t \in T).$$

In particular, $p \equiv 0$ *if* (x_0, u_0) *is normal relative to* S_1.

Proof: Let

$$F(t, u, \nu) := L(t, u) + \langle \nu, \varphi(t, u) \rangle.$$

By Lemma 5.1,

$$L(t, u) \geq L(t, u_0(t)) \quad (t \in T)$$

whenever $(t, u) \in \mathcal{A}$, and so there exists a unique $\nu \in \mathcal{U}_q$ such that

$$F_u(t, u_0(t), \nu(t)) =$$

$$L_u(t, u_0(t)) + \nu^*(t)\varphi_u(t, u_0(t)) = 0\ (t \in T).$$

Moreover,

$$\nu_\alpha(t) \geq 0 \text{ and } \nu_\alpha(t)\varphi_\alpha(t, u_0(t)) = 0$$

for all $\alpha \in R$ and $t \in T$, and

$$\langle h, F_{uu}(t, u_0(t), \nu(t))h \rangle \geq 0$$

for all $h \in \tau_1(t, u_0(t), \nu(t))$. Assume that $p \equiv 0$. By (b), we have

$$F_u(t, u_0(t), \mu(t)) =$$

$$L_u(t, u_0(t)) + \mu^*(t)\varphi_u(t, u_0(t)) = 0\ (t \in T)$$

and so, by uniqueness, $\mu \equiv \nu$. Since

$$H(t, x, u, p, \mu, 1) = \langle p, f(t, x, u)\rangle - L(t, u) - \langle \mu, \varphi(t, u)\rangle = -F(t, u, \mu)$$

we have

$$2\Omega(t, y, v) = \langle v, F_{uu}(t, u_0(t), \mu(t))v\rangle$$

and the first part follows.

To show that normality relative to S_1 implies $p \equiv 0$, note that

$$L_u(t, u_0(t)) = -\nu^*(t)\varphi_u(t, u_0(t)) \quad (t \in T)$$

and so, by (b),

$$p^*(t)B(t) = \sum_1^q (\mu_\alpha(t) - \nu_\alpha(t))\varphi_{\alpha u}(t, u_0(t)).$$

This implies that, for all $h \in \tau_1(t, u_0(t), \mu(t))$,

$$p^*(t)B(t)h = \sum_{\alpha \in N(t)} -\nu_\alpha(t)\varphi_{\alpha u}(t, u_0(t))h$$

where

$$N(t) = \{\alpha \in I_a(t, u_0(t)) \mid \mu_\alpha(t) = 0\}.$$

We conclude that

$$p^*(t)B(t)h \geq 0 \text{ for all } h \in \tau_1(t, u_0(t), \mu(t))$$

and therefore $-p$ is a solution to the system

$$\dot{z}(t) = -A^*(t)z(t), \quad z^*(t)B(t)h \leq 0$$

for all $h \in \tau_1(t, u_0(t), \mu(t))$ $(t \in T)$. Thus, if (x_0, u_0) is normal relative to S_1, then $p \equiv 0$. ▮

This last result yields in a natural way a fundamental question which we shall answer with one example (for a simpler class of problems, see also [6]). Note that Example 5.2 provides a solution (x_0, u_0) to the problem (P) for which u_0 does not solve (C). The solution is not weakly normal nor, in consequence, normal relative to S_1 or strongly normal. A natural question is if, in Theorem 5.4, the assumption that u_0 solves the problem (C) is redundant if the corresponding solution (x_0, u_0) to the original problem (P) is normal relative to S_1.

In the following example we provide a solution to (P) which is normal relative not only to S_1 but to S_0 and is not a solution to (C). In other words, it is a *sine qua non* assumption.

5.5 Example. Consider the problem (P) of minimizing

$$I(x, u) = \int_0^1 u(t)dt$$

subject to

$$\dot{x}(t) = u(t)x(t) \ (t \in [0, 1]),$$

$$x(0) = 1, \ x(1) = e, \ b(t)u(t) \geq 0 \ (t \in [0, 1])$$

where b is any positive piecewise continuous function mapping $T = [0, 1]$ to $\mathbf{R}$.

For this problem $\xi_0 = 1$, $\xi_1 = e$,

$$L(t, x, u) = u, \quad f(t, x, u) = ux,$$

$$\varphi(t, u) = -b(t)u$$

so that the set of constraints is given by

$$S = \{(x, u) \in D \mid b(t)u(t) \geq 0 \ (t \in T)\}$$

where

$$D = \{(x, u) \in Z \mid \dot{x}(t) = u(t)x(t), \ x(0) = 1, \ x(1) = e\}.$$

Let us show that one may have a solution to (P) which is normal relative to S_0 but is not a solution to (C). Note first that, if (x, u) is admissible, then

$$x(t) = \exp\Big(\int_0^t u(s)ds\Big)$$

and

$$x(1) = e = \exp\Big(\int_0^1 u(t)dt\Big)$$

and so

$$\int_0^1 u(t)dt = I(x, u) = 1.$$

Therefore

$$(x_0(t), u_0(t)) := (e^t, 1) \quad (t \in T)$$

is a solution to the problem (P) but not a solution to (C). The solution to (C) is clearly $u_0 \equiv 0$.

Now, we have

$$f_x(\tilde{x}_0(t)) = 1, \quad f_u(\tilde{x}_0(t)) = e^t,$$

and $\varphi(t, u_0(t)) = -b(t) < 0$. Since, for this solution, there are no active constraints, $S_0 = D$. Clearly (x_0, u_0) is a normal process of S_0 since, given (p, μ) satisfying

$$-\mu(t)b(t) = 0, \ \dot{p}(t) = -p(t),$$

$$e^t p(t) = -b(t)\mu(t) = 0 \quad (t \in [0,1])$$

then $p \equiv 0$.

Let us now show that, for this solution, one can find (p, μ) in $X \times \mathcal{U}_1$ such that (a)–(c) of Theorem 5.4 are satisfied, so that (x_0, u_0, p, μ) is an extremal and, moreover, (x_0, u_0) is not only normal relative to S_1 but also to S_0. However, the conclusions of Theorem 5.4 do not hold since neither $p \equiv 0$ nor

$$J((x_0, u_0, p, \mu); (y, v)) \geq 0$$

for all $(y, v) \in Z$ with $v(t) \in \tau_1(t, u_0(t), \mu(t))$ $(t \in T)$.

Indeed, if $p(t) = e^{-t}$ and $\mu \equiv 0$ then $(x_0, u_0, p, \mu) \in \mathcal{E}$ (see Definition 3.2) since

$$\dot{p}(t) = -p(t) \text{ and } e^t p(t) = 1 \quad (t \in T).$$

Therefore (a)–(c) hold. Also $p \not\equiv 0$. Moreover,

$$H = pux - u + \mu b(t)u$$

and so

$$H_u = px - 1 + \mu b(t), \quad H_x = pu,$$

$$H_{uu} = H_{xx} = 0, \quad H_{ux} = H_{xu} = p$$

all evaluated at $(t, x, u, p, \mu, 1)$. Thus

$$2\Omega(t, y, v) = -2p(t)yv$$

and so

$$J((x_0, u_0, p, \mu); (y, v)) = -2\int_0^1 e^{-t} y(t)v(t)dt.$$

Since there are no active constraints,

$$\tau_0(u_0(t)) = \tau_1(u_0(t), \mu(t)) = \tau(u_0(t)) = \mathbf{R}.$$

Thus, if $y \equiv v \equiv 1$, the above integral is negative.

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