Optimization of wireless networks performance: an approach based on a partial penalty method

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Abstract— We study an optimization problem for a wireless telecommunication network stated as a generalized transportation problem (TP), where m (the number of "sellers") is the number of network providers, and n (the number of "buyers") is the number of connections established at a given time moment. Since in practice initial data of such problems are, generally speaking, inexact and/or vary rather quickly, it is more important to obtain an approximate solution of the problem (with a prescribed accuracy) within a reasonable time interval rather than to solve it precisely (but in a longer time). We propose to solve this problem by a technique that explores the idea of penalty functions, namely, the so-called Partial Penalty Method (PPM, for short). As distinct from exact solution methods for TP (e.g., the method of potentials), our approach allows us to further extend the class of considered problems by including to it TP with nonlinear objective functions. As an example, we consider a TP, where the objective function (expenses connected with resource allocation) is such that the price of the unit amount of the resource is not constant but depends on the total purchase size. In addition, we study the limit behavior of solutions to TP whose data are subject to fading disturbances. Since in our approach the initial point is not necessarily admissible, we use an approximate solution of each problem as the initial point for the next one. As expected, under certain requirements to disturbances the sequence of solutions to "disturbed" problems tends to a solution of the limit problem. We prove experimentally that PPM is more efficient than the usual variant of the Penalty Function Method (the Full Penalty Method, or FPM). The preference of PPM over FPM is more evident for *n* much greater than *m*.

Keywords— Optimization, telecommunication networks, provider, network connection, open transportation problem, penalty function methods, nonlinear objective function, random disturbances, approximate solution, decaying perturbations, limit behavior.

I. Introduction

One of modern trends in the development of information technologies on a global scale is a radical modernization of economy through the ubiquitous implementation of wireless networks. Owing to sensors, electronics and software that provide the interaction of wireless devices with each other and with the environment, wireless networks open enormous opportunities for data collecting, storing, exchanging and processing. This gives new possibilities for the development of artificial intelligence, robotics, 3D printing, nanotechnology, biotechnology, quantum computing and other breakthrough technologies.

At the same time, increasing and variable demand of information services and users movement lead to serious congestion effects, whereas significant network resources may be utilized inefficiently, especially in the case when fixed allocation mechanisms are implemented. This situation forces us to apply more flexible and dynamical allocation mechanisms; see e.g. [1, 2]. For this reason, it seems more suitable to find an approximate solution of a proper resource allocation problem, which does not require high accuracy, within an acceptable time interval rather than to calculate the exact one. Usually, resource allocation problems are based on the utility maximization approach; see e.g. [3, 4, 5].

In this paper, we study a general problem of the optimal assignment of users to providers of wireless telecommunication networks that minimizes the total expenses under certain resource allocation restrictions. That is, each provider has a certain coverage area with the required level of service quality for each connection within this area, whereas users have lower bounds for their volume of the resource and their desired prices. We should also take into account expenses of providers for maintaining the required volume of service. We show that the problem allows the statement in the form of the transportation problem (TP for short) with bilateral constraints on variables. We propose a technique that implies the use of penalty functions but only for certain constraints, whereas the rest constraints form a set of points having a special structure. It is used as a feasible set for an auxiliary problem. The key moment is that in spite of the presence of binding constraints, the suggested auxiliary problem is solvable by a simple finite algorithm. We have performed extensive numerical experiments that confirmed the advantage of the proposed method in comparison with the custom one involving penalization of all the constraints. We consider both linear and nonlinear variants of the objective function of the problem and experimentally prove that the proposed technique (PPM) is applicable in both cases. In addition, we demonstrate the efficiency of the use of the PPM for finding an approximate solution of the limit problem obtained from a sequence of problems whose data are subject to fading disturbances.

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II. THE PROBLEM STATEMENT

Within a certain fixed planning time period we consider a region (territory) where wireless network services of several providers are used by mobile devices owners. Each of these users can be either a transmitter or a receiver of a signal. Denote by m the number of providers; let us numerate providers using the index i ($i=1,\ldots,m$). Within the given time period there arise connections (signal transmissions) between certain users. Denote by n the number of (pair) connections; let us numerate connections using the index j ($j=1,\ldots,n$). Signal transmissions require certain expenditures of providers' resources (say, the bandwidth or power of the wireless channel). It is natural to assume that the resource amount possessed by each provider i is bounded by some value γ_i. Let the symbol $x_{i,j}$ stand for the unknown amount of the resource allotted by provider i for pair connection j (below for brevity we just say "flow (i,j)"). Denote by $\alpha_{i,j}$ the upper bound for flow (i,j) and by β_j the lower bound for the total flow for connection j. Let b_j be the price (willingness to pay) proposed by pair j and let $a_{i,j}$ be expenses per unit for connection j incurred by provider i. Then the pure total expenses are given by the expression

$$\sum_{i=1}^{m}\sum_{j=1}^{n}a_{i,j}x_{i,j}-\sum_{j=1}^{n}b_j\left(\sum_{i=1}^{m}x_{i,j}\right)\equiv\sum_{i=1}^{m}\sum_{j=1}^{n}c_{i,j}x_{i,j},$$

where $c_{i,j}\equiv a_{i,j}-b_j$. The goal is to minimize the pure total expenses due to a proper distribution of the load upon network providers.

Note that any connection can be supported and accomplished at the proper service level only by selected providers in accordance with their quality service coverage areas. That is, each connection j can be accomplished by selected providers whose indices belong to the set P_j. However, for all $i\notin P_j$ we can set $\alpha_{i,j}=0$, which implies $x_{i,j}=0$. Therefore, without loss of generality we can consider only the case where $P_j=\{1,\ldots,m\}$ for each $j\in\{1,\ldots,n\}$. The problem takes the form

$$\min\to\sum_{i=1}^{m}\sum_{j=1}^{n}c_{i,j}x_{i,j}, \tag{1}$$

subject to

$$\sum_{i=1}^{m}x_{i,j}\geq\beta_j,\quad j=1,\ldots,n,$$

$$\sum_{j=1}^{n}x_{i,j}\leq\gamma_i,\quad i=1,\ldots,m,$$

$$0\leq x_{i,j}\leq\alpha_{i,j},\ i=1,\ldots,m,\ j=1,\ldots,n.$$

(4)

Problem (1)–(4) is nothing but the so-called open transportation problem with bilateral constraints on variables. It turns into the classical transportation problem if $\alpha_{i,j}=+\infty$ for all i,j; see [6] for more details and references.

In spite of the existence of finite solution methods for the TP (see, for example, [6]), we intend to apply some other iterative methods for this problem. The most influential factor that affects the applicability of exact methods for solving the TP, evidently, is the fast growth of the problem dimension, which, in turn, leads to the accumulation of computation errors and poor conditionality of the constraint coefficient matrix. Moreover, in practice, the feasible set of the open TP is not necessarily nonempty. In such cases one can find a solution close to the optimal (feasible) one only by approximate methods. Another factor that contributes to the relevance of the development of approximate solution methods for the TP is the appearance of new applications of the transportation model; for example, along with classical applications in the optimization of production, transportation, and sales of some commodity, this model appears to be applicable in the optimization of the performance of mobile networks. Such a problem usually has a large dimensionality, and its initial data are inexact and non-stationary. Moreover, in practice, problem (1)–(4) are often being solved in order to estimate certain characteristics of the network performance; in this case it is more important to find an appropriate solution of the problem within an acceptable time frame rather than to obtain a high accuracy solution.

In this paper we propose an approximate solution method for problem (1)–(4) which is based on application of penalty functions. Since we "fine" the violation of only one group of constraints (rather that all of them) we call the proposed approach the Partial Penalty Method (PPM). We show that the PPM allows us to essentially widen the class of solved problems, namely, to waive the requirement of the linearity of the objective function of the problem. We also study the application of the PPM to finding an approximate solution (with a given accuracy level) to the limit problem when disturbances of data tend to zero.

III. THE PARTIAL PENALTY METHOD

As distinct from the custom penalty method, in the partial penalty method (PPM for short) we impose penalties only on selected constraints. The set formed by the rest constraints has a special structure which allows us to solve the corresponding auxiliary problem by a simple finite algorithm. Thus we intend to attain higher quality of solutions.

First we introduce the so-called cut function

$$[t]_+=\max\{0,t\},$$

and then define the penalty function for the constraints in (3): (2)

$$\Phi(X)\equiv\sum_{i=1}^{m}\left[\sum_{j=1}^{n}x_{i,j}-\gamma_i\right]_+^2. \quad (3) \tag{5}$$

We take a positive penalty parameter τ and define the auxiliary function

$$\Psi(X,\tau)=\langle C,X\rangle+\tau\Phi(X). \tag{6}$$

Hereinafter C and X are $m\times n$-matrices and the denotation $\langle C, X\rangle$ stands for the double sum

$$\langle C, X\rangle \equiv \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}x_{i,j}. \tag{7}$$

We treat the matrix X as a point (in the space of $m\times n$-matrices). Denote the sets of points satisfying the inequalities in (4) and (2) by A and B, respectively, and set

$$X^*(\tau) \equiv \arg\min_{X\in A\cap B} \Psi(X,\tau). \tag{8}$$

Note that the function in (6) is continuous by definition and the set $A\cap B$ is closed and bounded, hence the point in (8) exists for any τ. Let us construct an iteration sequence $\{X_k^*(\tau_k)\}$, where k is the iteration number, such that the sequence $\{\tau_k\}$ is positive, increasing, and tending to $+\infty$ as $k\to\infty$, while each point $X_k^*(\tau_k)$ obeys formula (8) with $\tau=\tau_k$. Since the set $A\cap B$ is bounded, so is the sequence $\{X_k^*(\tau_k)\}$, which means that it has limit points as $k\to\infty$ and all these limit points X^* are solutions of problem (1) – (4) (see, for example, [7], Section 7.1). Moreover, this is the case for some approximations of points $x_k^*(\tau_k)$, $k=0,1,\ldots$ Let us now consider the technique for finding the points $X_k^*(\tau_k)$, $k=0,1,\ldots$

IV. Solution of Auxiliary Problems

Assume that certain real numbers $d_{i,j}$, $i=1,\ldots,m$; $j=1,\ldots,n$, are given (we concretize them below). Denote the corresponding $m\times n$-matrix by D. Let us use the denotation $\langle D, X\rangle$ in the sense of formula (7) with the symbol D in place of C. Let us describe an algorithm which solves the problem

$$\min_{X\in A\cap B} \to \langle D, X\rangle. \tag{9}$$

Let us show that in spite of the existence of constraints (2) which bound the problem variables, problem (9) falls into n independent problems which are solvable explicitly. Fix some connection $p\in\{1,\ldots n\}$ and describe the algorithm for finding components $x_{i,p}$, $i=1,\ldots,m$, of a solution x to problem (9). Since this algorithm solves the auxiliary problem, we call it " Algorithm A", for short.

Algorithm A.

Step 0. Given p, number providers in ascending order of $d_{i,p}$ and thus get a set of numbers $I\equiv\{i_1,\ldots,i_m\}$. Introduce a new variable s and put $s:=1$.

Step 1. If

$$\sum_{i=i_1}^{i_s}\alpha_{i,p} < \beta_p,$$

then put $x_{i_s,p} := \alpha_{i_s,p}$ and go to Step 2; otherwise put

$$x_{i_s,p} := \beta_p - \sum_{i=i_1}^{i_{s-1}}\alpha_{i,p},$$

do $x_{i_\nu,p} := 0$ for $\nu = s+1,\ldots,m$, and Algorithm A stops.

Step 2. If $s<m$, then put $s:=s+1$ and go to Step 1; otherwise Algorithm A stops.

Evidently, sequentially applying Algorithm A for $p=1,\ldots,n$, in n steps we get a point $\widetilde{X}(D)$, whose feasibility and optimality for problem (9) is evident, provided that $A\cap B\neq\varnothing$ (in what follows we assume that this condition is fulfilled).

Let us now consider the basic problem

$$\min_{X\in A\cap B} \to \Psi(X,\tau) \tag{10}$$

for finding a point satisfying (8) with some fixed $\tau>0$. We can solve problem (10) by the well-known conditional gradient method (CGM for short) (see, for example, [8]). Let us fix arbitrary indices $i_0\in\{1,\ldots,m\}$, $j_0\in\{1,\ldots,n\}$, and a number $\tau>0$ and write the partial derivative of the function in (6) at a point X with respect to the variable x_{i_0,j_0}:

$$\frac{\partial\Psi(X,\tau)}{\partial x_{i_0,j_0}} = c_{i_0,j_0} + 2\tau\left\lfloor \sum_{j=1}^{n} x_{i_0,j} - \gamma_{i_0}\right\rfloor_+ . \tag{11}$$

Denote by $\Psi'(X,\tau)$ the $m\times n$-matrix composed of elements (11) and treat it as the gradient of the function $\Psi(X,\tau)$ at the point X with fixed τ. Let us now describe CGM applied to problem (10).

(CGM).

Step 0. Given $\tau>0$, choose a point $X^0\in A\cap B$. Assume that a point X^l is known already; $l=0,1,\ldots$ Let us describe the way to find the next point X^{l+1}.

Step 1. Find a solution Z^l to the linear programming problem

$$\min_{X\in A\cap B} \to \langle \Psi'(X^l,\tau), X\rangle, \tag{12}$$

and go to Step 2.

Step 2. Calculate

$$\lambda_l := \arg\min_{\lambda\in[0,1]} \Psi(\lambda X^l + (1-\lambda)Z^l,\tau) \tag{13}$$

and put $X^{l+1} := \lambda_l X^l + (1-\lambda_l)Z^l$, $l:=l+1$ and go to Step 1.

For each $l=0,1,\ldots$ by putting $D:=\Psi'(X^l,\tau)$ we get problem (9) in (12) and solve it by Algorithm A. Problem (13) can be solved by any one-dimensional minimization method (see, for example, [7], Section 3.7). In numerical experiments we used the well-known golden section method (see, for example, [7], p. 84).

V. THE USUAL PENALTY METHOD

As distinct from the PPM, where the penalty function is introduced only for constraints in (3). In the usual (or full) penalty method (FPM for short) we define penalty functions for both groups of constraints, namely, for those in (3) and (2):

$$\tilde{\Phi}(X) \equiv \sum_{i=1}^{m}\left[\sum_{j=1}^{n} x_{i,j} - \gamma_i\right]_+^2 + \sum_{j=1}^{n}\left[\beta_j - \sum_{i=1}^{m} x_{i,j}\right]_+^2,$$

and

$$\tilde{\Psi}(X,\tau) \equiv \langle C, X\rangle + \tau\tilde{\Phi}(X), \qquad (14)$$

where τ is a positive penalty parameter. We now outline the main differences from the PPM.

The auxiliary problem which is solved at each step k of the FPM consists in finding the point

$$X^*(\tau_k) \equiv \arg\min_{X\in A} \tilde{\Psi}(X,\tau_k)$$

for $k = 0,1,\ldots$ Analogously, we can solve this auxiliary problem by the conditional gradient method (CGM). Its each iteration involves a solution to the linear programming problem

$$\min_{X\in A} \rightarrow \langle \tilde{\Psi}'(X^l,\tau), X\rangle \qquad (15)$$

with $\tau = \tau_k$. The components of the gradient $\tilde{\Psi}'(X,\tau)$ in view of (14) obey the formula

$$\frac{\partial\Phi(X,\tau)}{\partial x_{i_0,j_0}} = c_{i_0,j_0} + 2\tau\left[\sum_{j=1}^{n} x_{i_0,j} - \gamma_{i_0}\right]_+ - 2\tau\left[\beta_{j_0} - \sum_{i=1}^{m} x_{i,j_0}\right]_+.$$

Since its feasible set A represents a rectangle, problem (15) falls into $m\times n$ independent one-dimensional problems, each of them is solved explicitly. The other parts are implemented similarly.

VI. RESULTS OF NUMERICAL EXPERIMENTS

We have numerically tested the described methods via the package Wolfram Research Mathematica 9.0.1.0 by using a computer with Processor Intel Core^{TM} i5-430M (4M Cache, 2.26 GHz). In order to prove the efficiency of the new method (PPM) we compared the results of solving problem (1) – (4) with those of (FPM). We used the same rule for decreasing values of accuracy of inner problems. For changing the penalty parameter we used the rule $\tau_{k+1} := 2\tau_k$.

We modeled the initial data of the problem so as to know its optimum point (and, correspondingly, the exact optimal value of the objective function F^*). We stopped the process when either the absolute value of the relative deviation of the current approximation to the optimal value of the objective function from F_{opt} was not greater than 10% or the norm of the difference of neighboring points was less than some predefined value ε (we put $\varepsilon := 0.001$). For each concrete problem (i.e. concrete collection of initial data) we performed 10 tests for both methods, randomly choosing an initial point. In what follows the subscript h stands for the test number (within a series of 10 tests); symbols $F_{h(FPM)}$ and $F_{h(PPM)}$ denote, respectively, approximate values of the objective function of problem (1) – (4) calculated by FPM and PPM at test number h; symbols $\overline{F}^*_{PPM}$ and $\overline{F}^*_{PPM}$ stand, respectively, for average values of $F_{h(FPM)}$ and $F_{h(PPM)}$ in each series of 10 tests, i.e.

$$\overline{F}^*_{FPM} = \frac{\sum_{h=1}^{10} F_{h(FPM)}}{10};\overline{F}^*_{PPM} = \frac{\sum_{h=1}^{10} F_{h(PPM)}}{10};$$

the relative approximation errors

$$\frac{\overline{F}^*_{FPM} - F_{opt}}{F_{opt}} \quad and \quad \frac{\overline{F}^*_{PPM} - F_{opt}}{F_{opt}};$$

and values $\bar{t}_{FPM}$ and $\bar{t}_{PPM}$ are average time consumptions in a series of 10 tests. These values are given in Table 1 (see the Appendix).

According to results shown in Table 1, with small m (not greater than 20) PPM attains the given accuracy with respect to the value of the objective function (in our tests the allowed error was 10 $\%$) much faster than FPM. Moreover, the actual error introduced by PPM has never exceeded 2.17%; mainly it was even less than 0.5%, whereas the the actual error introduced by PPM was mostly greater than 3%, sometimes approaching (or even attaing) the limit admissible value of 10%. We also calculated the ratios

$$\frac{F_{max(FPM)} - F_{min(FPM)}}{\overline{\overline{F}}_{(FPM)}} \quad and$$

$$\frac{F_{max(PPM)} - F_{min(PPM)}}{\overline{\overline{F}}_{(PPM)}}$$

(after performing a series of 10 tests) in order to study the sensitivity of these methods to the choice of the initial point.

As appeared, both methods are insensitive to the choice of an initial point (not necessarily a feasible one), since these characteristics always equaled zero. It is evident that PPM gives better results both with respect to time and to the solution accuracy (which was much less than the allowed value of 10%). As expected, the advantage of PPM over FPM was more evident when m is small (not greater than 3) and n is very large (up to 3000), whereas the growth of m (with fixed n) impairs the performance of both methods at approximately the same rate. In certain cases time consumption of PPM was even greater than that of FPM. For example, the case when $m = 2$ and $n = 2000$ (i.e., the number of variables equals 4000) the time consumption equals 9.89 and 2.63 sec. for FPM and PPM, respectively, (see row 16 in Table 1). There were some examples with $m = 20$ and $n = 20$, where PPM showed better performance. In general, PPM appeared more efficient than FPM in most examples and is suitable for calculations. Nevertheless, due to the necessity of tuning several parameters, its convergence needs further investigations.

VII. Generalization of the problem: the case of a nonlinear objective function

In previous sections we have described the application of the penalty function method (both full and partial versions) to the transportation problem (1)–(4), i.e. under the assumption that the objective function is linear. However, as one can easily see, the linearity of the objective function is not essential for the use of the penalty function method. More precisely, we need only the auxiliary problems to be linear (namely, problem (8) in the PPM and (15) in the FPM). This fact allows us to replace the linear function in (1) with any function $G(X)$ which is defined and continuously differentiable on the space of $m\times n$-matrices X and satisfies the condition

$$\frac{\partial G(X)}{\partial x_{i,j}} < \infty$$

at any point X with any $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$. Consider a more general statement of the transportation problem, namely, assume that, as distinct from (1), prices depend on variable values $x_{i,j}$ nonlinearly. Let, for definiteness, for arbitrary numbers $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$ the total cost of the purchase of $x_{i,j}$ equal

$$g_{i,j}(x_{i,j}) \equiv \Big(c_{i,j} + \frac{\Delta_{i,j}}{\sqrt{1+x_{i,j}}}\Big)x_{i,j}; \tag{16}$$

here $c_{i,j}$ and $\Delta_{i,j}$ are some positive constants; the unit in the radical is used only for the sake of convenince, namely, it ensures the finiteness of the derivative of function (16) at zero. According to formula (16), the price of the unit amount of product depends on the size of the purchase; evidently, it tends to $c_{i,j} + \Delta_{i,j}$ as $x_{i,j} \to 0$ and does to $c_{i,j}$ as $x_{i,j} \to \infty$. Note that this model is more realistic than that with constant prices, because usually the greater is the purchase volume, the lower is the price of the unit amount of the commodity. Thus, the problem takes the form

$$min \to G(X) \tag{17}$$

subject to (2)–(4), where

$$G(X) \equiv \sum_{i=1}^{m}\sum_{j=1}^{n} g_{i,j}(x_{i,j}), \tag{18}$$

functions $g_{i,j}(x_{i,j})$ obey formula (16), $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$. This leads to certain modifications in the PPM and FPM.

In particular, in the PPM, formula (6) turns into

$$\Psi(X,\tau) = G(X) + \tau\Phi(X), \tag{19}$$

and formula (11) does to

$$\frac{\partial \Psi(X,\tau)}{\partial x_{i_0,j_0}} = g'_{i_0,j_0}(x_{i_0,j_0}) + 2\tau\Big\lfloor \sum_{j=1}^{n} x_{i_0,j} - \gamma_{i_0}\Big\rfloor_{+} \tag{20}$$

with

$$g'_{i_0,j_0}(x_{i_0,j_0}) = c_{i_0,j_0} + \Delta_{i_0,j_0}(1+x_{i_0,j_0})^{-\frac{1}{2}} + \\ + \frac{1}{2}\cdot\Delta_{i_0,j_0}\cdot x_{i_0,j_0}(1+x_{i_0,j_0})^{-\frac{3}{2}}.$$

Correspondingly, now we understand $\Psi'(X,\tau)$ as the $m\times n$-matrix composed of elements (20) and treat it as the gradient of the function $\Psi(X,\tau)$ at the point X with fixed τ.

Analogously, in the FPM, formula (14) takes the form

$$\tilde{\Psi}(X,\tau) \equiv G(X) + \tau\tilde{\Phi}(X), \tag{22}$$

and components of the gradient in formula (15) obey the formula

$$\frac{\partial \Phi(X,\tau)}{\partial x_{i_0,j_0}} = g'_{i_0,j_0}(x_{i_0,j_0}) + \\ + 2\tau\Big\lfloor \sum_{j=1}^{n} x_{i_0,j} - \gamma_{i_0}\Big\rfloor_{+} - 2\tau\Big\lfloor \beta_{j_0} - \sum_{i=1}^{m} x_{i,j_0}\Big\rfloor_{+}$$

with $g'_{i_0,j_0}(x_{i_0,j_0})$ satisfying (21).

For experimentally proving the efficiency of the PPM in application to problem (17), (2)–(4), similarly to cases considered above, we performed numerical tests with modelled data. In these tests, we compared time consumptions by the PPM and FPM for attaining the necessary accuracy (i.e., for obtaining a solution whose error with respect to the known optimal value of the objective function does not exceed 10%). Results of numerical tests are given in Table 2 (see the Appendix). Note that negative errors of solutions mean their infeasibility, but since absolute values of errors are small, solutions are located very close to the admissible domain. As a whole, according to results of numerical tests, PPM proved to be more efficient than FPM, especially when n was much greater than m. This can be easily explained by the fact that PPM exactly solves auxiliary problems by Algorithm A within a finite number of steps.

VIII. The limit behavior of approximate solutions of the transportation problem obtained by PPM and FPM

In above sections we have studied the efficiency of PPM (in comparison with that the FPM) for solving the extended transportation problem in form (1)–(4) or (17), (1)–(4) with increasing dimensions of the problem. In this section we consider the behavior of approximate solutions of this problem obtained by both methods when the initial data of the problem are subject to random decaying perturbations.

Since, as was shown in the previous section, both variants of the penalty function method are insensitive to the form of the objective function (i.e., it can be either linear or not), here we restrict ourselves to considering the limit behavior of a sequence of solutions to problems in form (1)–(4) with disturbed data. In numerical experiments, as above, for each fixed values of m and n we first generated initial data of problem (1)–(4) so as to know its exact solution, i.e., an optimal point X^* and the optimal value of the objective function $F^* \equiv \langle C, X^* \rangle$. Then we assumed that coefficients of the function in (1) were subject to random perturbations with zero mean value and some standard deviations eventually tending to zero. Namely, for fixed $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$, and fixed time moment k, $k = 0,1,\ldots$,

$$c_{i,j}^k \equiv c_{i,j} + \xi_{i,j}^k, \quad (23)$$

where $\xi_{i,j}^k$ is a gaussian random variable such that $M(\xi_{i,j}^k) = 0$, and $\sigma(\xi_{i,j}^k) \equiv \sigma_{i,j}^k \to 0$ as $k \to \infty$. So at each time moment k we get the problem

$$min \to \sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}^k x_{i,j} \quad (24)$$

subject to (2)–(4).

The computational experiment was organized as follows: we repeatedly solved problem (24), (2)–(4) first by the FPM and then by the PPM till obtaining the desired accuracy (the maximal absolute error was to be no greater than 10% with respect to the optimal value of the objective function). Note that unlike experiments described above, now we continiued the solution process untill we attain the desired accuracy with respect to the solution of the *limit* problem rather than the current one. The obtained results are given in Table 3 (see the Appendix). According to obtained results, PPM usually solves gets a solution to the limit problem, satisfying the accuracy requirement, in a shorter time interval.

IX. Conclusion

We studied a general optimization problem that occurs in wireless telecommunication networks, namely, the problem of the assignment of users to providers minimizing the total sum of the corresponding expenses. We have demonstrated that the mentioned problem can be stated as an extended open transportation problem, whose objective function is continuously differentiable but not necessarily linear. We have proposed an approach based on exploring the idea of the penalty function method (PFM) for solving the mentioned problem. We have developed a variant of the PFM called the partial penalty method (PPM) which has an essential distinction from its general scheme. Namely, according to the PPM, instead of imposing penalties on all constraints in the problem, we “fine” only the violation of certain linear inequalities, and exactly solve the auxiliary problem (i.e., an optimization problem with respect to the “fined” objective function and the rest constraints) within a finite number of steps. The algorithm for solving the auxiliary problem is described in the paper as Algorithm A. It essentially uses the specific structure of the admissible set of the auxiliary problem. We have performed an extensive numerical experiment for studying the efficiency of PPM in comparison with the classical (full) penalty function method (treated by us as the FPM). As expected, results of numerical tests have proved preferences of the PPM over the FPM in speed especially when n is much greater than n. Since this is just the case in wireless telecommunication networks, where the number of connections n essentially exceeds the number of providers m, this characteristic of the PPM is especially important for wireless networks optimization. The applicability for nonlinear problems (even not necessarily quadratic or convex) is the second strong feature of the PPM as a technique for solving wireless networks optimization problems, because in real world economy the dependence of the unit cost of goods usually is not constant but demonstrates an inverse relationship on the purchase size. Finally, the use of PPM seems most promising for getting an approximate solution to the *limit* problem by eventually solving problems with inexact and/or varying data.

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APPENDIX

Table 1: Comparison of FPM and PPM in application to problem (1)–(4)

m	n	F_{opt}	Avg. F_{opt}		Avg.err. (%)		Avg. t (sec)	
			$\overline{F}^*_{FPM}$	$\overline{F}^*_{PPM}$	$\overline{E}_{FPM}$	$\overline{E}_{PPM}$	$\bar{t}_{FPM}$	$\bar{t}_{PPM}$
3	20	230.22	236.78	230.22	2.85	0.00	0.11	0.03
3	20	422.03	431.31	422.03	2.19	0.00	0.13	0.03
3	20	376.24	397.35	376.24	5.60	0.00	0.04	0.03
10	20	1895.37	2057.44	1902.22	8.55	0.36	0.78	0.44
10	20	1596.32	1614.69	1600.98	1.15	0.29	0.37	0.08
10	20	1159.42	1201.77	1162.29	3.65	0.24	0.13	0.11
10	50	2089.31	2269.17	2089.31	8.61	0.00	1.30	0.13
10	50	1097.54	1986.86	1940.41	4.16	1.72	0.84	0.13
10	50	1856.6	1944.87	1892.4	4.75	1.93	0.86	0.56
10	100	6108.86	6335.65	6108.86	3.71	0.00	8.55	0.26
10	100	6907.88	7598.67	6943.07	10.00	0.52	15.75	3.97
10	100	7570.59	8326.72	7678.28	9.98	1.42	1.18	0.87
10	1000	46905.2	50378.4	47923.77	7.40	2.17	400.38	3.72
10	1000	55520.4	57728.4	55520.4	3.97	0.00	61.53	4.99
10	1000	49232.6	51894.4	49312.4	5.41	0.16	150.96	5.86
2	2000	22248.8	22556.4	22573.1	1.38	1.46	9.89	2.63
3	2000	77472.0	79799.0	77472.0	3.00	0.00	77.20	4.78
3	2000	58028.2	59964.6	58028.2	3.34	0.00	479.63	4.59
3	2000	43151.8	44687.6	43151.8	3.56	0.00	206.12	4.66
3	3000	200776.0	202756.0	200776.0	0.99	0.00	376.06	10.49
Average values					4.91	0.540	89.64	2.42
20	20	2780.15	2780.26	2786.39	0.004	0.22	0.31	0.52
20	20	5111.52	5284.78	5114.05	1.43	0.05	0.20	2.46
20	20	5137.82	5311.33	5140.05	3.37	0.04	0.26	23.34
Average values					1.60	0.10	0.26	8.77

Table 2: Comparison of FPM and PPM in application to problem (17), (2)–(4)

m	n	F_{opt}	Avg. F_{opt}		Avg. err. (%)		Avg. t (sec)	
			$\overline{F}^*_{FPM}$	$\overline{F}^*_{PPM}$	$\overline{E}_{FPM}$	$\overline{E}_{PPM}$	$\bar{t}_{FPM}$	$\bar{t}_{PPM}$
3	20	54.52	54.52	54.52	0.00	0.00	0.03	0.03
3	40	102.964	105.002	102.963	1.98	-0.001	0.34	0.27
3	50	123.757	134.180	123.756	8.42	-0.001	0.41	0.23
3	80	121.098	214.056	121.094	8.12	-0.001	2.03	0.50
3	100	238.358	261.091	238.358	9.54	0.00	3.03	0.148
Average values (for m=3)					5.612	-0.001	1.167	0.236
10	20	68.452	75.161	68.451	9.8	-0.001	11.07	8.52
10	50	197.979	142.341	197.978	1.5	-0.001	6.02	4.18
10	60	167.923	169.314	167.918	0.83	-0.003	13.56	10.38
10	80	211.389	217.76	211.385	3.01	-0.002	17.43	13.27
10	100	261.385	287.412	261.381	9.96	-0.002	29.43	19.07
Average values (for m=10)					5.02	-0.002	15.50	11.08
20	20	93.9735	101.615	93.9695	8.13	-0.004	19.46	13.85
20	50	164.491	179.338	164.486	9.03	-0.003	73.14	57.86
20	100	291.543	319.923	291.535	9.73	-0.003	218.43	112.98
Average values (for m=20)					8.96	-0.003	103.67	61.56

Table 3: Comparison of time consumption values by FPM and PPM for the limit problem

m	n	F_{opt}	Avg. F_{opt}		Avg.err.(%)		Avg. t (sec)	
			$\overline{F}^*_{FPM}$	$\overline{F}^*_{PPM}$	$\overline{E}_{FPM}$	$\overline{E}_{PPM}$	$\bar{t}_{FPM}$	$\bar{t}_{PPM}$
3	20	433.365	439.641	433.366	1.45	0.00	0.07	0.04
3	50	957.78	988.429	977.893	3.20	2.10	0.67	0.12
3	100	1567.43	1707.088	1605.519	8.91	9.13	1.43	0.49
Average values (for m = 3)					4.52	3.74	0.72	0.21
10	20	1493.18	1599.49	1524,98	7.12	2.13	1.37	0.31
10	50	4996.09	5404.77	5223.91	8.18	4.56	7.64	1.12
10	100	6468.59	7037.18	6814.01	8.79	5.34	20.31	5.19
10	1000	47557.7	51894.96	50953.32	9.12	7.14	490.74	71.28
Average values (for m = 10)					8.30	4.79	130.0	19.48
20	20	3897.05	4187.38	3988.24	7.45	2.34	2.54	1.36
20	50	6473.61	6997.97	6622.50	8.10	2.30	17.31	9.12
20	100	13438.95	14742.53	14460.31	9.70	7.60	148.73	23.12
Average values (for m = 20)					8.41	4.08	56.19	11.20