Design of Linear Functional Observers with H_∞ Performance

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Abstract—**The paper solves the problem of parameter designing for one class of linear functional observers. To solve this problem, a simple design procedure for providing the generalized structure framework based on H_∞ norm principle is presented. Related to estimation of given function of system state, the design steps is given out in the example to illustrate the properties of the functional observer.**

Keywords—**linear dynamical systems, state estimation, functional observers, linear matrix inequalities, bounded real lemma.**

I. Introduction

To implement a linear state feedback control law it is suitable to estimate only a linear function of the system state vector. Such observers are called function (functional) observers and realize linear control laws whose rank is the number of system inputs [17]. Functional observers for linear tim-invariant systems have been studied by many authors (see, e.g., [4], [14], [29] and the references therein), however, the most fundamental problem of finding the observer minimal order still remains unsolved [24]. This set of observers can be regarded as forming a part of a linear feedback control scheme used to generate the system state-dependent control law value estimation.

The observers subjected to a given linear state vector function allow implementation with a lover order of the observer dynamics [1], [25], can be constructed for linear time-varying system [20], [21] as well as used to disturbance attenuation for systems with unknown input disturbances [22], [23], [28]. This has narrowed the scope to which they can effectively be applied in practical cases. The associated theory is related to the fundamental linear system concepts of controllability, observability, and stability while the design procedures for observing a linear function of the state of a multiple input, multi output (MIMO) system are concerned with the minimal realization theory [10].

There exist different structures employed in the studies of linear functional observers [8]. Using the matrix pseudoinverse approach, a framework providing existence conditions for the r-order functional observer has been reported in [6]. Providing the set of design formulas [6], a derived structure is used in [7] to exploit a freedom in the output signal gain tuning in the functional observer structures. Since potential application in fault tolerant control (FTC) structures does not require an estimate of the entire system state, a new perspective of functional observers is convenient.

The paper is concerned with the problem of determining the functional observer for a given control law. The problem formulation, preliminary results and the separation principle in design of functional observers appear in Section II and Section III presents the pseudoinverse technique in design of given class of functional observers where the estimation error in the state function is treated as an additional disturbance input. Once the relations of the functional observer parameters are determined, a constructive procedure is given in Section IV for determining the observer dynamics, formulating the design conditions needed to ensure the existence of a functional observer by using linear matrix inequality (LMI) techniques. The necessary modifications, reflecting the forced mode control law structure, are additionally outlined in Section V. In response, Section VI shows the performance of the proposed approach using an application example and Sec. VII gives some concluding remarks.

Throughout the paper, the following notations are used: $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and the matrix $\boldsymbol{X}$, respectively, $\boldsymbol{Y}^{\ominus 1}$ designates Moore-Penrose pseudoinverse of the non-square matrix $\boldsymbol{Y}$, for a square matrix $\boldsymbol{X} < 0$ means that $\boldsymbol{X}$ is symmetric negative definite matrix, the symbol $\boldsymbol{I}_n$ indicates the n-th order unit matrix, $I\!R$ denotes the set of real numbers and $I\!R^n$, $I\!R^{n\times r}$ refers to the set of all n-dimensional real vectors and $n \times r$ real matrices.

II. System Description

In the paper is considered the class of linear dynamic systems which state-space description is

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{d}(t)\,, \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t)\,, \tag{2}$$

$$\boldsymbol{z}(t) = \boldsymbol{L}\boldsymbol{q}(t)\,, \tag{3}$$

where $\boldsymbol{q}(t) \in I\!R^n$, $\boldsymbol{u}(t) \in I\!R^r$, $\boldsymbol{y}(t) \in I\!R^m$ stand for the state, control input and measurable output, $\boldsymbol{z}(t) \in I\!R^h$ is the vector to be estimated, $\boldsymbol{d}(t) \in I\!R^p$ is unknown disturbance, $\boldsymbol{A} \in I\!R^{n\times n}$, $\boldsymbol{B} \in I\!R^{n\times r}$, $\boldsymbol{C} \in I\!R^{m\times n}$, $\boldsymbol{L} \in I\!R^{h\times n}$, $\boldsymbol{E} \in I\!R^{n\times p}$ are known real matrices.

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Throughout the paper is considered the following standard assumptions that the pair $(\boldsymbol{A},\boldsymbol{B})$ is controllable and the pair $(\boldsymbol{A},\boldsymbol{C})$ is observable. Moreover, the following basic propositions are introduced, which will play an important role in the proof of main theorem presented below.

Proposition 1: [11] (Eigenvalue matrix decomposition) Let $\boldsymbol{Y} \in I\!R^{m\times n}$ is a real square matrix with n linearly independent eigenvectors $\boldsymbol{n}_i$, $i = 1,2,\ldots,n$. Then $\boldsymbol{Y}$ can be factorized as

$$\boldsymbol{Y} = \boldsymbol{N}\boldsymbol{Z}\boldsymbol{N}^{-1}, \tag{4}$$

where $\boldsymbol{N} \in I\!R$ is the square matrix whose i-th column is the eigenvector $\boldsymbol{n}_i$ of $\boldsymbol{Y}$ and $\boldsymbol{Z}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues.

Proposition 2: [2] (Matrix pseudoinverse) Let $\boldsymbol{\Theta}$ is a matrix and $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{\Lambda}$ are known non-square matrices of appropriate dimensions such that there yields the equality

$$\boldsymbol{X}\boldsymbol{\Theta}\boldsymbol{Y} = \boldsymbol{\Lambda}\,. \tag{5}$$

Then all solution to $\boldsymbol{\Theta}$ means

$$\boldsymbol{\Theta} = \boldsymbol{X}^{\ominus 1}\boldsymbol{\Lambda}\boldsymbol{Y}^{\ominus 1} + \boldsymbol{\Theta}^{\circ} - \boldsymbol{X}^{\ominus 1}\boldsymbol{X}\boldsymbol{\Theta}^{\circ}\boldsymbol{Y}\boldsymbol{Y}^{\ominus 1}, \tag{6}$$

where

$$\boldsymbol{X}^{\ominus 1} = \boldsymbol{X}^{T}(\boldsymbol{X}\boldsymbol{X}^{T})^{-1}, \quad \boldsymbol{Y}^{\ominus 1} = (\boldsymbol{Y}^{T}\boldsymbol{Y})^{-1}\boldsymbol{Y}^{T}, \tag{7}$$

is Moore-Penrose pseudoinverse of $\boldsymbol{X}$, $\boldsymbol{Y}$, respectively, and $\boldsymbol{\Theta}^{\circ}$ is an arbitrary matrix of appropriate dimension.

Proposition 3: [5] (Lyapunov inequality) Autonomous system (1), (2) with a bounded disturbance is asymptotically stable if there exist a symmetric positive definite matrix $\boldsymbol{X} \in I\!R^{n\times n}$ such that

$$\boldsymbol{X} = \boldsymbol{X}^{T} > 0\,, \tag{8}$$

$$\boldsymbol{X}\boldsymbol{A} + \boldsymbol{A}^{T}\boldsymbol{X} < 0\,. \tag{9}$$

Proposition 4: [9] (quadratic performance) If the matrix $\boldsymbol{A}$ of system (1), (2) is stable and the disturbance $\boldsymbol{d}(t)$ is bounded then

$$\gamma_{\infty}^{-1}\int_{0}^{\infty}(\boldsymbol{y}^{T}(t)\boldsymbol{y}(t) - \gamma_{\infty}\boldsymbol{d}^{T}(t)\boldsymbol{d}(t))\mathrm{d}t > 0\,, \tag{10}$$

where $\gamma_{\infty} \in I\!R$ is the H_{∞} norm of the disturbance transfer function matrix.

Proposition 5: [3], [26] (Bounded real lemma (BRL)) Autonomous system (1), (2) with a bounded disturbance is asymptotically stable if there exist a symmetric positive definite matrix $\boldsymbol{X} \in I\!R^{n\times n}$ and a positive scalar $\gamma_{\infty} \in I\!R$ such that

$$\boldsymbol{X} = \boldsymbol{X}^{T} > 0\,, \quad \gamma_{\infty} > 0\,, \tag{11}$$

$$\begin{bmatrix} \boldsymbol{X}\boldsymbol{A} + \boldsymbol{A}^{T}\boldsymbol{X} & * & * \\ \boldsymbol{E}^{T}\boldsymbol{X} & -\gamma_{\infty}\boldsymbol{I}_{p} & * \\ \boldsymbol{C} & \boldsymbol{0} & -\gamma_{\infty}\boldsymbol{I}_{m} \end{bmatrix} < 0\,. \tag{12}$$

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

III. Functional Observer

In order to estimate $\boldsymbol{z}(t)$ it is proposed a functional observer of the form [7]

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{P}\boldsymbol{p}(t) + \boldsymbol{Q}\boldsymbol{u}(t) + \boldsymbol{J}\boldsymbol{y}(t)\,, \tag{13}$$

$$\boldsymbol{z}_{e}(t) = \boldsymbol{R}\boldsymbol{p}(t) + \boldsymbol{O}\boldsymbol{y}(t)\,, \tag{14}$$

where the functional observer initial state $\boldsymbol{p}(0)$ is arbitrary, $\boldsymbol{z}_e(t) \in I\!R^{h}$ is an estimate of $\boldsymbol{z}(t)$, $\boldsymbol{p}(t) \in I\!R^{h}$ is the state vector of the functional observer and the matrices $\boldsymbol{P} \in I\!R^{h\times h}$, $\boldsymbol{Q} \in I\!R^{h\times r}$, $\boldsymbol{J} \in I\!R^{h\times m}$, $\boldsymbol{O} \in I\!R^{h\times m}$, $\boldsymbol{R} \in I\!R^{h\times h}$ are the observer matrix parameters to bee designed.

In general, the main objective is to design the functional observer parameters such that the following errors in estimation

$$\boldsymbol{e}_{z}(t) = \boldsymbol{z}(t) - \boldsymbol{z}_{e}(t)\,, \tag{15}$$

$$\boldsymbol{e}(t) = \boldsymbol{S}\boldsymbol{q}(t) - \boldsymbol{p}(t)\,, \tag{16}$$

converge asymptotically towards the zero equilibrium vectors when $t \to \infty$.

Note that if the pair $(\boldsymbol{A},\boldsymbol{C})$ is unobservable, it is impossible to estimate all system states. However, state function $\boldsymbol{z}(t) = \boldsymbol{L}\boldsymbol{q}(t)$ can be estimated.

Lemma 1: The system state estimation error $\boldsymbol{e}(t)$ can be described by the equation

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{P}\boldsymbol{e}(t) + \boldsymbol{S}\boldsymbol{E}\boldsymbol{d}(t) \tag{17}$$

if it is satisfied

$$\boldsymbol{S}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{S} - \boldsymbol{J}\boldsymbol{C} = \boldsymbol{0}\,, \tag{18}$$

$$\boldsymbol{S}\boldsymbol{B} - \boldsymbol{Q} = \boldsymbol{0}\,, \tag{19}$$

and, additionally,

$$\boldsymbol{L} = \boldsymbol{R}\boldsymbol{S} + \boldsymbol{O}\boldsymbol{C}\,. \tag{20}$$

Proof: If the structure of the error vector (16) it is prescribed, then $\boldsymbol{e}(t)$ propagates as

$$\begin{aligned} \dot{\boldsymbol{e}}(t) &= \boldsymbol{S}\dot{\boldsymbol{q}}(t) - \dot{\boldsymbol{p}}(t) = \\ &= \boldsymbol{S}(\boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{d}(t)) - \\ &\quad -(\boldsymbol{P}\boldsymbol{p}(t) + \boldsymbol{Q}\boldsymbol{u}(t) + \boldsymbol{J}\boldsymbol{y}(t)) = \\ &= (\boldsymbol{S}\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{q}(t) + (\boldsymbol{S}\boldsymbol{B} - \boldsymbol{Q})\boldsymbol{u}(t) - \\ &\quad -\boldsymbol{P}\boldsymbol{p}(t) + \boldsymbol{S}\boldsymbol{E}\boldsymbol{d}(t)\,. \end{aligned} \tag{21}$$

Since (16) gives

$$\boldsymbol{p}(t) = \boldsymbol{S}\boldsymbol{q}(t) - \boldsymbol{e}(t)\,, \tag{22}$$

substituting (22) in (21) the expression to (21) may be re written in terms of the FO matrices as

$$\begin{aligned} \dot{\boldsymbol{e}}(t) &= \boldsymbol{P}\boldsymbol{e}(t) + \boldsymbol{S}\boldsymbol{E}\boldsymbol{d}(t) + \\ &+(\boldsymbol{S}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{S} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{q}(t) + (\boldsymbol{S}\boldsymbol{B} - \boldsymbol{Q})\boldsymbol{u}(t)\,, \end{aligned} \tag{23}$$

which implies an autonomous dynamics (17) for the disturbance free system, if (18), (19) are satisfied. This concludes the proof. ■

Remark 1: Substituting (22) in (14) entails consequently also that

$$\boldsymbol{z}_e(t) = -\boldsymbol{R}\boldsymbol{e}(t) + (\boldsymbol{R}\boldsymbol{S} + \boldsymbol{O}\boldsymbol{C})\boldsymbol{q}(t)\,, \tag{24}$$

which, using (20), gives

$$\boldsymbol{z}_e(t) = -\boldsymbol{R}\boldsymbol{e}(t) + \boldsymbol{L}\boldsymbol{q}(t)\,. \tag{25}$$

Then, immediately, the evolution of the estimation error $\boldsymbol{e}_z(t)$ is described by the equation

$$\boldsymbol{e}_z(t) = \boldsymbol{L}\boldsymbol{q}(t) + \boldsymbol{R}\boldsymbol{e}(t) - \boldsymbol{L}\boldsymbol{q}(t) = \boldsymbol{R}\boldsymbol{e}(t)\,, \tag{26}$$

which means that if $\boldsymbol{e}(t)$ converges to the equilibrium point, also $\boldsymbol{e}_z(t)$ converges to its equilibrium.

Lemma 2: (separation principle) The equivalent state-space description of the system (1)-(3) with the functional observer (13), (14) is of the form

$$\begin{bmatrix} \dot{\boldsymbol{q}}(t) \\ \dot{\boldsymbol{e}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}+\boldsymbol{B}\boldsymbol{L} & -\boldsymbol{B}\boldsymbol{R} \\ \boldsymbol{0} & \boldsymbol{P} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{e}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{S}\boldsymbol{E} \end{bmatrix} \boldsymbol{d}(t)\,, \tag{27}$$

$$\boldsymbol{z}_e(t) = \begin{bmatrix} \boldsymbol{L} & -\boldsymbol{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \boldsymbol{e}(t) \end{bmatrix}. \tag{28}$$

Evidently, (27), (28) establish a separation principle in design of an observer-based feedback stabilization scheme and the functional observer dynamics, implying that the observer matrix $\boldsymbol{P}$ can be designed autonomously.

Proof: Prescribing that

$$\boldsymbol{u}(t) = \boldsymbol{z}_e(t) = \boldsymbol{L}\boldsymbol{q}(t) - \boldsymbol{R}\boldsymbol{e}(t) \tag{29}$$

and inserting (29) in (1), then it yields

$$\dot{\boldsymbol{q}}(t) = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{L})\boldsymbol{q}(t) - \boldsymbol{B}\boldsymbol{R}\boldsymbol{e}(t) + \boldsymbol{E}\boldsymbol{d}(t)\,. \tag{30}$$

Combining (1), (29) with (17) and (25) implies the structure (27), (28). This concludes the proof. ■

Lemma 3: The system matrix $\boldsymbol{U} \in I\!R^{h\times h}$ is conditioned by the relation

$$\boldsymbol{U} = \boldsymbol{U}_1 - \boldsymbol{Z}\boldsymbol{U}_2\,, \tag{31}$$

where

$$\boldsymbol{U}_1 = \left(\boldsymbol{L}\boldsymbol{A} - \boldsymbol{L}\boldsymbol{A}\boldsymbol{N}_L\boldsymbol{\Sigma}^{\ominus 1}\boldsymbol{\Xi}\right)\boldsymbol{L}^{\ominus 1}\,, \tag{32}$$

$$\boldsymbol{U}_2 = (\boldsymbol{I}_{2m} - \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\ominus 1})\boldsymbol{\Xi}\boldsymbol{L}^{\ominus 1}\,, \tag{33}$$

$$\boldsymbol{N}_L = \boldsymbol{I}_n - \boldsymbol{L}^{\ominus 1}\boldsymbol{L}\,, \tag{34}$$

$$\boldsymbol{\Xi} = \begin{bmatrix} \boldsymbol{C}\boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix}, \quad \boldsymbol{\Sigma} = \boldsymbol{\Xi}\boldsymbol{N}_L\,, \tag{35}$$

$$\boldsymbol{U} = \boldsymbol{R}\boldsymbol{P}\boldsymbol{R}^{-1}, \tag{36}$$

while

$$\boldsymbol{L}^{\ominus 1} = \boldsymbol{L}^T(\boldsymbol{L}\boldsymbol{L}^T)^{-1} \tag{37}$$

is the right Moore-Penrose pseudoinverse of the non-square matrix $\boldsymbol{L}$,

$$\boldsymbol{\Sigma}^{\ominus 1} = \boldsymbol{\Sigma}^T(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T)^{-1} \tag{38}$$

is the generalized Moore-Penrose pseudoinverse of the matrix $\boldsymbol{\Sigma}$ and $\boldsymbol{Z} \in I\!R^{h\times 2m}$ is an arbitrary matrix.

Proof: Starting from the relation (20) and despatching $\boldsymbol{S}$ as follows

$$\boldsymbol{S} = \boldsymbol{R}^{-1}\boldsymbol{L} - \boldsymbol{R}^{-1}\boldsymbol{O}\boldsymbol{C}\,, \tag{39}$$

it may be deduced that (18) is described by the following compact form

$$\boldsymbol{S}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{R}^{-1}(\boldsymbol{L} - \boldsymbol{O}\boldsymbol{C}) - \boldsymbol{J}\boldsymbol{C} = \boldsymbol{0} \tag{40}$$

and, subsequently, it follows directly that

$$\boldsymbol{P}\boldsymbol{R}^{-1}\boldsymbol{L} = \boldsymbol{S}\boldsymbol{A} - (\boldsymbol{J} - \boldsymbol{P}\boldsymbol{R}^{-1}\boldsymbol{O})\boldsymbol{C}\,. \tag{41}$$

Nominating the notation

$$\boldsymbol{R}^{-1}\boldsymbol{H} = \boldsymbol{J} - \boldsymbol{P}\boldsymbol{R}^{-1}\boldsymbol{O}\,, \tag{42}$$

(41) can be represented as

$$\boldsymbol{P}\boldsymbol{R}^{-1}\boldsymbol{L} = \boldsymbol{S}\boldsymbol{A} - \boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C}\,. \tag{43}$$

Multiplying the right side of (42) by the transpose of the gain matrix $\boldsymbol{L}^T$ leads to

$$\boldsymbol{P}\boldsymbol{R}^{-1}\boldsymbol{L}\boldsymbol{L}^T = (\boldsymbol{S}\boldsymbol{A} - \boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C})\boldsymbol{L}^T, \tag{44}$$

which implies the particular solution of the matrix product $\boldsymbol{P}\boldsymbol{R}^{-1}$ of the form

$$\boldsymbol{P}\boldsymbol{R}^{-1} = (\boldsymbol{S}\boldsymbol{A} - \boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C})\boldsymbol{L}^{\ominus 1}, \tag{45}$$

with the right Moore-Penrose pseudoinverse $\boldsymbol{L}^{\ominus 1}$ defined in (20).

Moreover, multiplying the right side of (39) by the system matrix $\boldsymbol{A}$ gives

$$\boldsymbol{S}\boldsymbol{A} = \boldsymbol{R}^{-1}\boldsymbol{L}\boldsymbol{A} - \boldsymbol{R}^{-1}\boldsymbol{O}\boldsymbol{C}\boldsymbol{A} \tag{46}$$

and it yields

$$\begin{aligned} &\boldsymbol{S}\boldsymbol{A} - \boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C} = \\ &= \boldsymbol{R}^{-1}(\boldsymbol{L}\boldsymbol{A} - \boldsymbol{O}\boldsymbol{C}\boldsymbol{A} - \boldsymbol{H}\boldsymbol{C}) = \\ &= \boldsymbol{R}^{-1}(\boldsymbol{L}\boldsymbol{A} - \begin{bmatrix} \boldsymbol{O} & \boldsymbol{H} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}\boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix}, \end{aligned} \tag{47}$$

as well as, using (35),

$$\boldsymbol{S}\boldsymbol{A} - \boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C} = \boldsymbol{R}^{-1}\left(\boldsymbol{L}\boldsymbol{A} - \begin{bmatrix} \boldsymbol{O} & \boldsymbol{H} \end{bmatrix}\boldsymbol{\Xi}\right). \tag{48}$$

Then, substituting (48) in (45) and post-multiplying the left side by the matrix $\boldsymbol{R}$, a solution takes the form

$$\boldsymbol{R}\boldsymbol{P}\boldsymbol{R}^{-1} = \left(\boldsymbol{L}\boldsymbol{A} - \begin{bmatrix} \boldsymbol{O} & \boldsymbol{H} \end{bmatrix}\boldsymbol{\Xi}\right)\boldsymbol{L}^{\ominus 1}. \tag{49}$$

Multiplying the right side of (43) by $\boldsymbol{L}$ means, with respect to (43) and (44) that

$$\boldsymbol{P}\boldsymbol{R}^{-1}\boldsymbol{L} = \boldsymbol{S}\boldsymbol{A} - \boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C} = (\boldsymbol{S}\boldsymbol{A} - \boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C})\boldsymbol{L}^{\ominus 1}\boldsymbol{L}, \tag{50}$$

which can be interpreted as

$$(\boldsymbol{S}\boldsymbol{A} - \boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C})(\boldsymbol{I}_n - \boldsymbol{L}^{\ominus 1}\boldsymbol{L}) = (\boldsymbol{S}\boldsymbol{A} - \boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C})\boldsymbol{N}_L = \boldsymbol{0}, \tag{51}$$

where $\boldsymbol{N}_L$ is an orthogonal projector of $\boldsymbol{L}$.

It is evident that with (48) it has to be

$$(\boldsymbol{R}^{-1}\boldsymbol{L}\boldsymbol{A} - \boldsymbol{R}^{-1}\boldsymbol{O}\boldsymbol{C}\boldsymbol{A} - \boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C})\boldsymbol{N}_L = \boldsymbol{0}\,, \tag{52}$$

which implies using the same notation as in (49) that

$$\boldsymbol{R}^{-1}\left(\boldsymbol{L}\boldsymbol{A}-\left[\boldsymbol{O}\ \ \boldsymbol{H}\right]\boldsymbol{\Xi}\right))\boldsymbol{N}_L=\boldsymbol{0}\,, \tag{53}$$

$$\boldsymbol{L}\boldsymbol{A}\boldsymbol{N}_L=\left[\boldsymbol{O}\ \ \boldsymbol{H}\right]\boldsymbol{\Xi}\boldsymbol{N}_L=\left[\boldsymbol{O}\ \ \boldsymbol{H}\right]\boldsymbol{\Sigma}\,, \tag{54}$$

respectively.

Considering in general that $n \geq 2h$, then (54) can be rewritten as

$$\left[\boldsymbol{O}\ \ \boldsymbol{H}\right]\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T=\boldsymbol{L}\boldsymbol{A}\boldsymbol{N}_L\boldsymbol{\Sigma}^T \tag{55}$$

that is, similar to (49), the particular solution of $[\ \boldsymbol{O}\ \boldsymbol{H}\ ]$ takes the following form

$$\left[\boldsymbol{O}\ \ \boldsymbol{H}\right]=\boldsymbol{L}\boldsymbol{A}\boldsymbol{N}_L\boldsymbol{\Sigma}^{\ominus 1}, \tag{56}$$

where the generalized Moore-Penrose pseudoinverse of the matrix $\boldsymbol{\Sigma}$ is given in (37).

Multiplying the right side (56) by the composed matrix $\boldsymbol{\Sigma}$ gives

$$\left[\boldsymbol{O}\ \ \boldsymbol{H}\right]\boldsymbol{\Sigma}=\boldsymbol{L}\boldsymbol{A}\boldsymbol{N}_L\boldsymbol{\Sigma}^{\ominus 1}\boldsymbol{\Sigma}=\left[\boldsymbol{O}\ \ \boldsymbol{H}\right]\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\ominus 1}\boldsymbol{\Sigma}\,, \tag{57}$$

which can be interpreted as

$$\left[\boldsymbol{O}\ \ \boldsymbol{H}\right](\boldsymbol{I}_{2m}-\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\ominus 1})\boldsymbol{\Sigma}=\boldsymbol{0} \tag{58}$$

and, using an arbitrary matrix $\boldsymbol{Z}$, in the sense of Proposition 2, the general solution of $[\ \boldsymbol{O}\ \boldsymbol{H}\ ]$ can be considered as

$$\left[\boldsymbol{O}\ \ \boldsymbol{H}\right]=\boldsymbol{L}\boldsymbol{A}\boldsymbol{N}_L\boldsymbol{\Sigma}^{\ominus 1}+\boldsymbol{Z}(\boldsymbol{I}_{2m}-\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\ominus 1})\,. \tag{59}$$

Since (49) constitutes the relation

$$\boldsymbol{U}=\boldsymbol{R}\boldsymbol{P}\boldsymbol{R}^{-1}=\left(\boldsymbol{L}\boldsymbol{A}-\left[\boldsymbol{O}\ \ \boldsymbol{H}\right]\boldsymbol{\Xi}\right)\boldsymbol{L}^{\ominus 1}, \tag{60}$$

substituting (60) then naturally (59) implies (31)-(33). This concludes the proof. ■

Given Lemma 3 provides the basic guideline for functional observer design using matrix pseudoinverse approach. It is not suitable for testing the functional observer existence since, depending on the matrix $\boldsymbol{Z}$, all the matrices in the above equations are unknown. Moreover, it has to be satisfied the following necessary and sufficient condition for the existence of the functional observer (13), (14).

Proposition 6: [6] The linear functional observer (13), (14) associated with the system closed-loop system (27), (28) exists if

$$\operatorname{rank}\begin{bmatrix}\boldsymbol{L}\boldsymbol{A}\\ \boldsymbol{C}\boldsymbol{A}\\ \boldsymbol{C}\\ \boldsymbol{L}\end{bmatrix}=\operatorname{rank}\begin{bmatrix}\boldsymbol{C}\boldsymbol{A}\\ \boldsymbol{C}\\ \boldsymbol{L}\end{bmatrix}. \tag{61}$$

Because this condition reflects the properties of the system, they can be checked for existence of a functional observer.

IV. Functional Observer Design

The following corollary is proposed for the functional observer design to select a matrix $\boldsymbol{R}$.

Corollary 1: If a matrix $\boldsymbol{U} \in I\!R^{h\times h}$ is a real square matrix with h linearly independent real eigenvectors $\boldsymbol{r}_i$, $i = 1, 2, \dots, h$, then, using (4), $\boldsymbol{U}$ can be factorized as

$$\boldsymbol{U}=\boldsymbol{R}\boldsymbol{P}\boldsymbol{R}^{-1}, \tag{62}$$

where $\boldsymbol{R} \in I\!R^{h\times h}$ is the square matrix whose i-th column is the eigenvector $\boldsymbol{r}_i$ of $\boldsymbol{U}$ and $\boldsymbol{P}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues.

Let a matrix $\boldsymbol{U} \in I\!R^{h\times h}$ is a real square matrix with h linearly independent eigenvectors $\boldsymbol{r}_i$, $i = 1, 2, \dots, h$ from which two are complex conjugated (that is they are associated wit a pair of complex conjugated eigenvalues). Considering, for example, (62) as follows

$$\boldsymbol{R}^{-1}\boldsymbol{U}\boldsymbol{R}=\boldsymbol{P}\,, \tag{63}$$

where

$$\boldsymbol{P}=\begin{bmatrix}s_1 & \cdots & 0 & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & \cdots & s_{h-2} & 0 & 0\\ 0 & \cdots & 0 & s_\alpha+js_\omega & 0\\ 0 & \cdots & 0 & 0 & s_\alpha-js_\omega\end{bmatrix}. \tag{64}$$

$$\boldsymbol{R}=\left[\boldsymbol{r}_1\ \ \cdots\ \ \boldsymbol{r}_{h-2}\ \ \boldsymbol{r}_{\alpha h}+j\boldsymbol{r}_{\omega h}\ \ \boldsymbol{r}_{\alpha h}-j\boldsymbol{r}_{\omega h}\right], \tag{65}$$

then it can write

$$\boldsymbol{U}\boldsymbol{R}\boldsymbol{T}=\boldsymbol{R}\boldsymbol{T}\boldsymbol{T}^{-1}\boldsymbol{P}\boldsymbol{T}\,, \tag{66}$$

$$\boldsymbol{T}=\operatorname{diag}\left[\boldsymbol{I}_{h-2}\ \ 0.5\boldsymbol{T}_q\right],\quad \boldsymbol{T}_q=\begin{bmatrix}1 & j\\ 1 & -j\end{bmatrix}, \tag{67}$$

$$\boldsymbol{T}^{-1}=\operatorname{diag}\left[\boldsymbol{I}_{h-2}\ \ \boldsymbol{T}_p\right],\quad \boldsymbol{T}_p=\begin{bmatrix}1 & 1\\ -j & j\end{bmatrix}. \tag{68}$$

$$0.5\boldsymbol{T}_p\boldsymbol{T}_q=\boldsymbol{I}_h\,. \tag{69}$$

Since it can be easily verified that

$$\boldsymbol{P}_s=\boldsymbol{T}^{-1}\boldsymbol{P}\boldsymbol{T}=\begin{bmatrix}s_1 & \cdots & 0 & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & \cdots & s_{h-2} & 0 & 0\\ 0 & \cdots & 0 & s_\alpha & -s_\omega\\ 0 & \cdots & 0 & s_\omega & s_\alpha\end{bmatrix}, \tag{70}$$

$$\boldsymbol{R}_s=\boldsymbol{R}\boldsymbol{T}=\left[\boldsymbol{r}_1\ \ \cdots\ \ \boldsymbol{r}_{h-2}\ \ \boldsymbol{r}_{\alpha h}\ \ -\boldsymbol{r}_{\omega h}\right], \tag{71}$$

the relation (66) takes the form

$$\boldsymbol{U}\boldsymbol{R}_s=\boldsymbol{R}_s\boldsymbol{P}_s\,, \tag{72}$$

where $\boldsymbol{R}_s$, $\boldsymbol{P}_s$ are real matrices and the relation

$$\boldsymbol{U}=\boldsymbol{R}_s\boldsymbol{P}_s\boldsymbol{R}_s^{-1} \tag{73}$$

gives the possibility to use in FO design $\boldsymbol{R}_s$, $\boldsymbol{P}_s$.

The approach can also be applied in the case of occurrence of further pairs of complex conjugated eigenvalues in $\boldsymbol{U}$ [16].

Exploiting (62) or (73), the following theorems provide conditions for convergence of the linear system state function.

Theorem 1: The functional observer (13), (14) is asymptotically stable if there exist a positive definite symmetric matrix $\boldsymbol{X} \in I\!R^{h\times h}$, and a matrix $\boldsymbol{Y} \in I\!R^{h\times n}$ such that

$$\boldsymbol{X} = \boldsymbol{X}^T > 0\,, \tag{74}$$

$$\boldsymbol{X}\boldsymbol{U}_1 + \boldsymbol{U}_1^T\boldsymbol{X} - \boldsymbol{Y}\boldsymbol{U}_2 - \boldsymbol{U}_2^T\boldsymbol{Y} < 0\,. \tag{75}$$

If the above inequalities are satisfied, the auxiliary matrix variables can be computed as follows

$$\boldsymbol{Z} = \boldsymbol{X}^{-1}\boldsymbol{Y}, \tag{76}$$

$$\boldsymbol{U} = \boldsymbol{U}_1 - \boldsymbol{Z}\boldsymbol{U}_2\,, \tag{77}$$

$$\begin{bmatrix}\boldsymbol{O} & \boldsymbol{H}\end{bmatrix} = \boldsymbol{L}\boldsymbol{A}\boldsymbol{N}_L\boldsymbol{\Sigma}^{\ominus 1} + \boldsymbol{Z}(\boldsymbol{I}_{2m} - \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\ominus 1})\,. \tag{78}$$

where the matrices $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, $\boldsymbol{N}_L$ and $\boldsymbol{\Sigma}$ are given as in (32)–(35).

Consequently, using (62) or (73) to construct the regular matrices $\boldsymbol{P}$, $\boldsymbol{R}$ for the obtained Hurwitz matrix $\boldsymbol{U}$, then the observer parameters are solved as

$$\boldsymbol{S} = \boldsymbol{R}^{-1}(\boldsymbol{L} - \boldsymbol{O}\boldsymbol{C})\,, \tag{79}$$

$$\boldsymbol{Q} = \boldsymbol{S}\boldsymbol{B}\,, \tag{80}$$

$$\boldsymbol{J} = \boldsymbol{R}^{-1}\boldsymbol{H} + \boldsymbol{P}\boldsymbol{R}^{-1}\boldsymbol{O}\,. \tag{81}$$

Proof: Adapting (8), (9), with respect to the matrix dimensions and the actually given notations, then for the autonomous part of (17) it can write using a positive definite matrix $\boldsymbol{X} \in I\!R^{h\times h}$ that

$$\boldsymbol{X}\boldsymbol{U} + \boldsymbol{U}^T\boldsymbol{X} < 0\,. \tag{82}$$

Therefore, substituting (77) the inequality (82) takes the form

$$\boldsymbol{X}(\boldsymbol{U}_1 - \boldsymbol{Z}\boldsymbol{U}_2) + (\boldsymbol{U}_1 - \boldsymbol{Z}\boldsymbol{U}_2)^T\boldsymbol{X} < 0 \tag{83}$$

and using the notation

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{Z}, \tag{84}$$

then (83) implies (75) and (84) gives (77).

Constructing the composed matrix (78) and separating the observer parameter $\boldsymbol{O}$, then the relation (79) can be set using (20) and, subsequently, the condition (19) specifies (80).

Consequently, separating the auxiliary matrix variable $\boldsymbol{H}$ from (78), then the relation (42) implies (81). This concludes the proof. ■

Note, the problem is reduced to find a Hurwitz matrix $\boldsymbol{U}$ such that the observer is stable. Evidently, for an arbitrary regular matrix $\boldsymbol{R}$, the condition (62) implies that the eigenvalues of $\boldsymbol{U}$ and $\boldsymbol{P}$ are identical. In this sense, to tune the functional observer responses, an arbitrary positive (negative) definite square matrix $\boldsymbol{R}$ of appropriate dimension can be used in design, while an acceptable trivial choice is the identity matrix.

Theorem 2: The functional observer (13), (14) is asymptotically stable with a quadratic constraints γ_∞ if there exist a positive definite symmetric matrix $\boldsymbol{X} \in I\!R^{h\times h}$, a matrix $\boldsymbol{Y} \in I\!R^{h\times n}$ and a positive scalar $\gamma_\infty \in I\!R$ such that

$$\boldsymbol{X} = \boldsymbol{X}^T > 0\,, \quad \gamma_\infty > 0\,, \tag{85}$$

$$\begin{bmatrix}\boldsymbol{X}\boldsymbol{U}_1 + \boldsymbol{U}_1^T\boldsymbol{X} - \boldsymbol{Y}\boldsymbol{U}_2 - \boldsymbol{U}_2^T\boldsymbol{Y} & * & * \\ \boldsymbol{X} & -\gamma_\infty\boldsymbol{I}_h & * \\ \boldsymbol{I}_h & \boldsymbol{0} & -\gamma_\infty\boldsymbol{I}_h\end{bmatrix} < 0\,. \tag{86}$$

If the inequalities are satisfied, $\boldsymbol{Z}$ can be computed as

$$\boldsymbol{Z} = \boldsymbol{X}^{-1}\boldsymbol{Y} \tag{87}$$

and exploiting the solution of (87), the FO parameters can be computed by using (77)–(81).

Proof: To adapt BRL structure (12) for FO parameter design, it is necessary at first to reformulate the quadratic constraint (10). Using (17) and defining

$$\boldsymbol{e}(t) = \boldsymbol{R}^{-1}\boldsymbol{e}_o(t)\,, \tag{88}$$

it can write using (62) that

$$\begin{aligned}&\dot{\boldsymbol{e}}_o(t) = \\ &= \boldsymbol{R}\boldsymbol{P}\boldsymbol{R}^{-1}\boldsymbol{e}_o(t) + \boldsymbol{R}\boldsymbol{S}\boldsymbol{E}\boldsymbol{d}(t) = \boldsymbol{U}\boldsymbol{e}_o(t) + \boldsymbol{E}_o\boldsymbol{d}_o(t)\,,\end{aligned} \tag{89}$$

where

$$\boldsymbol{E}_o = \boldsymbol{I}_h\,, \tag{90}$$

$$\boldsymbol{d}_o(t) = \boldsymbol{R}\boldsymbol{S}\boldsymbol{E}\boldsymbol{d}(t) \tag{91}$$

and to prescribe

$$\boldsymbol{y}_o(t) = \boldsymbol{e}_o(t) = \boldsymbol{C}_o\boldsymbol{e}_o(t)\,, \tag{92}$$

where

$$\boldsymbol{C}_o = \boldsymbol{I}_h\,. \tag{93}$$

Thus, replacing in (12) the matrix $\boldsymbol{A}$ by $\boldsymbol{U}$, the matrix $\boldsymbol{E}$ by $\boldsymbol{E}_o$ and the matrix $\boldsymbol{C}$ by $\boldsymbol{C}_o$, as well as adequate modifying the dimensions of the matrix inequality blocks, then it is obtained

$$\begin{bmatrix}\boldsymbol{X}\boldsymbol{U} + \boldsymbol{U}^T\boldsymbol{X} & \boldsymbol{X} & \boldsymbol{I}_h \\ \boldsymbol{X} & -\gamma_\infty\boldsymbol{I}_h & \boldsymbol{0} \\ \boldsymbol{I}_h & \boldsymbol{0} & -\gamma_\infty\boldsymbol{I}_h\end{bmatrix} < 0\,. \tag{94}$$

Finally, substituting (77) and using (84) then (94) implies (86). This concludes the proof. ■

Note, using the design conditions (85), (86), the obtained functional observer is stable and satisfies the condition

$$\gamma_\infty^{-1}\int_0^\infty (\boldsymbol{e}_o^T(t)\boldsymbol{e}_o(t) - \gamma_\infty\boldsymbol{d}_o^T(t)\boldsymbol{d}_o(t))\mathrm{d}t > 0\,, \tag{95}$$

where $\gamma_\infty \in I\!R$ is the H_∞ norm of the generalized disturbance transfer function matrix $\boldsymbol{G}_{od}(s)$. Since

$$\boldsymbol{G}_d(s) = \boldsymbol{R}^{-1}(s\boldsymbol{I}_h - \boldsymbol{U})^{-1}\boldsymbol{R}\boldsymbol{S}\boldsymbol{E} = (s\boldsymbol{I}_h - \boldsymbol{P})^{-1}\boldsymbol{S}\boldsymbol{E}, \tag{96}$$

it is evident that by selecting the matrix $\boldsymbol{R}$ can not change dynamics of the disturbance action on the estimator.

V. Forced Mode Control

In practice, the case with $r = m$ (square plants) is often encountered, where it is associated with each output signal a reference signal. Such regime is usually called the forced regime.

Definition 1: The forced regime for (1)-(3) is given by the control policy

$$\boldsymbol{u}(t) = \boldsymbol{L}\boldsymbol{q}(t) + \boldsymbol{W}\boldsymbol{w}(t)\,, \tag{97}$$

where $r = m$, $\boldsymbol{w}(t) \in I\!R^m$ is desired output signal vector, and $\boldsymbol{W} \in I\!R^{m\times m}$ is the signal gain matrix.

Theorem 3: If the system (1)-(3) is stabilizable by the control policy (97) and [27]

$$\text{rank}\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = n + m\,, \tag{98}$$

then the matrix $\boldsymbol{W}$ takes the form

$$\boldsymbol{W} = -(\boldsymbol{C}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{O}_c)^{-1}\boldsymbol{U}^{-1}\boldsymbol{H}\boldsymbol{C})^{-1}\boldsymbol{B})^{-1}, \tag{99}$$

where

$$\boldsymbol{O}_c = \boldsymbol{I}_r + \boldsymbol{U}^{-1}(\boldsymbol{L} - \boldsymbol{O}\boldsymbol{C})\boldsymbol{B}\,. \tag{100}$$

Proof: In a steady-state, the disturbance-free system equations (1)-(3) and the functional observer equations (13), (14) imply

$$\boldsymbol{0} = \boldsymbol{A}\boldsymbol{q}_o + \boldsymbol{B}\boldsymbol{u}_o\,, \tag{101}$$

$$\boldsymbol{0} = \boldsymbol{P}\boldsymbol{p}_o + \boldsymbol{Q}\boldsymbol{u}_o + \boldsymbol{J}\boldsymbol{C}\boldsymbol{q}_o\,, \tag{102}$$

where $\boldsymbol{q}_o$, $\boldsymbol{u}_o$, $\boldsymbol{p}_o$ are the steady-state values vectors of $\boldsymbol{q}(t)$, $\boldsymbol{u}(t)$, $\boldsymbol{p}(t)$, respectively.

Since (16) and (29) in a steady-state give

$$\boldsymbol{u}_o = \boldsymbol{L}\boldsymbol{q}_o - \boldsymbol{R}\boldsymbol{e}_o = (\boldsymbol{L} - \boldsymbol{R}\boldsymbol{S})\boldsymbol{q}_o - \boldsymbol{R}\boldsymbol{p}_o\,, \tag{103}$$

where $\boldsymbol{e}_o$ is the steady-state values vector of $\boldsymbol{e}(t)$, and (102) implies

$$\boldsymbol{p}_o = -\boldsymbol{P}^{-1}(\boldsymbol{Q}\boldsymbol{u}_o + \boldsymbol{J}\boldsymbol{C}\boldsymbol{q}_o)\,, \tag{104}$$

then, substituting (104) into (103) to eliminate the steady-state value of the functional observer state vector,

$$\boldsymbol{u}_o = (\boldsymbol{L} - \boldsymbol{R}\boldsymbol{S})\boldsymbol{q}_o - \boldsymbol{R}\boldsymbol{P}^{-1}(\boldsymbol{Q}\boldsymbol{u}_o + \boldsymbol{J}\boldsymbol{C}\boldsymbol{q}_o) \tag{105}$$

and, consequently, using (20)

$$(\boldsymbol{I}_r + \boldsymbol{R}\boldsymbol{P}^{-1}\boldsymbol{Q})\boldsymbol{u}_o = (\boldsymbol{O}\boldsymbol{C} - \boldsymbol{R}\boldsymbol{P}^{-1}\boldsymbol{J}\boldsymbol{C})\boldsymbol{q}_o\,. \tag{106}$$

To eliminate from a solution the derived functional observer matrix parameters $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, it can write by using (79), (80) and (60)

$$\begin{aligned}\boldsymbol{R}\boldsymbol{P}^{-1}\boldsymbol{Q} = \boldsymbol{R}\boldsymbol{P}^{-1}\boldsymbol{S}\boldsymbol{B} = \boldsymbol{R}\boldsymbol{P}^{-1}\boldsymbol{R}^{-1}\boldsymbol{R}\boldsymbol{S}\boldsymbol{B} = \\ = \boldsymbol{R}\boldsymbol{P}^{-1}\boldsymbol{R}^{-1}(\boldsymbol{L}-\boldsymbol{O}\boldsymbol{C})\boldsymbol{B} = \boldsymbol{U}^{-1}(\boldsymbol{L} - \boldsymbol{O}\boldsymbol{C})\boldsymbol{B}\,,\end{aligned} \tag{107}$$

where, evidently,

$$\boldsymbol{U} = \boldsymbol{R}\boldsymbol{P}\boldsymbol{R}^{-1}, \quad \boldsymbol{U}^{-1} = \boldsymbol{R}\boldsymbol{P}^{-}\boldsymbol{R}^{-1}. \tag{108}$$

Analogously, to eliminate the derived parameter $\boldsymbol{J}$, then considering (81) it yields

$$\begin{aligned}\boldsymbol{O}\boldsymbol{C} - \boldsymbol{R}\boldsymbol{P}^{-1}\boldsymbol{J}\boldsymbol{C} = \\ = \boldsymbol{O}\boldsymbol{C} - \boldsymbol{R}\boldsymbol{P}^{-1}(\boldsymbol{R}^{-1}\boldsymbol{H} + \boldsymbol{P}\boldsymbol{R}^{-1}\boldsymbol{O})\boldsymbol{C} = \\ = -\boldsymbol{R}\boldsymbol{P}^{-1}(\boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{C} = -\boldsymbol{U}^{-1}\boldsymbol{H}\boldsymbol{C}\,.\end{aligned} \tag{109}$$

Therefore, inserting (107) and (109) into (106) it can obtain

$$\begin{aligned}\boldsymbol{u}_o = \\ = -(\boldsymbol{I}_r + \boldsymbol{U}^{-1}(\boldsymbol{L} - \boldsymbol{O}\boldsymbol{C})\boldsymbol{B})^{-1}\boldsymbol{U}^{-1}\boldsymbol{H}\boldsymbol{C}\boldsymbol{q}_o = \\ = -\boldsymbol{O}_c^{-1}\boldsymbol{U}^{-1}\boldsymbol{H}\boldsymbol{C}\boldsymbol{q}_o\,,\end{aligned} \tag{110}$$

where the matrix $\boldsymbol{O}_c$ is defined in (100).

Substituting (110) into (101) it can write, with respect to the control law (97),

$$(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{O}_c^{-1}\boldsymbol{U}^{-1}\boldsymbol{H}\boldsymbol{C})\boldsymbol{q}_o + \boldsymbol{B}\boldsymbol{W}\boldsymbol{w}_o = \boldsymbol{0}\,, \tag{111}$$

where $\boldsymbol{w}_o$ is the steady-state values vector of $\boldsymbol{w}(t)$, and evidently

$$\boldsymbol{q}_o = -(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{O}_c^{-1}\boldsymbol{U}^{-1}\boldsymbol{H}\boldsymbol{C})^{-1}\boldsymbol{B}\boldsymbol{W}\boldsymbol{w}_o\,. \tag{112}$$

Since (2) implies

$$\boldsymbol{y}_o = \boldsymbol{C}\boldsymbol{q}_o\,, \tag{113}$$

where $\boldsymbol{y}_o$ is the steady-state values vector of $\boldsymbol{y}(t)$, substituting (112) then (113) implies

$$\boldsymbol{y}_o = -\boldsymbol{C}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{O}_c^{-1}\boldsymbol{U}^{-1}\boldsymbol{H}\boldsymbol{C})^{-1}\boldsymbol{B}\boldsymbol{W}\boldsymbol{w}_o\,. \tag{114}$$

Therefore, considering $\boldsymbol{y}_o = \boldsymbol{w}_o$, then (114) implies (99). This concludes the proof. ■

Remark 2: Evidently, for given $\boldsymbol{L}$ the matrix $\boldsymbol{W}$ depends only on the primary defined auxiliary matrix variables, specified by the solutions of (76), (77) and (78).

Proposition 7: [15] If the system (1), (2) is stabilizable by the control policy (97), full system state vector $\boldsymbol{q}(t)$ is available for control and the condition (98) is satisfied, then the matrix $\boldsymbol{W}$ in (97), designed by using the static decoupling principle, takes the form

$$\boldsymbol{W} = -(\boldsymbol{C}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{L})^{-1}\boldsymbol{B})^{-1}. \tag{115}$$

The $\boldsymbol{W}$ matrix is nothing else than the inverse of the closed-loop static gain matrix. Note, the static gain realized by the $\boldsymbol{W}$ matrix is ideal in control only if the plant parameters, on which the value of $\boldsymbol{W}$ depends, are known and do not vary with time.

The forced regime is basically designed for constant references and is very closely related to shift of origin. If the command value $\boldsymbol{w}(t)$ is changed "slowly enough," the above scheme can do a reasonable job of tracking, i.e., making $\boldsymbol{y}(t)$ follow $\boldsymbol{w}(t)$ [12], [18].

VI. Illustrative Example

To illustrate the effectiveness of the algorithms, the linear time-invariant system is considered [13], supporting the model (1), (2) by the matrix parameters

$$\boldsymbol{A} = \begin{bmatrix} 1.380 & -0.208 & 6.715 & -5.676 \\ -0.581 & -4.290 & 0.000 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} 0.000 & 0.000 \\ 5.679 & 0.000 \\ 1.136 & -3.146 \\ 1.136 & 0.000 \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 1.046 \\ 2.600 \\ 2.186 \\ 0.739 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{L} = \begin{bmatrix} 0.1014 & 0.2357 & -0.0147 & -0.1030 \\ 1.1721 & 0.2466 & -0.1472 & 0.4907 \end{bmatrix}.$$

where an unknown input $\boldsymbol{d}(t)$ is assumed to be Gaussian noise with with variance $\sigma_d^2 = 10^{-4}$ and zero mean value. It can be seen that the system is controllable and observable.

Thus, the secondary matrix parameters entering the design conditions are computed as follows

$$\boldsymbol{L}^{\ominus 1} = \begin{bmatrix} 0.1879 & 0.6764 \\ 3.2429 & -0.1003 \\ -0.0529 & -0.0827 \\ -2.0948 & 0.4478 \end{bmatrix},$$

$$\boldsymbol{N}_L = \begin{bmatrix} 0.1881 & -0.2111 & 0.1023 & -0.3125 \\ -0.2111 & 0.2604 & 0.0329 & 0.3833 \\ 0.1023 & 0.0329 & 0.9870 & 0.0352 \\ -0.3125 & 0.3833 & 0.0352 & 0.5645 \end{bmatrix},$$

$$\boldsymbol{\Xi} = \begin{bmatrix} 6.5870 & 3.4410 & 20.2060 & -16.8110 \\ 0.0480 & 4.2730 & 1.3430 & -2.1040 \\ 4.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix},$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 7.8345 & -6.2745 & 20.1404 & -9.5187 \\ -0.0981 & 0.3401 & 1.3970 & 0.4825 \\ 0.8548 & -0.8116 & 1.3964 & -1.2150 \\ -0.3125 & 0.3833 & 0.0352 & 0.5645 \end{bmatrix},$$

$$\boldsymbol{\Sigma}^{\ominus 1} = \begin{bmatrix} 0.0106 & -0.1542 & 0.0760 & -0.0788 \\ -0.0069 & 0.2332 & -0.1140 & 0.1188 \\ 0.0379 & 0.2954 & -0.1396 & 0.1490 \\ -0.0106 & 0.3398 & -0.1661 & 0.1731 \end{bmatrix},$$

$$\boldsymbol{U}_1 = \begin{bmatrix} -1.7486 & -0.5145 \\ -0.2561 & -1.9341 \end{bmatrix},$$

$$\boldsymbol{U}_2 = \begin{bmatrix} -0.4859 & -0.1860 \\ 4.8517 & 0.4405 \\ 2.3383 & 2.2100 \\ -7.3073 & 1.2446 \end{bmatrix}.$$

Solving (85), (86) using SeDuMi package for Matlab [19], the design problem is feasible with the resulting LMI matrix variables

$$\boldsymbol{Y} = \begin{bmatrix} 0.0210 & 0.0168 & -0.0833 & 0.0724 \\ 0.0156 & -0.0889 & -0.3862 & -0.0064 \end{bmatrix},$$

$$\boldsymbol{X} = \begin{bmatrix} 1.8452 & -0.0075 \\ -0.0075 & 1.8026 \end{bmatrix} > 0,$$

while the H_∞ norm upper bound is $\gamma_\infty = 3.8604$.

Thus, (87) gives

$$\boldsymbol{Z} = \begin{bmatrix} 0.0114 & 0.0089 & -0.0460 & 0.0392 \\ -0.0087 & -0.0493 & -0.2144 & -0.0034 \end{bmatrix},$$

and the implying auxiliary matrix parameters are computed as follows

$$\boldsymbol{U} = \begin{bmatrix} -1.3919 & -0.4634 \\ 0.4642 & -1.4327 \end{bmatrix},$$

$$[\,\boldsymbol{O}\ \boldsymbol{H}\,] = \begin{bmatrix} 0.0338 & -0.2377 & 0.1618 & -0.1571 \\ 0.4433 & -0.1238 & 0.3451 & -0.0998 \end{bmatrix}.$$

$$\boldsymbol{O} = \begin{bmatrix} 0.0338 & -0.2377 \\ 0.4433 & -0.1238 \end{bmatrix}, \ \boldsymbol{H} = \begin{bmatrix} 0.1618 & -0.1571 \\ 0.3451 & -0.0998 \end{bmatrix},$$

respectively. Since (64), (65) takes the form

$$\boldsymbol{P}_u = \begin{bmatrix} -1.4123 + 0.4633\,\mathrm{i} & \\ & -1.4123 - 0.4633\,\mathrm{i} \end{bmatrix},$$

$$\boldsymbol{R}_u = \begin{bmatrix} 0.0311 - 0.7061\,\mathrm{i} & 0.0311 + 0.7061\,\mathrm{i}i \\ 0.7074 & 0.7074 \end{bmatrix},$$

respectively, the parameters of (73) are

$$\boldsymbol{R}_s = \begin{bmatrix} 0.0311 & -0.7061 \\ 0.7074 & 0.0000 \end{bmatrix},$$

$$\boldsymbol{P}_s = \begin{bmatrix} -1.4123 & -0.4633 \\ 0.4633 & -1.4123 \end{bmatrix}$$

and setting in the design condition $\boldsymbol{P} = \boldsymbol{P}_s$ and $\boldsymbol{R} = \boldsymbol{R}_s$ then, consequently, it yields

$$\boldsymbol{S} = \begin{bmatrix} -0.8498 & 0.3486 & -0.8348 & 0.8686 \\ 0.0106 & -0.3184 & 0.0319 & -0.1525 \end{bmatrix},$$

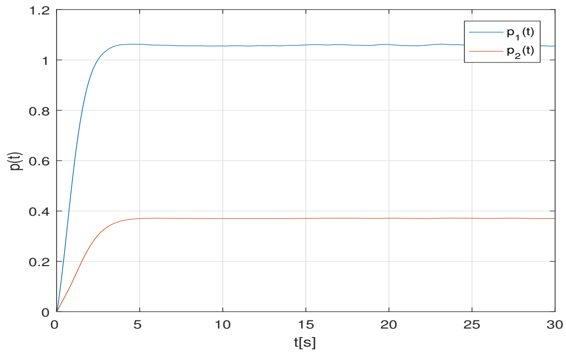
$$\boldsymbol{Q} = \begin{bmatrix} 2.0184 & 2.6263 \\ -1.9454 & -0.1005 \end{bmatrix},$$

$$\boldsymbol{J} = \begin{bmatrix} -0.3879 & -0.0464 \\ 0.1113 & -0.3294 \end{bmatrix}.$$

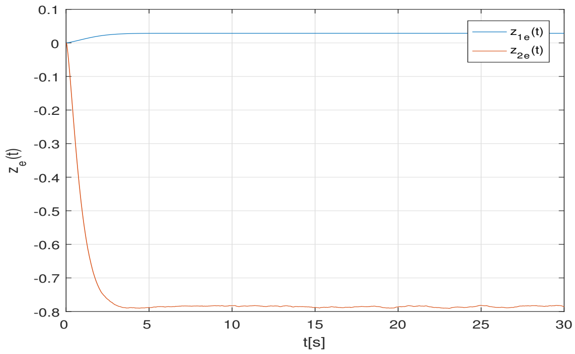
To support the advantage arguments of the H_∞ approach, the set of linear matrix inequalities (74), (75) is also solved using SeDuMi package. As result the following LMI matrix variables are received:

$$\boldsymbol{Y} = \begin{bmatrix} 0.0053 & -0.0426 & -0.0350 & 0.0503 \\ 0.0227 & -0.0805 & -0.2443 & -0.0750 \end{bmatrix},$$

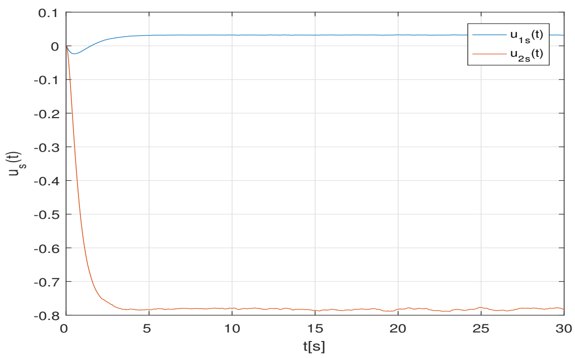
$$\boldsymbol{X} = \begin{bmatrix} 0.6648 & -0.0105 \\ -0.0105 & 0.6202 \end{bmatrix} > 0.$$



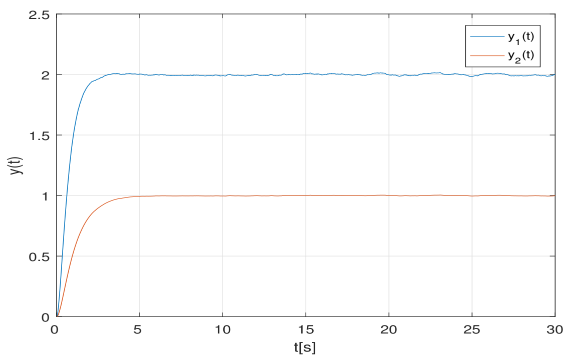
a) Functional observer state variables response.



b) Closed-loop system input estimate.

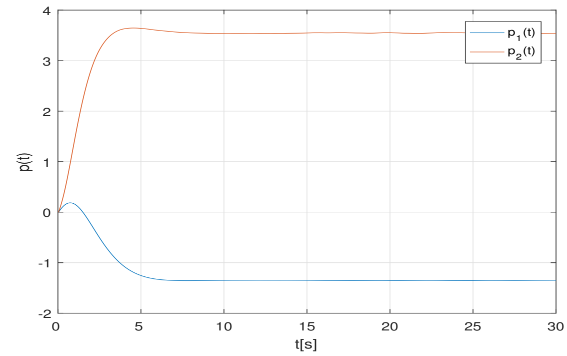


c) Closed-loop system input response.

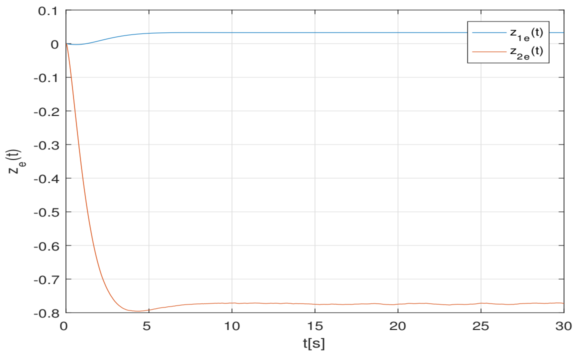


d) Closed-loop system output response.

Fig. 1. Responses of system and functional observer



a) Functional observer state variables response.



b) Closed-loop system input estimate.

Fig. 2. Responses of functional observer

Applying (87), the observer regularizing matrix $\boldsymbol{Z}$ is computed as

$$\boldsymbol{Z} = \begin{bmatrix} 0.0085 & -0.0661 & -0.0589 & 0.0737 \\ 0.0367 & -0.1309 & -0.3948 & -0.1197 \end{bmatrix},$$

and the imbedded auxiliary matrix parameters to be

$$\boldsymbol{U} = \begin{bmatrix} -0.7476 & -0.4454 \\ 0.4454 & -0.8480 \end{bmatrix},$$

$$[\,\boldsymbol{O}\;\boldsymbol{H}\,] = \begin{bmatrix} 0.0291 & -0.1974 & 0.1907 & -0.2089 \\ 0.4262 & -0.0773 & 0.5436 & -0.0017 \end{bmatrix}.$$

$$\boldsymbol{O} = \begin{bmatrix} 0.0291 & -0.1974 \\ 0.4262 & -0.0773 \end{bmatrix}, \quad \boldsymbol{H} = \begin{bmatrix} 0.1907 & -0.2089 \\ 0.5436 & -0.0017 \end{bmatrix}.$$

The following matrices $\boldsymbol{P}_u$, $\boldsymbol{R}_u$, satisfying the requirements (64), (65), are found

$$\boldsymbol{P}_u = \begin{bmatrix} -0.7978 + 0.4425\,\mathrm{i} & \\ & -0.7978 - 0.4425\,\mathrm{i} \end{bmatrix},$$

$$\boldsymbol{R}_u = \begin{bmatrix} 0.7071 & 0.7071 \\ 0.0797 + 0.7026\,\mathrm{i} & 0.0797 - 0.7026\,\mathrm{i}i \end{bmatrix},$$

so that the parameters $\boldsymbol{R}_s$, $\boldsymbol{P}_s$ which satisfies (73) are

$$\boldsymbol{R}_s = \begin{bmatrix} 0.7071 & 0.0000 \\ 0.0797 & 0.7026 \end{bmatrix},$$

$$\boldsymbol{P}_s = \begin{bmatrix} -0.7978 & -0.4425 \\ 0.4425 & -0.7978 \end{bmatrix}.$$

Finally, setting $\boldsymbol{P} = \boldsymbol{P}_s$, $\boldsymbol{R} = \boldsymbol{R}_s$ and exploiting the above given intermediate data, the functional observer matrix parameters have to be

$$\boldsymbol{S} = \begin{bmatrix} -0.0213 & 0.3333 & -0.0619 & 0.1334 \\ -0.7555 & 0.3132 & -0.8091 & 0.7932 \end{bmatrix},$$

$$\boldsymbol{Q} = \begin{bmatrix} 1.9742 & 0.1948 \\ 1.7607 & 2.5453 \end{bmatrix},$$

$$\boldsymbol{J} = \begin{bmatrix} -0.0295 & -0.0381 \\ 0.2812 & -0.0299 \end{bmatrix}.$$

All simulations are done in the forced mode, conditioned by availability of the full system state vector $\boldsymbol{q}(t)$ to apply the control law (97), where the associated signal gain matrix $\boldsymbol{W}$ is computed by using (115) as

$$\boldsymbol{W} = \begin{bmatrix} 0.0024 & 0.1055 \\ -0.0957 & 0.0401 \end{bmatrix},$$

the desired steady state vector of the output variables is $\boldsymbol{w}_o^T = [\ 2 \ \ 1\ ]$ and the starting system and observer initial vectors values are $\boldsymbol{q}(0) = \boldsymbol{0}$, $\boldsymbol{p}(0) = \boldsymbol{0}$.

When the system inputs and outputs are obtained as shown in Fig. 1c) and Fig. 1d), and the applied functional observer parameters are those resulted from the conditions defined by Theorem 2, the functional observer state and the reconstruction of the system inputs are as shown in Fig. 1a) and Fig. 1b), respectively. In this case, it is clear that the functional observer exhibits very good disturbance decoupling properties and successfully estimates the evolution of the control law.

Considering the same inputs and outputs as shown in Fig. 1c) and Fig. 1d), but using the functional observer parameters accomplished by solving the inequalities defined in Theorem 1, the trajectories of the functional observer state are drawn in the Fig. 2a) and the system inputs estimates are shown in Fig. 2b). Even in this case, the estimation exactitude is sufficiently accurate, but the functional observer dynamic properties, determined by this set of matrix parameters, are worse. It is consistent with the results implying from the set of eigenvalues of the functional observer matrix $\boldsymbol{P}$, when the H_∞ approach determines $\rho(\boldsymbol{P}) = \{-1.4123 \pm 0.4633\,\mathrm{i}\}$ while the Lyapunov method leads to values $\rho(\boldsymbol{P}) = \{-0.7978 \pm 0.4425\,\mathrm{i}\}$.

In terms of suppressing the impact of unknown disturbance on the state function estimate error both approaches are comparable since, if (96) is used for evaluation, the H_∞ approach implies $\|\boldsymbol{S}\|_F = 1.5266$ and the Lyapunov method gives $\|\boldsymbol{S}\|_F = 1.4120$. Here $\|*\|_F$ means the Frobenius norm of a non-square matrix.

It can be concluded from the above simulation results that also for forced modes the asymptotic convergence of functional state estimation errors can be achieved using both two methods in accordance with the above theoretical analysis, but the proposed H_∞ approach improves the rapidity of estimation evidently.

VII. Concluding Remarks

The design problem for one class of the functional observer structures is investigated in the paper. The newly formulated design conditions, allowing enough flexibility to guarantee asymptotic stability and dynamics for observer structures, are proven in the sense of the bounded real lemma and Lyapunov method. The design conditions are accounted in terms of LMIs, and use the standard numerical optimization operations to manipulate the matrix inequalities. Formulation in dependency on the forced mode and solution of the design problem by applying H_∞ norm techniques support solutions founded by applying H_∞ optimization within an LMI formulation.

Assuming the existence of the suitable matrices, it is demonstrate that the reconstruction of the input vector of the system is adequate. The proposed methods do not use adjustment technique to set some prescribed matrix variables and to calculate others in an iterative way.

Further research works include aspects of the system parameter fault detection and isolation using the functional observers strategy, unknown input functional observer design to suppress disturbance action and computational algorithms to address robustness and performance specifications.

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