One approach for circuit optimization process

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***Abstract*—**The possibility of applying the maximum principle of Pontryagin to the problem of optimization of electronic circuits is analyzed. It is shown that in spite of the fact that the problem of optimization is formulated as a nonlinear task, and the maximum principle in this case isn't a sufficient condition for obtaining a minimum of the functional, it is possible to obtain the decision in the form of local minima. The analysis of optimization process for some circuits showed that application of the maximum principle really allows finding the optimum structure of the control vector by means of iterative procedure. The theoretical justification is given for the earlier discovered effect of acceleration of the process of circuit optimization in the conditions of a new methodology of design. The relative acceleration of the CPU time for the best strategy found by means of maximum principle compared with the traditional approach is equal two to three orders of magnitude.

***Keywords*—**Control theory approach, generalized optimization methodology, maximum principle, set of optimization strategies.

I. Introduction

TO improve the overall quality of electronic circuit designs, it is very important to reduce their design time. Many works devoted to this problem focus on how to reduce the number of operations when solving two main problems: circuit analysis and numerical optimization. By solving these problems successfully, one can reduce the total time required for analogue circuit optimization and this fact serves as a basis for improving design quality.

The methods used to analyse complex systems are being improved continuously. Some well-known ideas related to the use of a method of sparse matrixes [1] and decomposition methods [2] are used for the reduction of time for the analysis of circuits. Some alternative methods such as homotopy methods [3] were successfully applied to circuit analysis.

Practical methods of optimization were developed for circuit designing, timing, and area optimization [4]. However, classical deterministic optimization algorithms may have a number of drawbacks: they may require that a good initial point be selected in the parameter space, they may reach an unsatisfactory local minimum, and they require that the cost function be continuous and differentiable. To overcome these issues, special methods were applied to determine the initial point of the process by centring [5] or applying geometric programming methods [6].

A more general formulation of the circuit optimization problem was developed on a heuristic level some decades ago [7]. This approach ignored Kirchhoff's laws for all or part of a circuit during the optimization process. The practical aspects of this idea were developed for the optimization of microwave circuits [8] and for the synthesis of high-performance analogy circuits [9] in an extreme case where all the equations of the circuit were not solved during the optimisation process.

In work [10] the problem of circuit optimization is formulated in terms of the theory of optimal control. Thus, the process of circuit optimization was generalised and defined as the dynamic controllable system. In this case, the basic element is the control vector that changes the structure of the equations of the system of optimization process. Thus, there is a set of strategies of optimization that have different number of operations and different computing times. The introduction and analysis of the function of Lyapunov of the optimization process [11-12] allows comparison of various strategies of optimization and choosing the best of them having minimum processor time. At the same time, the problem of searching for the optimal strategy and the corresponding optimal trajectory can be solved most appropriately by the maximum principle of Pontryagin [13].

The main complexity of application of the maximum principle consists of the search of initial values for auxiliary variables at the solution of the conjugate system of equations. Application of the maximum principle in case of linear dynamic systems is based on the creation of an iterative process [14-15].

In case of nonlinear systems, the convergence of this process is not guaranteed. However, application of the additional approximating procedures [16] allows constructing sequence of the solutions converging to a limit under certain conditions.

The first step in the problem of possibility of application of maximum principle for circuit optimization was presented in [17]. In the present work, the application of the maximum principle for circuit optimization was investigated with a sufficient accuracy.

II. Problem Formulation

In accordance with the conventional approach, the process of electronic circuit optimization is defined as the problem of minimizing an objective function $C(\mathbf{X})$, $\mathbf{X} \in R^N$, with constraints given by a system of the circuit´s equations based

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on Kirchhoff's laws. We assume that, by minimizing $C(\mathbf{X})$, we achieve all our design goals. The circuit optimization problem can be generalized by introducing a special control vector $\mathbf{U}=(u_1,u_2,...,u_M)$ and a special generalized objective function $F(\mathbf{X},\mathbf{U})$. The electronic circuit optimization process can be defined as the problem of minimizing the generalized objective function $F(\mathbf{X},\mathbf{U})$ based on the vector equation (1) with the constraints (2). The system of constraints is the mathematical model of the electronic circuit.

$$\mathbf{X}^{s+1}=\mathbf{X}^{s}+t_s\mathbf{H}^{s}, \tag{1}$$

$$(1-u_j)g_j(\mathbf{X})=0,\ \ j=1, 2,..., M, \tag{2}$$

where $\mathbf{X}\in R^N$, $\mathbf{X}=(\mathbf{X}',\mathbf{X}'')$, $\mathbf{X}'\in R^K$ is a vector of independent variables, $\mathbf{X}''\in R^M$ is a vector of dependent variables, M is the number of the circuit's dependent variables, K is the number of independent variables, N is the total number of variables ($N=K+M$) and t_s is an iteration parameter. The equation (1) describes a two-step minimization procedure, and the function $H\equiv H(\mathbf{X},\mathbf{U})$ determines the direction in which the generalized objective function $F(\mathbf{X},\mathbf{U})$ decreases. The functions $g_j(\mathbf{X})$ for all j define the equations of the circuit model. The components of control vector $\mathbf{U}$ are a set of control functions: $\mathbf{U}=(u_1,u_2,...,u_M)$, where $u_j\in\Omega$, $\Omega=\{0;1\}$. The vector $\mathbf{U}$ leads to redistribution of expenses of computing time between the block of procedure of optimization (1) and the block of the analysis of the scheme (2). The complete set of different optimization strategies (structural basis) includes 2^M strategies. The generalized objective function $F(\mathbf{X},\mathbf{U})$ can be defined, for example, as follows:

$$F(\mathbf{X},\mathbf{U})=C(\mathbf{X})+\varphi(\mathbf{X},\mathbf{U}), \tag{3}$$

where $C(\mathbf{X})$ is a non-negative ordinary objective function of the optimization process and $\varphi(\mathbf{X},\mathbf{U})$ is a penalty function. The structure of the penalty function must potentially include all the equations from the system (2) and can be defined, for example, as follows:

$$\varphi(\mathbf{X},\mathbf{U})=\frac{1}{\delta}\sum_{j=1}^{M}u_j g_j^2(\mathbf{X}), \tag{4}$$

where δ is an additional coefficient used to adapt the penalty function. In our context, δ equals 1.

This definition of the circuit optimization problem allows us to redistribute the computing time between the problems (1) and (2). A control function u_j has the following meaning: if $u_j=0$, the jth equation is present in the system (2) and the term $g_j^2(\mathbf{X})$ is removed from the equation (4); and, the other way around, if $u_j=1$, the jth equation is removed from the system (2) and the term $g_j^2(\mathbf{X})$ is present in the equation (4). We can define two special strategies: **U**=(0,0,…,0) and **U**=(1,1,…,1). The first strategy can be named as Traditional Strategy of Optimization (TSO) and corresponds to the solution of system (2) at each point of optimization process. The second strategy can be named as Modified Traditional Strategy of Optimization (MTSO) and corresponds to the elimination of the system (2), but in this case all the information on the circuit is included in the penalty function (4). The vector **U** is the main tool of this methodology: it controls the dynamic process of minimizing the objective functions $F(\mathbf{X},\mathbf{U})$ and $C(\mathbf{X})$ in the possible minimum time. This definition allows us to express the problem of searching for the optimal strategy as the typical problem of minimizing a functional, where the functional is the CPU time. When defining the optimization process as a dynamical system, a more standard approach is to use differential equations, in continuous form. We can rewrite the main system of the optimization procedure (1) in continuous form as the following system of differential equations:

$$\frac{dx_i}{dt}=f_i(\mathbf{X},\mathbf{U}),\ \ i=1, 2,..., N, \tag{5}$$

Together with the equations (2), (3) and (4), this system specifies the continuous form of the optimization process. The structure of the functions $f_i(\mathbf{X},\mathbf{U})$ is defined by a concrete optimization method. For example, for the gradient method, it takes the following form:

$$f_i(\mathbf{X},\mathbf{U})=-\frac{\delta}{\delta x_i}F(\mathbf{X},\mathbf{U}),\ i=1, 2,\ldots,K$$
$$f_i(\mathbf{X},\mathbf{U})=-u_{i-K}\frac{\delta}{\delta x_i}F(\mathbf{X},\mathbf{U})+(1-u_{i-K})\alpha_i,\ i=K+1,..., N, \tag{6}$$

where α_i is the additional parameter defining an increment of the value of the dependent variables x_i in the course of optimization and computed by the formula $\alpha_i=\frac{1}{\tau}\left[\eta_i(\mathbf{X}^{s+1})-x_i^s\right]$, and $\eta_i(\mathbf{X})$ is the implicit function defining the component number i of a vector $\mathbf{X}$ at the solution of system (2), τ is the step of integration and s is the step number of the procedure of numerical integration of system (5). The operator $\delta/\delta x_i$ is defined by the formula $\frac{\delta}{\delta x_i}\rho(\mathbf{X})=\frac{\partial\rho(\mathbf{X})}{\partial x_i}+\sum_{p=K+1}^{K+M}\frac{\partial\rho(\mathbf{X})}{\partial x_p}\frac{\partial x_p}{\partial x_i}$ and determines the application of the gradient method for a complex function that has both independent and dependent variables.

By using formulas (2)–(6), we formulate the circuit optimization process as a controllable process or as a controllable dynamical system. The vector **U** defines the right hand parts of the system (5) and gives the possibility of changing the optimization strategies and operation's number. Such approach contains optimization strategies much more optimum than TSO. So, the vector **U** is the principal tool for searching and constructing the optimization process with a minimal computing time. Control functions u_j, so and functions $f_i(\mathbf{X},\mathbf{U})$ are piecewise continuous. The optimal control problem for the system (5) with the non-continuous right hand parts can be solved most correctly using Pontryagin maximum principle.

Let's analyse an example of the optimization of the simplest nonlinear circuit for which the solution was obtained on the basis of the maximum principle. We will consider the simplest nonlinear circuit of a voltage divider in Fig. 1.

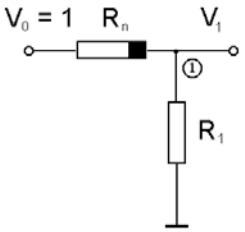

Fig. 1. Simplest nonlinear voltage divider

Let us consider that the nonlinear element has the following dependence:

$$R_n = a + b(V_1 - V_0), \tag{7}$$

where $a>0$, $b>0$, $a>b$, V_0 and V_1 the voltages on an input and an output of circuit.

We will consider that V_0 is equal 1. We will define the variables x_1, x_2. $x_1 = R$, $x_2 = V_1$. Thus the vector of phase variables $\mathbf{X} \in R^2$. In this case the formula (7) can be replaced with the following expression:

$$R_n = a + b(x_2 - 1). \tag{8}$$

We can present the equation of a circuit in the form:

$$g_1(x_1, x_2) \equiv x_2[x_1 + a + b(x_2 - 1)] - x_1 = 0 \tag{9}$$

The circuit optimization is formulated as a problem of obtaining at the exit of a circuit of the defined voltage *w*. We will determine the cost function of the optimization process by the formula:

$$C(\mathbf{X}) = (x_2 - w)^2. \tag{10}$$

In this case, the problem of circuit optimization is converted to minimization of the cost function $C(\mathbf{X})$. Following theoretical bases that were developed in [10], we formulate the problem for circuit optimization as a task of search of the optimization strategy with a minimum possible CPU time. For this purpose, we define the functional, which is subject to minimization, by the following expression:

$$J = \int_0^T f_0(\mathbf{X})\ dt, \tag{11}$$

where $f_0(\mathbf{X})$ is the function that is conditionally determining the density of a number of arithmetic operations in a unit of time *t*. In that case, the integral (11) defines total number of operations necessary for circuit optimization and is proportional to the total CPU time.

The structure of function $f_0(\mathbf{X})$ cannot be defined. However, we can compute CPU time using the possibilities of the compiler. We will further identify the integral (11) with CPU time, and therefore, the problem of minimization of CPU time corresponds to a problem of minimization of this integral.

According to [10], we introduce the control vector **U** that consists of only one component *u(t)* for the reviewed example.

The process of circuit optimization thus can be described by the system (12) with restrictions (13):

$$\frac{dx_i}{dt} = f_i(x_1, x_2, u), \quad i=1, 2, \tag{12}$$

$$(1-u)g_1(x_1, x_2) = 0, \tag{13}$$

where functions $f_i(x_1, x_2, u)$ are defined by a concrete numerical method of optimization. When using a gradient method, these functions are defined by the following formulas:

$$f_i(x_1, x_2, u) = -\frac{\delta}{\delta x_i} F(\mathbf{X}), \ i=1,2, \tag{14}$$

where the operator $\delta / \delta x_i$ is defined by the expression: $\frac{\delta}{\delta x_i}\sigma(\mathbf{X}) = \frac{\partial \sigma(\mathbf{X})}{\partial x_i} + \sum_{p=K+1}^{K+M} \frac{\partial \sigma(\mathbf{X})}{\partial x_p}\frac{\partial x_p}{\partial x_i}$.

The value *u(t)*=0 corresponds to the traditional strategy of optimization (TSO). In this case in the system (12), there is only one equation for the independent x_1 variable, whereas the variable x_2 is defined from the equation (13). The value *u(t)*=1 corresponds to the modified traditional strategy of optimization (MTSO) when both x_1 and x_2 variables are independent. In this case, the system (12) includes two equations for the independent variables x_1 и x_2, and the equation (13) disappears. A change in the value of function *u(t)* with 0 on 1 and back can be made at any moment and generates a set of various strategies of optimization. Two main strategies are defined as follows:

1) TSO, u=0. The equations (12)–(14) are replaced with the following equations:

$$\frac{dx_1}{dt} = -\frac{\partial C}{dx_2}\frac{dx_2}{dx_1}, \quad (15)$$

$$\frac{dx_2(x_1,t)}{dt} = \frac{\partial x_2}{\partial x_1}\frac{dx_1}{dt}, \quad (16)$$

where the derivative dx_2/dx_1 is defined from the equation (13) and can be calculated by the formula:

$$\frac{dx_2}{dx_1} = \frac{1}{2b}\left[-1+\frac{x_1+c+2b}{\sqrt{(x_1+c)^2+4bx_1}}\right], \; c=a-b.$$

2) MTSO, u=1. The equations (12) are transformed to the next one:

$$\frac{dx_i}{dt} = -\frac{\delta}{\delta x_i}\left[C(\mathbf{X})+g_1^2(\mathbf{X})\right], \quad i=1, 2. \quad (17)$$

In a general case, the right-hand parts of the equations (12) can be presented in the form:

$$\begin{aligned} f_1(x_1,x_2,u) &= (1-u)\cdot f_{11}(x_1,x_2)+u\cdot f_{12}(x_1,x_2),\\ f_2(x_1,x_2,u) &= (1-u)\cdot f_{21}(x_1,x_2)+u\cdot f_{22}(x_1,x_2), \end{aligned} \quad (18)$$

where the functions $f_{ij}(x_1,x_2)$ are determined by the following formulas:

$$\begin{aligned} f_{11}(x_1,x_2) &= \frac{(w-x_2)}{b}\left[-1+\frac{x_1+c+2b}{\sqrt{(x_1+c)^2+4bx_1}}\right]\\ f_{12}(x_1,x_2) &= -2(x_2-1)\{(x_2-1)x_1+[a+b(x_2-1)]x_2\} \end{aligned} \quad (19)$$

$$\begin{aligned} f_{21}(x_1,x_2) &= \frac{(w-x_2)}{2b^2}\left[-1+\frac{x_1+a+b}{\sqrt{(x_1+c)^2+4bx_1}}\right]^2\\ f_{22}(x_1,x_2) &= -2(x_2-w)-2(c+x_1+2bx_2)\\ &\quad\cdot[(x_2-1)x_1+ax_2+b(x_2-1)x_2] \end{aligned}$$

According to methodology of the maximum principle, the system of the conjugate equations for additional variables ψ_1, ψ_2 has the next form:

$$\begin{aligned} \frac{d\psi_1}{dt} &= -\frac{\partial f_1(x_1,x_2,u)}{\partial x_1}\cdot\psi_1-\frac{\partial f_2(x_1,x_2,u)}{\partial x_1}\cdot\psi_2,\\ \frac{d\psi_2}{dt} &= -\frac{\partial f_1(x_1,x_2,u)}{\partial x_2}\cdot\psi_1-\frac{\partial f_2(x_1,x_2,u)}{\partial x_2}\cdot\psi_2. \end{aligned} \quad (20)$$

The Hamiltonian is expressed by the following formula:

$$H = \psi_1\cdot f_1(x_1,x_2,u)+\psi_2\cdot f_2(x_1,x_2,u) \quad (21)$$

Substituting (18) in (21) and doing identical transformations, we obtain the following expression for the Hamiltonian:

$$\begin{aligned} H = \psi_1\cdot f_{11}(x_1,x_2)+\psi_2\cdot f_{21}(x_1,x_2)\\ +u\cdot\Phi(x_1,x_2,\psi_1,\psi_2) \end{aligned}, \quad (22)$$

where

$$\begin{aligned} \Phi(x_1,x_2,\psi_1,\psi_2) = \psi_1\cdot[f_{12}(x_1,x_2)-f_{11}(x_1,x_2)]\\ +\psi_2\cdot[f_{22}(x_1,x_2)-f_{21}(x_1,x_2)] \end{aligned}. \quad (23)$$

According to the maximum principle, we obtain the next main condition for the control function u:

$$u = \begin{cases} 0, & \Phi<0\\ 1, & \Phi>0 \end{cases} \quad (24)$$

The behaviour of the control function $u(t)$ that corresponds to the maximum principle is also defined by the functions $\psi_1(t)$ and $\psi_2(t)$, which are computed from the Eq. (20).

III. Numerical Results

The solution of the equations (20) depends on the initial values ψ_{10} и ψ_{20}, which are defined within the precision of the common multiplier. One of these constants can be taken arbitrarily. Let us define the constant $\psi_{10}=-1$. The value of the constant ψ_{20}, which corresponds to the correct solution of a task in the conditions of the maximum principle ψ_{20c}, can be obtained by iterative procedure. We use the iterative procedure for minimizing the functional (11).

The analysis of the process of optimization for a similar example, which is carried out in work [18], showed that the TSO (u=0) is the optimal one when both initial values of variables x_1 and x_2, (x_{10}, x_{20}) are positive. In this case the number of iterations is equal to 3898, and CPU time is equal to 42.88 msec for the initial point x_{10}=1, x_{20}=2. At the same time, the negative initial values of the variable x_2 significantly lead to other results. In the case of negative initial values of the variable x_2, emergence of effect of acceleration of the process of circuit optimization is possible [18]. This effect accelerates the optimization process. It is interesting to check if this result corresponds to the maximum principle.

Fig. 2 shows the trajectories of the process of circuit optimization with the negative initial value of coordinate x_{20}, (x_{10}=1, x_{20} = –2).

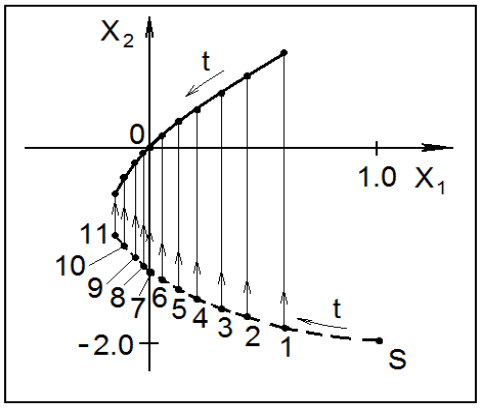

Fig. 2. Trajectories of optimisation process with initial point (x_{10} =1, x_{20} = –2) and different values of ψ_{20}.

The structure of function *u(t)* that was obtained automatically and corresponds to a condition of the maximum principle (24) has one or two points of a rupture that corresponds to switching from the trajectory corresponding to MTSO (*u*=1, a dotted curve) on trajectory corresponding to TSO (*u*=0, a continuous curve). Coordinates of a point of switching of t_{sw} depend on the value of ψ_{20}. The data corresponding to the different points of switching from 1 to 11 in Fig. 2 are presented in Table 1.

Table 1. Data of some strategies with different initial values of variable $\psi_2(t)$.

N	ψ_{20}	Control function structure	Switching points	Total iterations number	CPU time (msec)
1	7.27	1; 0; 1	198; 199	2606	14.34
2	7.265	1; 0; 1	200; 201	2464	13.56
3	7.26	1; 0; 1	202; 203	2274	12.52
4	7.255	1; 0; 1	203; 204	2148	11.82
5	7.25	1; 0; 1	205; 206	1759	9.68
6	7.245	1; 0	206	207	1.14
7	7.24	1; 0	209	620	5.67
8	7.235	1; 0	211	711	6.66
9	7.23	1; 0	214	785	7.46
10	7.225	1; 0	216	818	7.81
11	7.22	1; 0	219	855	8.21

A change in the value of ψ_{20} from 7.27 to 7.245 leads to reduction of iterations number and CPU time from 14.34 msec to 1.14 msec, but the CPU time is increasing later on. That is visible also in Fig. 3, where the dependence of CPU time of a task from initial value ψ_{20} is shown.

The value ψ_{20opt} = 7.245 corresponds to the minimum CPU time T_{min} and in this case the integral *J* and the initial value of variable $\psi_2(t)$ provides the maximum value of a Hamiltonian according to the maximum principle.

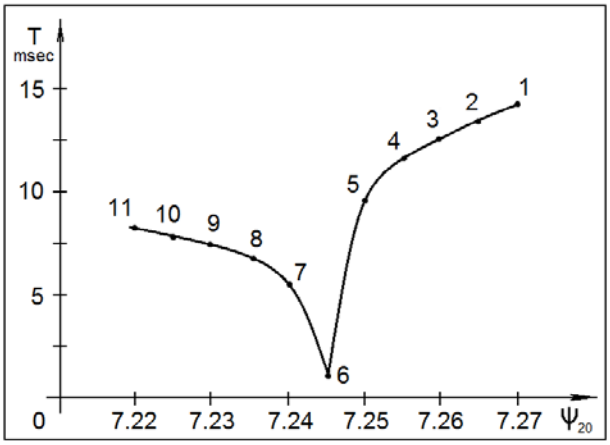

Fig. 3. CPU time for different initial values of $\psi_2(t)$

The gain in time computed as time relation for TSO by the minimum time of T_{min} thus equal to 37.6 times.

Let us define partial Hamiltonians $H_{(0)}$, $H_{(1)}$ by formulas:

$$H_{(0)} = \psi_1 \cdot f_1(x_1, x_2, 0) + \psi_2 \cdot f_2(x_1, x_2, 0), \quad (25)$$

$$H_{(1)} = \psi_1 \cdot f_1(x_1, x_2, 1) + \psi_2 \cdot f_2(x_1, x_2, 1). \quad (26)$$

Dependencies of the functions $H_{(0)}(t)$, $H_{(1)}(t)$ and $\Phi(t)$ for various values of parameter ψ_{20} are presented in Fig. 4 – Fig. 6. Optimum value of a constant ψ_{20} is equal to 7.245 and corresponds to the results presented in Fig. 4.

In this case the function $H_{(1)}(t)$ passes above the function $H_{(0)}(t)$ from the beginning of the process until the point T_{sw}. At this point both functions become equal, function $\Phi(t)$ changes a sign, and according to condition (24), value of the control function *u* is changing to 1 on 0. Then, the iterative process comes to the end because the criterion for the end of the optimization process is satisfied.

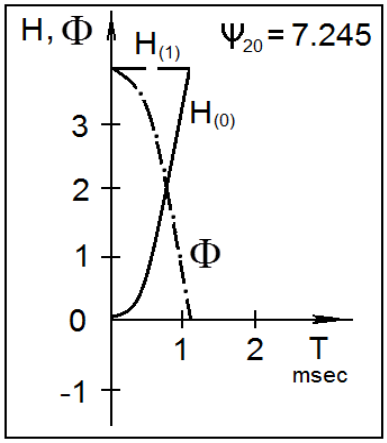

Fig. 4. Time dependency of functions $H_{(0)}(t)$, $H_{(1)}(t)$ and $\Phi(t)$ for optimal parameter ψ_{20}.

We can analyse the behaviour of the functions $H_{(0)}(t)$, $H_{(1)}(t)$ and $\Phi(t)$ with non-optimal initial value ψ_{20}. The point of switching of the control function u from 1 on 0 is not satisfying the optimum point. The behaviour of functions $H_{(0)}(t)$, $H_{(1)}(t)$ and $\Phi(t)$ is shown in Fig. 5 for ψ_{20}=7.249.

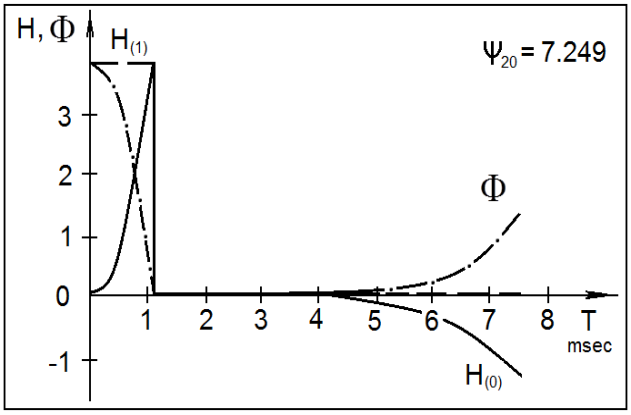

Fig. 5. Time dependency of functions $H_{(0)}(t)$, $H_{(1)}(t)$ and $\Phi(t)$ for non-optimal value of parameter ψ_{20}, $\psi_{20} > \psi_{20opt}$.

The control function switching happens before an optimum point and the computing time grows till 7.55 msec.

The behaviour of these functions is given in Fig. 6 at ψ_{20} = 7.24. In this case the control function switching happens after an optimum point and the time of computing grows again to 5.67 msec.

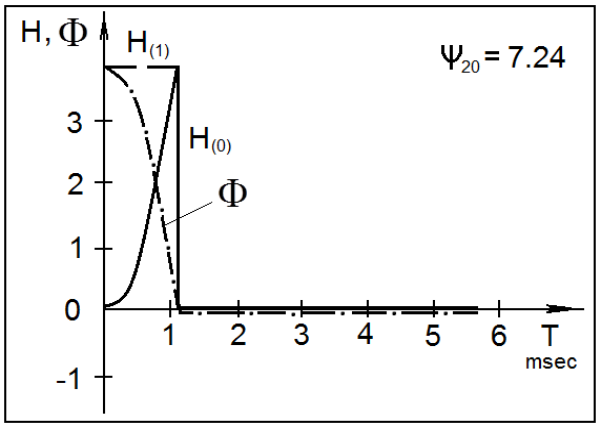

Fig. 6. Time dependency of functions $H_{(0)}(t)$, $H_{(1)}(t)$ and $\Phi(t)$ for non-optimal value of parameter ψ_{20}, $\psi_{20} < \psi_{20opt}$.

It is clear that when the point of switching differs from the optimal one, the value of the Hamiltonian is changing over time.

The analysis of the optimization process for the considered circuit has shown that use of the maximum principle really allows for the finding of the optimum structure of the control function u(t) by means of the iterative procedure. At the same time the considerable reduction of the processor time in comparison with the traditional approach is observed. The interesting question is whether it is possible to extend the obtained analytical result to the numerical solution of the optimization problem for nonlinear circuit of any dimension. The next section is devoted to this problem.

IV. N – Dimensional Case

We need to extend earlier obtained result on the solution of N-dimensional problem of circuit optimization. Let's consider the problem of optimization of a nonlinear circuit with two nodes shown on Fig. 7.

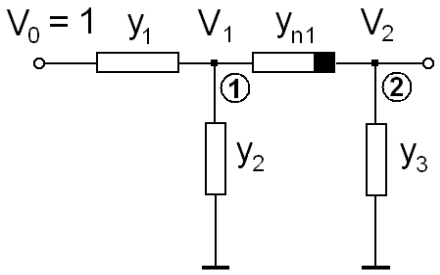

Fig 7. Nonlinear two-node voltage divider.

The given circuit is a nonlinear divider of voltage. There are three independent parameters (K=3) and two dependent ones (M=2). The nonlinear element has the following dependency: $y_{n1} = y_0 + a(V_1 - V_2)^2$. Here and further, all physical variables are presented in the normalized, unitless form. We define the voltage V_0 as 1, and the variables x_1, x_2, x_3, x_4 and x_5 as: $x_1^2 = y_1$, $x_2^2 = y_2$, $x_3^2 = y_3$, $x_4 = V_1$, and $x_5 = V_2$. By defining the components x_1, x_2, x_3 using the above formulas, we automatically obtain positive values of the conductance, which eliminates the issue of positive definiteness for each resistance and conductance and allows us to carry out optimization in the full space of the values of these variables without any restrictions. The vector of the phase variables of the circuit is $X \in R^5$. In this case the nonlinear element is defined by the following expression: $y_{n1} = y_0 + a(x_4 - x_5)^2$.

Let's determine function of the purpose of process of optimization by the formula:

$$C(\mathbf{X}) = (x_5 - w)^2, \tag{27}$$

where w – the required value of output voltage.

The model of a circuit is defined by the following system:

$$\begin{aligned} g_1(\mathbf{X}) &\equiv (1 - x_4)x_1^2 - \left[y_0 + a(x_4 - x_5)^2\right](x_4 - x_5) - x_4 x_2^2 = 0 \\ g_2(\mathbf{X}) &\equiv \left[y_0 + a(x_4 - x_5)^2\right](x_4 - x_5) - x_5 x_3^2 = 0 \end{aligned} \tag{28}$$

The system of equations of the optimization procedure is presented by the system (5) with right hand parts (6).

Let's obtain the main expressions corresponding to the maximum principle. The conjugate system of the equations for the additional variables ψ_i has a form:

$$\frac{d\psi_i}{dt} = -\sum_{k=1}^{N} \frac{\partial f_k(\mathbf{X},\mathbf{U})}{\partial x_i} \cdot \psi_k , \quad (29)$$

The Hamiltonian is determined by the following formula:

$$H(\mathbf{X},\boldsymbol{\Psi},\mathbf{U}) = \sum_{i=1}^{N} \psi_i \cdot f_i(\mathbf{X},\mathbf{U}) = \sum_{i=1}^{K} \psi_i \cdot f_i(\mathbf{X},\mathbf{U}) + \sum_{i=K+1}^{N} \psi_i \cdot f_i(\mathbf{X},\mathbf{U}) \quad (30)$$

where the first and second sums are defined by the following expressions:

$$\sum_{i=1}^{K} \psi_i \cdot f_i(\mathbf{X},\mathbf{U}) = -\sum_{i=1}^{K} \psi_i \cdot \frac{\delta C}{\delta x_i} - \sum_{i=1}^{K} \psi_i \sum_{k=1}^{M} u_k \cdot \frac{\delta (g_k(\mathbf{X}))^2}{\delta x_i}$$

$$\sum_{i=K+1}^{N} \psi_i \cdot f_i(\mathbf{X},\mathbf{U}) = \sum_{i=K+1}^{N} (1-u_{i-K}) \psi_i \alpha_i - \sum_{i=K+1}^{N} u_{i-K} \psi_i \left[\frac{\delta C}{\delta x_i} + \sum_{k=1}^{M} u_k \frac{\delta (g_k(\mathbf{X}))^2}{\delta x_i} \right] \quad (31)$$

As a result the Hamiltonian can be expressed as follows:

$$H(\mathbf{X},\boldsymbol{\Psi},\mathbf{U}) = h_c + h_0 + h_1 + h_2 \quad (32)$$

where h_c is the part of a Hamiltonian that does not depend on the control vector,

$$h_c(\mathbf{X},\boldsymbol{\Psi}) = -\sum_{i=1}^{K} \psi_i \cdot \frac{\delta C}{\delta x_i} + \sum_{i=K+1}^{N} \psi_i \alpha_i . \quad (33)$$

Other components of a Hamiltonian depend on the control vector **U**:

$$h_0(\mathbf{X},\boldsymbol{\Psi},\mathbf{U}) = -\sum_{i=K+1}^{N} u_{i-K} \psi_i \alpha_i , \quad (34)$$

$$h_1(\mathbf{X},\boldsymbol{\Psi},\mathbf{U}) = -\sum_{i=1}^{K} \psi_i \sum_{k=1}^{M} u_k \cdot \frac{\delta (g_k(\mathbf{X}))^2}{\delta x_i} , \quad (35)$$

$$h_2(\mathbf{X},\boldsymbol{\Psi},\mathbf{U}) = -\sum_{i=K+1}^{N} u_{i-K} \psi_i \left[\frac{\delta C}{\delta x_i} + \sum_{k=1}^{M} u_k \frac{\delta (g_k(\mathbf{X}))^2}{\delta x_i} \right] . \quad (36)$$

Let's designate the sum of these three components as $h_v(\mathbf{X},\boldsymbol{\Psi},\mathbf{U})$ (h_v=h_0+h_1+h_2). Formulas (5)-(6) and (27)-(36) define the process of system optimization and the process of computing a Hamiltonian in case of a *K* independent variable and *M* dependent variables. In the case of the circuit presented in Fig. 7, formulas are used for *K*=3 and *M*=2. In this case the control vector of **U** contains two components (u_1, u_2). The supremum of the function $H(\mathbf{X},\boldsymbol{\Psi},\mathbf{U})$ in the parameter **U** will be designated $H_{\max}$:

$$H_{\max}(\mathbf{X},\boldsymbol{\Psi}) = h_c(\mathbf{X},\boldsymbol{\Psi}) + \sup_{u \in \mathbf{U}} h_v(\mathbf{X},\boldsymbol{\Psi},\mathbf{U}). \quad (37)$$

For the circuit in Fig. 7 this function is defined by the following expression:

$$H_{\max}(\mathbf{X},\boldsymbol{\Psi}) = h_c(\mathbf{X},\boldsymbol{\Psi}) + \max \begin{Bmatrix} h_v(\mathbf{X},\boldsymbol{\Psi},(0,0)), h_v(\mathbf{X},\boldsymbol{\Psi},(0,1)), \\ h_v(\mathbf{X},\boldsymbol{\Psi},(1,0)), h_v(\mathbf{X},\boldsymbol{\Psi},(1,1)) \end{Bmatrix}. \quad (38)$$

The structure of the control vector providing this maximum in each point of optimization process represents the result of the use of the maximum principle. This optimal structure ensures the minimal value of the functional (11) and the minimal computing time.

V. Numerical Results and Discussion

The analysis of the process of optimization for a circuit with two nodes allows for the finding of the optimum structure of the control vector.

The possibility of applying the maximum principle of Pontryagin to the problem of optimization of electronic circuits is analyzed. It is shown that in spite of the fact that the problem of optimization is formulated as a nonlinear task, and the maximum principle in this case isn't a sufficient condition for obtaining a maximum of the functional, it is possible to obtain the decision in the form of local minima. Local minima of the functional, which is defined as the processor time necessary for the procedure of optimization, provide a rather low value of the functional. The relative acceleration of the CPU time for the best strategy found by means of maximum principle compared with the traditional approach is equal two to three orders of magnitude.

The behaviour of Hamiltonian for four possible options of the control vector **U**: (00), (01), (10), and (11), with the correct initial value of an auxiliary vector $\boldsymbol{\Psi}$, ($\boldsymbol{\Psi}_{0c} = (0.3,-1.85,-0.35,-1.9,0.32,2.5)$) is presented in Fig. 8.

The value of $\boldsymbol{\Psi}_{0c}$ has been obtained by the additional optimizing procedure on the basis of a gradient method for the following initial point in process of designing $\mathbf{X}^0$: (x_{10} =1.0, x_{20} = 1.0, x_{30} =1.0, x_{40} = -1.5, x_{50} = -1.6). Four possible combinations of the components of the control vector **U** define four various dependencies for Hamiltonian: $(H_{(00)}, H_{(01)}, H_{(10)}, H_{(11)})$. The Hamiltonian corresponding to the control vector **U**=(11) has the greatest value of all possible.

Therefore, the optimum trajectory corresponds to this vector and defines the first part of a trajectory in the space of parameters. Some two-dimensional projections of a trajectory of optimization process in the space of variables **X** are presented in Fig. 9.

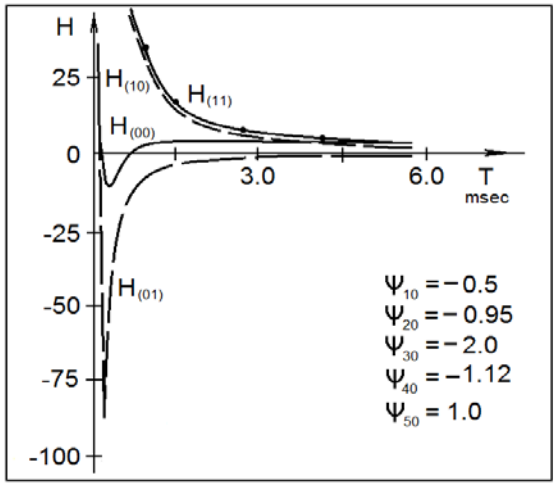

Fig. 8. Time dependency of functions $H_{(00)}$, $H_{(01)}$, $H_{(10)}$, $H_{(11)}$ for correct value of parameter Ψ_{0c}.

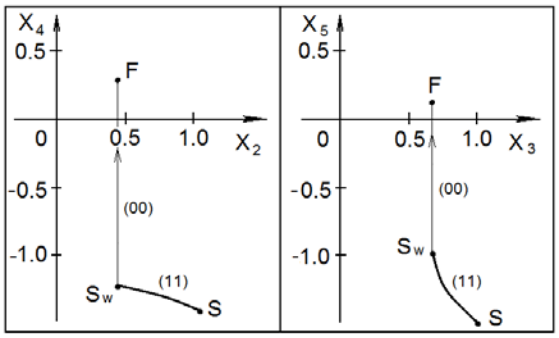

Fig. 9. Projections of trajectory of optimization process with initial point $\mathbf{X}^0$: $(x_{10}=1, x_{20}=1, x_{30}=1, x_{40}=-1.5, x_{50}=-1.6)$.

A part of the trajectory from point S to point S_w corresponds to the control vector (11). Then, at a time point of 5.753 msec, which corresponds to 121 steps of integration of the system (5), the Hamiltonian corresponding to the strategy with the control vector (00) becomes the greatest of all, and at this moment the vector (00) becomes the optimum control vector. The trajectory includes the jump and the current point of the optimization process instantly moves to the final point of the solution of a problem of F. This effect was named as a special effect of acceleration of the circuit optimization process.

The data of optimization process for the presented circuit for four strategies of structural basis and the same initial point $\mathbf{X}^0$ are provided in Table 2 for comparison.

Table 2. Data of all strategies of structural basis.

N	Control vector	Iterations number	Total CPU time (sec)
1	(0 0)	116973	16.081
2	(0 1)	139143	8.897
3	(1 0)	133154	11.241
4	(1 1)	170953	7.934

All the strategies provide the same solution for the independent variables x_1, x_2, x_3, minimizing the objective function $C(\mathbf{X})$, but they have the different iteration numbers and different total CPU time.

The most rapid strategy is MTSO, which corresponds to the control vector (11) has a processor time of 7.934 sec. Acceleration for the obtained optimum strategy in comparison with MTSO with control vector (11) of Table 2 is equal to 1,379 times and 2,795 times in comparison with TSO with control vector (00).

The behaviour of the Hamiltonian that corresponds to another choice for the initial point of optimization process of circuit $\mathbf{X}^0$: (x_{10} =1.0, x_{20} = 1.0, x_{30} =1.0, x_{40} = -2.5, x_{50} = -2.5) for four possible values of the control vector $\mathbf{U}$: (00), (01), (10), and (11) is shown in Fig. 10.

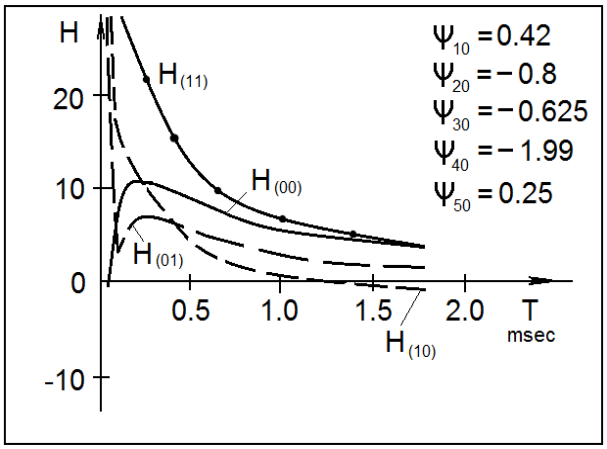

Fig. 10. Time dependency of functions $H_{(00)}$, $H_{(01)}$, $H_{(10)}$, $H_{(11)}$ for other correct value of parameter $\Psi_{0c} = (0.42, -0.8, -0.625, -1.99, 0.25)$.

In this case the following correct value of an auxiliary vector $\Psi_{0c} = (0.42, -0.8, -0.625, -1.99, 0.25)$ providing the minimum value of processor time has been obtained. In this example, the optimization procedure is defined by the control vector (11) from T=0 to T=1.761msec because the Hamiltonian of $H_{(11)}$ has the maximum value for this control vector of the four possible. Then, at a time of T=1.761 msec, which corresponds to the 35th step of process of integration of system (5) the Hamiltonian corresponding to the control vector (00) has the maximum value ($H_{(00)}$>=$H_{(11)}$) and the switching to the TSO is observed. The movement corresponding to the strategy (00) is carried out on one step of integration and the current point of the optimization process moves to the final point of F with the given accuracy. It is clear from the behaviour of the projections of the optimization trajectory shown in Fig. 11.

It is important to emphasize that the numerical algorithm automatically switches from one strategy to another on the basis of ratio (38), corresponding to the maximum principle.

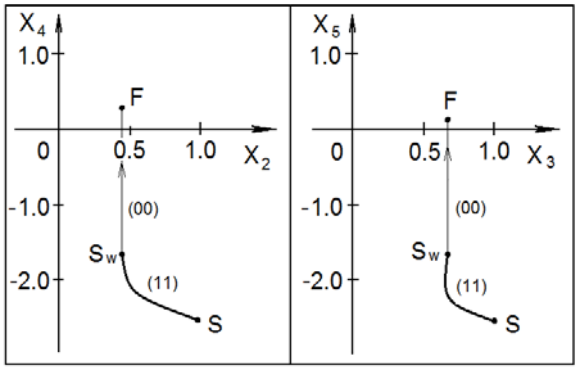

Fig. 11. Projections of trajectory of optimization process with initial point $\mathbf{X}^0$: $(x_{10}=1, x_{20}=1, x_{30}=1, x_{40}=-2.5, x_{50}=-2.5)$.

It would be desirable to note that the obtained decision is not the only local minimum of the target function of (11). Other local minimum has been reached with the other vector $\boldsymbol{\Psi}$.

The behaviour of the function of Hamilton for the same initial point of the optimization process of the circuit $\mathbf{X}^0$: $(x_{10}=1, x_{20}=1, x_{30}=1, x_{40}=-2.5, x_{50}=-2.5)$ but containing other initial value for auxiliary vector $\boldsymbol{\Psi}$ is presented in Fig. 12. The correct value of $\boldsymbol{\Psi}_{0c}$ obtained by the additional optimizing procedure is next $\boldsymbol{\Psi}_{0c}=(0.1,-0.02,-0.5,-0.2,0.6)$. In this case the other strategy is optimal one.

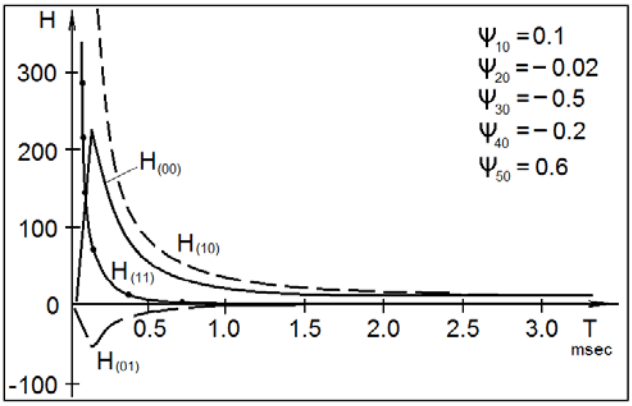

Fig. 12. Time dependency of functions $H_{(00)}$, $H_{(01)}$, $H_{(10)}$, $H_{(11)}$ for other correct value of parameter $\boldsymbol{\Psi}_{0c}=(0.1,-0.02,-0.5,-0.2,0.6)$.

An additional optimization by means of parameter $\boldsymbol{\Psi}_0$ leads to other local minimum that is visible from the obtained dependencies. The Hamiltonian corresponding to the strategy with control vector (10) accepts the maximum value and this strategy is the first part of the optimum strategy. The Hamiltonian corresponding to TSO with the control vector (00) is greater than for all other strategy from the point corresponding to the 38th step of integration, and a switching to TSO takes place. That is also clear from the behaviour of projections of the optimization trajectory in Fig. 13.

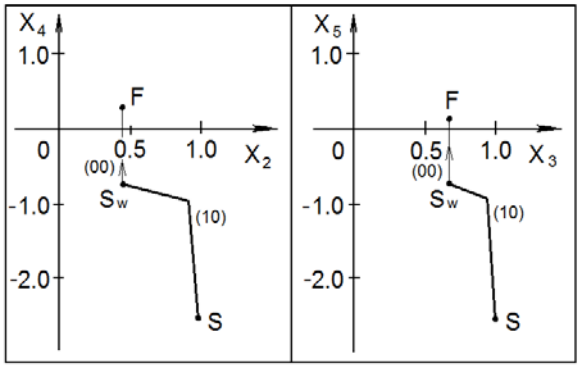

Fig. 13. Projections of trajectory of optimization process for other correct value of $\boldsymbol{\Psi}_{0c}=(0.1,-0.02,-0.5,-0.2,0.6)$.

The total time of optimization is equal 3.345 msec. This decision represents other local minimum of the functional (11).

It is important to note that the strategy found from the conditions of the maximum principle and corresponding to the control vector with two parts (10) and (00) and switching between them on a 73rd step of integration has not been predicted in previous research. In earlier executed analysis it was supposed that the optimum strategy must be constructed on the basis of the combination of MTSO and TSO. This assumption is not always fulfilled as shown in the present analysis.

This example shows that earlier predicted optimum structure consisting of only of MTSO and TSO is not always optimum.

VI. Conclusion

Analysis of the application of maximum principle to a problem of circuit optimization proves that the formerly studied effect of acceleration on the process of optimization appears owing to this principle. This means that the maximum principle of Pontryagin provides a theoretical justification for the acceleration effect that appears when we use the generalized formulation of process of circuit optimization. It is confirmed that the maximum principle allows for finding one or several local minima of the functional that is defined as the processor time. Aside from that, the use of the maximum principle provides the chance to significantly reduce the computing time for circuit optimization.

The analysis of optimization process of the presented circuits showed that application of the maximum principle really allows finding the optimum structure of the control vector **U**(t) by means of iterative procedure. These results were obtained for *N*-dimensional space of parameters.

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