Optimality conditions and duality results for a class of differentiable vector optimization problems with the multiple interval-valued objective function

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***Abstract*—In this paper, a differentiable interval-valued vector optimization problem with the multiple objective function and with both inequality and equality constraints is considered. The Karush-Kuhn-Tucker necessary optimality conditions are established for a weak LU-Pareto solution in the considered vector optimization problem with the multiple interval-objective function under the Kuhn-Tucker constraint qualification. Further, the sufficient optimality conditions for a (weak) LU-Pareto solution and several duality results in Mond-Weir sense are proved under assumptions that the functions constituting the considered differentiable vector optimization problem with the multiple interval-objective function are (F,ρ)-convex.**

***Index Terms*—differentiable multiobjective programming problem with the multiple interval-objective function; Karush-Kuhn-Tucker necessary optimality conditions; Kuhn-Tucker constraint qualification; LU-Pareto solution; (F,ρ)-convex function; Mond-Weir duality.**

AMS Classification: 90C29; 90C30; 90C46; 90C26.

I. Introduction

MOST of the real-life problems are frequently characterized by multiple and conflicting criteria. Such conditions are normally estimated by optimizing multiple objective functions. In the conventional vector optimization problems, the coefficients are all assumed as real numbers. However, uncertainty always occurs in the real world. Among many types of methodologies usually used to solve vector optimization models, the interval-valued multiobjective optimization problems have been of much interest in recent past and thus explored the extent of optimality conditions and duality applicability in different areas (see, for example, [1], [2], [13], [16], and the references therein). The Karush-Kuhn-Tucker optimality conditions play an important role in the area of optimization theory and have been studied for over a century. For interval-valued vector optimization problems, the Karush-Kuhn-Tucker necessary optimality conditions are also studied in many recent publications. Ishibuchi and Tanaka [6] considered multiobjective programming problems with interval-valued objective functions and proposed the ordering relation between two closed intervals by considering the maximization and minimization problems separately. Urli and Nadeau [14] used an interactive method for solving the linear multiobjective programming problems with interval coefficients. To do this, they also proposed a methodology in which a nondeterministic problem is transformed into a deterministic problem. Chanas and Kuchta [2] generalized the concept of optimality introduced by lshihuchi and Tanaka [6] for vector optimization problems with interval-valued objective functions to the case of the linear multiobjective programming problem with interval coefficients in the objective function based on preference relations between intervals. Wu [16] studied the Karush-Kuhn-Tucker necessary optimality conditions for multiobjective programming problems with interval-valued objective functions. Similar to the concept of a nondominated solution in vector optimization problems, Wu has proposed a solution concept for optimization problems with an interval-valued objective function based on a partial ordering on the set of all closed intervals. By using gH-derivative of interval valued functions, Singh et al. [13] established the Karush-Kuhn-Tucker necessary optimality conditions for multiobjective programming problems with interval valued objective functions considering order relationship between two closed intervals. Hosseinzade and Hassanpour [5] established the optimality conditions for convex multiobjective programming problems with interval valued objective functions and with inequality constraints only. Jana and Panda [7] considered a nonlinear vector optimization problem with both linear and nonlinear interval-valued functions in the objective function as well as in the constraints. They proposed a methodology to find efficient solutions and they named them as preferable efficient solutions. Karmakar and Bhunia [8] proposed an alternative optimization technique via multiobjective programming for constrained optimization problems with interval-valued objectives. Recently, Singh et al. [12] developed a theoretical and practical solution method for convex multiobjective programming problems with interval valued objective functions by considering order relationship between two closed intervals.

Another main part in optimization theory is establishing sufficient optimality conditions. In the optimization literature, it is possible to find a few articles devoted on this issue only. Wu [16] established the sufficiency of the KKT necessary optimality conditions under various convexity and pseudoconvexity hypotheses. Recently, Zhang et al. [17] studied the Karush-Kuhn-Tucker optimality conditions in a class of nonconvex optimization problems with an interval-valued objective function and derived for LU-preinvex and invex optimization problems with an interval-valued objective function under the conditions of weakly continuous differentiability and Hukuhara differentiability

Most of the works on optimality conditions and duality results for interval-valued optimization problems concerns scalar optimization problems of such a type. The purpose of this work is, therefore, to study optimality conditions and duality for a new class of differentiable interval-valued multiobjective programming problems with multiple interval-valued objective function. Namely, the Karush-Kuhn-Tucker necessary optimality conditions are proved under Kuhn-Tucker constraint qualification for a differentiable vector optimization problem with the multiple objective function and with both equality and inequality constraints. Further, the sufficiency of these necessary optimality conditions are established for the considered differentiable vector optimization problem with the multiple objective function and with both equality and inequality constraints under assumption that the involved functions are (F,ρ)-convex, not necessarily with respect to the same ρ. Further, for the considered differentiable vector optimization problem with the multiple objective function and with both equality and inequality constraints, its interval-valued vector dual problem in the sense of Mond-Weir is defined and several duality results are established between these two interval-valued vector optimization problems with multiple objective functions also under (F,ρ)-convexity hypotheses. The optimality results established in the paper are illustrated by examples of differentiable vector optimization problems with the multiple interval-valued objective functions.

II. Notations and preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper.

For any vectors $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ in R^n, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;
(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;
(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$;
(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

Let $I(R)$ be a class of all closed and bounded intervals in R. Throughout this paper, when we say that A is a closed interval, we mean that A is also bounded in R. If A is a closed interval, we use the notation $A = [a^L, a^U]$, where a^L and a^U mean the lower and upper bounds of A, respectively. In other words, if $A = [a^L, a^U] \in I(R)$, then $A = [a^L, a^U] = \{x \in R : a^L \leqq x \leqq a^U\}$. If $a^L = a^U = a$, then $A = [a, a] = a$ is a real number.

Let $A = [a^L, a^U]$, $B = [b^L, b^U]$, then, by definition, we have:

i) $A + B = \{a + b : a \in A \text{ and } b \in B\} = [a^L + b^L, a^U + b^U]$,
ii) $A - B = A + (-B) = \{a - b : a \in A \text{ and } b \in B\} = [a^L - b^U, a^U - b^L]$,
iii) $-A = \{-a : a \in A\} = [-a^U, -a^L]$.
iv) $k + A = \{k + a : a \in A\} = [k + a^L, k + a^U]$, where k is a real number,
v) $kA = \begin{cases} [ka^L, ka^U] & \text{if} \quad k > 0, \\ [ka^U, ka^L] & \text{if} \quad k \leqq 0 \end{cases}$ where k is a real number.

In interval mathematics, an order relation is often used to rank interval numbers and it implies that an interval number is better than another but not that one is larger than another.

For $A = [a^L, a^U]$ and $B = [b^L, b^U]$, we write

$$A \leqq_{LU} B \text{ if and only if } \begin{cases} a^L \leqq b^L \\ a^U \leqq b^U \end{cases}. \tag{1}$$

It means that A is inferior to B, or B is superior to A. It is easy to see that $\leqq_{LU}$ is a partial ordering on $I(R)$.

Further, we can write $A <_{LU} B$ if and only if $A \leqq_{LU} B$ and $A \neq B$. Equivalently,

$$A <_{LU} B \text{ if and only if}$$

$$\begin{cases} a^L < b^L \\ a^U \leqq b^U \end{cases}, \text{ or} \begin{cases} a^L \leqq b^L \\ a^U < b^U \end{cases} \text{ or} \begin{cases} a^L < b^L \\ a^U < b^U \end{cases}. \tag{2}$$

Throughout this section, let X be a nonempty subset of R^n. A function $\psi : X \to I(R)$ is called an interval-valued function if $\psi(x) = [\psi^L(x), \psi^U(x)]$ with ψ^L, $\psi^U : X \to R$ such that $\psi^L(x) \leqq \psi^U(x)$ for each $x \in X$.

Now, we shall consider the differentiation of an interval-valued function. Namely, we use a very straightforward concept of differentiation introduced by Wu [15].

Definition 1: Let S be a nonempty open set in R. An interval-valued function $\psi : S \to I(R)$ with $f(x) = [f^L(x), f^U(x)]$ is called weakly differentiable at u if the real-valued functions f^L and f^U are differentiable at u (in the usual sense).

Now, we recall the definition of a sublinear functional (with respect to the third component).

Definition 2: A functional $F : X \times X \times R^n \to R$ is sublinear (with respect to the third component) if, for all $x, u \in X \subset R^n$,

i) $F(x, u; q_1 + q_2) \leqq F(x, u; q_1) + F(x, u; q_2)$, $\forall q_1, q_2 \in R^n$,
ii) $F(x, u; \alpha q) = \alpha F(x, u; q)$, $\forall \alpha \in R_+$, $\forall q \in R^n$.

The concept of the sublinear functional was given by Hanson and Mond [4] (see also Preda [11]). By ii), it is clear that

$$F(x, u; 0) = 0. \tag{3}$$

Several generalizations of the definition of a convex function have been introduced to optimization theory in order to weak the assumption of convexity for establishing optimality and duality results for new classes of nonconvex optimization problems, including vector optimization problems. One of such generalizations in the smooth vectorial case is the definition of a vector-valued (F, ρ)-convex function introduced by Preda [11]. Now, we recall it for a common reader. Let $d(\cdot, \cdot)$ be a pseudometric on R^n.

Definition 3: Let $f = (f_1, ..., f_p) : X \to R^p$ be a differentiable vector-valued function defined on X and $\overline{x} \in X$ be given. If there exist a sublinear function $F : X \times X \times R^{n+1} \to R$ with respect to the third component and $\rho = (\rho_1, ..., \rho_p) \in R^p$ such that, the following inequalities

$$f_i(x) - f_i(\overline{x}) \geqq F(x, \overline{x}; \nabla f_i(\overline{x})) + \rho_i d^2(x, \overline{x}) \quad (>), \quad i = 1, ..., p, \tag{4}$$

hold for all $x \in X$, then f is said to be a (vector) (F, ρ)-convex (strictly (F, ρ)-convex) at $\overline{x}$ on X.
Each function f_i, $i = 1, ..., p$, satisfying (4) is said to be a (vector) (F, ρ_i)-convex (strictly (F, ρ_i)-convex) at $\overline{x}$ on X.
If inequalities (4) are satisfied at any point $\overline{x}$, then f is said to be a vector (F, ρ)-convex (vector strictly (F, ρ)-convex) function on X.

Remark 4: In the case $\rho = 0$, the function f satisfying (4) is said to be F-convex at $\overline{x}$ on X (see Hanson and Mond [4] in a scalar case and Gulati and Islam [3] in a vectorial case). In the case $\rho > 0$, the function f satisfying (4) is said to be strongly F-convex at $\overline{x}$ on X, whereas when $\rho < 0$, the function f satisfying (4) is said to be weakly F-convex at $\overline{x}$ on X (see Preda [11]).

In order to define an analogous class of differentiable vector (strictly) (F, ρ)-concave functions, the direction of the inequality in the definition of these functions should be changed to the opposite one.

III. (WEAK) LU-PARETO OPTIMALITY

In this section, we consider the following differentiable vector optimization problem with the multiple interval-valued objective function:

$$\begin{aligned} & f(x) = (f_1(x), ..., f_p(x)) \to V\text{-min} \\ & g(x) = (g_1(x), ..., g_m(x)) \leqq 0, \\ & h(x) = (h_1(x), ..., h_q(x)) = 0, \\ & x \in X, \end{aligned} \tag{IVP}$$

where V-min denotes the (weak) LU-Pareto minimization, each $f_i : R^n \to I(R)$, $i \in I = \{1, ..., p\}$ is an interval-valued function, that is,

$$f_i(x) = \left[f_i^L(x), f_i^U(x)\right], \; i \in I,$$

and, moreover, $g : X \to R^m$, $h : X \to R^q$, X is a nonempty open convex subset of R^n. We will assume, moreover, that f_i^L, $f_i^U : R^n \to R$, $i \in I$, $g_j : R^n \to R$, $j \in J$, and $h_k : R^n \to R$, $k \in K$, are differentiable functions on X. For the purpose of simplifying our presentation, we will introduce the following notations $f^L = \left(f_1^L, ..., f_p^L\right)^T$, $f^U = \left(f_1^U, ..., f_p^U\right)^T$. Further, let us denote by Ω the set of all feasible solutions in the considered interval-valued multiobjective optimization problem (IVP), that is, the set $\Omega = \{x \in R^n : g(x) \leqq 0, \, h(x) = 0\}$ and, moreover, by $J(x)$, the set of constraint indices that are active at a feasible solution x, that is, $J(x) = \{j \in J : g_j(x) = 0\}$.

Since each of objective values f_i is a closed interval, we need to provide an ordering relation between any two closed intervals. The most direct way is to invoke the ordering relation $\leqq_{LU}$ that was defined above. However, $\leqq_{LU}$ is a partial ordering relation, not a total ordering, on $I(R)$, we shall follow the similar concept of a nondominated solution used in multiobjective programming problem to investigate the solution concepts.

For such interval-valued multicriterion optimization problems, Wu [16] proposed the following different concepts of (weak) Pareto optimal solutions in terms of a weak LU-Pareto (weakly LU-efficient) solution and a LU-Pareto (LU-efficient) solution in the following sense:

Definition 5: A feasible point $\overline{x}$ is said to be a weak LU-Pareto (weakly LU-efficient) solution for (IVP) if and only if there exists no feasible point x such that, for each $i \in I$,

$$f_i(x) <_{LU} f_i(\overline{x}).$$

Definition 6: A feasible point $\overline{x}$ is said to be a LU-Pareto (LU-efficient) solution for (IVP) if and only if there exists no feasible point x such that, for each $i \in I$,

$$f_i(x) \leqq_{LU} f_i(\overline{x})$$

and

$$f_i(x) <_{LU} f_i(\overline{x}) \text{ for at least one } i \in I.$$

In order to prove the Karush-Kuhn-Tucker necessary optimality conditions for a weak LU-Pareto solution in the multiobjective programming problem (VP), we extend the Kuhn-Tucker constraint qualification given by Mangasarian [9] to the case of optimization problems with both inequality and equality constraints.

Definition 7: Let the constraint functions $g = (g_1, ..., g_m)$ and $h = (h_1, ..., h_q)$ be differentiable at $\overline{x} \in \Omega$. It is said that the Kuhn-Tucker constraint qualification is satisfied at $\overline{x}$ if, for any $d \in R^n$, $d \neq 0$, such that $\nabla g_j(\overline{x})^T d \leqq 0$ for all $j \in J(\overline{x})$, and $\nabla h_k(\overline{x})^T d = 0$, $k \in K$, there exist a function $\varphi : [0, 1] \to R^n$ which is continuously differentiable at 0, and some real scalar $\beta > 0$, such that

$$\varphi(0) = \overline{x}, \quad \varphi(\alpha) \in \Omega \text{ for all } \alpha \in [0, 1] \text{ and } \varphi'(0) = \beta d. \tag{5}$$

Before we establish the Karush-Kuhn-Tucker necessary optimality conditions for problem (VP), we re-call the Motzkin's theorem of the alternative.

Theorem 8: [9] (Motzkin's theorem of the alternative). Let A, C, D be given matrices, with A being nonvacuous. Then either the system of inequalities

$$Ax < 0, \;\; Cx \leqq 0, \;\; Dx = 0$$

has a solution x, or the system

$$A^T y_1 + C^T y_2 + D^T y_3 = 0, \quad y_1 \geq 0, \, y_2 \geqq 0$$

has solution y_1, y_2 and y_3, but never both.

In [15], Wu proved the Karush-Kuhn-Tucker necessary optimality conditions for a scalar optimization problem with the multiple interval-valued objective function under the Kuhn-Tucker constraint qualification. Now, we extend this result for a differentiable vector optimization problem with the interval-valued objective function and with both inequality and equality constraints.

Theorem 9: (Karush-Kuhn-Tucker necessary optimality conditions). Let $\overline{x} \in \Omega$ be a weak LU-Pareto solution in the vector optimization problem (IVP) with the multiple interval-valued objective function and the Kuhn-Tucker constraint qualification be satisfied at $\overline{x}$. Then there exist Lagrange multipliers $\overline{\lambda}^L \in R^p$, $\overline{\lambda}^U \in R^p$, $\overline{\mu} \in R^m$ and $\overline{\xi} \in R^q$ such that

$$\sum_{i=1}^{p} \overline{\lambda}_i^L \nabla f_i^L(\overline{x}) + \sum_{i=1}^{p} \overline{\lambda}_i^U \nabla f_i^U(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(\overline{x}) + \sum_{k=1}^{q} \overline{\xi}_k \nabla h_k(\overline{x}) = 0, \tag{6}$$

$$\overline{\mu}_j g_j(\overline{x}) = 0, \ \ j \in J, \tag{7}$$

$$\overline{\lambda}^L \geq 0, \, \overline{\lambda}^U \geq 0, \ \ \overline{\mu} \geqq 0. \tag{8}$$

Proof. Let $\overline{x} \in \Omega$ be a weak LU-Pareto solution in the vector optimization problem (VP) with the multiple interval-valued objective function and the Kuhn-Tucker constraint qualification be satisfied at $\overline{x}$. Now, we prove that there does not exist $d \in R^n$, $d \neq 0$, satisfying the following system of inequalities:

$$\nabla f_i^L(\overline{x})^T d < 0, \, \nabla f_i^U(\overline{x})^T d < 0, \quad i \in I, \tag{9}$$

$$\nabla g_j(\overline{x})^T d \leqq 0, \ \ j \in J\left(\overline{x}\right), \tag{10}$$

$$\nabla h\left(\overline{x}\right)^T d = 0. \tag{11}$$

By means of contradiction, suppose that there exists any $d \in R^n$, $d \neq 0$, satisfying (9), (10) and (11). By the Kuhn-Tucker constraint qualification, there exists a function $\varphi : [0, 1] \to R^n$ which is continuously differentiable at 0, and some real scalar $\beta > 0$ such that (5) is satisfied. Since f_i^L and f_i^U $i \in I$, are differentiable, we can approximate $f_i^L\left(\varphi\left(\alpha\right)\right)$ and $f_i^U\left(\varphi\left(\alpha\right)\right)$ linearly as follows:

$$\begin{aligned} f_i^L\left(\varphi\left(\alpha\right)\right) = f_k^L\left(\overline{x}\right) + \nabla f_k^L(\overline{x})^T\left(\varphi\left(\alpha\right) - \overline{x}\right) + \\ \left\|\varphi\left(\alpha\right) - \overline{x}\right\| \theta_i^L\left(\varphi\left(\alpha\right), \overline{x}\right) = \\ f_i^L\left(\overline{x}\right) + \alpha \nabla f_i^L(\overline{x})^T \left(\frac{\varphi(\alpha) - \varphi(0)}{\alpha}\right) + \\ \left\|\varphi\left(\alpha\right) - \varphi\left(0\right)\right\| \theta_i^L\left(\varphi\left(\alpha\right), \varphi\left(0\right)\right), \end{aligned} \tag{12}$$

where $\theta_i^L\left(\varphi\left(\alpha\right), \varphi\left(0\right)\right) \to 0$ as $\left\|\varphi\left(\alpha\right) - \varphi\left(0\right)\right\| \to 0$ and

$$\begin{aligned} f_i^U\left(\varphi\left(\alpha\right)\right) = f_i^U\left(\overline{x}\right) + \nabla f_i^U(\overline{x})^T\left(\varphi\left(\alpha\right) - \overline{x}\right) \\ + \left\|\varphi\left(\alpha\right) - \overline{x}\right\| \theta_i^U\left(\varphi\left(\alpha\right), \overline{x}\right) = \\ f_i^U\left(\overline{x}\right) + \alpha \nabla f_i^U(\overline{x})^T \left(\frac{\varphi(\alpha) - \varphi(0)}{\alpha}\right) \\ + \left\|\varphi\left(\alpha\right) - \varphi\left(0\right)\right\| \theta_i^U\left(\varphi\left(\alpha\right), \varphi\left(0\right)\right), \end{aligned} \tag{13}$$

where $\theta_i^U\left(\varphi\left(\alpha\right), \varphi\left(0\right)\right) \to 0$ as $\left\|\varphi\left(\alpha\right) - \varphi\left(0\right)\right\| \to 0$. As $\alpha \to 0$, then $\left\|\varphi\left(\alpha\right) - \varphi\left(0\right)\right\| \to 0$ and, moreover,

$$\frac{\varphi\left(\alpha\right) - \varphi\left(0\right)}{\alpha} \to \varphi'\left(0\right) = \beta d. \tag{14}$$

Since (9) is assumed to hold, by (12), (13) and (14) yield, respectively, that the following inequalities

$$f_i^L\left(\varphi\left(\alpha\right)\right) < f_i^L\left(\overline{x}\right), \quad i \in I,$$

$$f_i^U\left(\varphi\left(\alpha\right)\right) < f_i^U\left(\overline{x}\right), \quad i \in I,$$

hold for sufficiently small α. This is a contradiction to the assumption that $\overline{x} \in \Omega$ is a weak LU-Pareto solution in the considered vector optimization problem (IVP). This means that there does not exist any $d \in R^n$ satisfying the system of inequalities (9)-(11). Therefore, by Motzkin's theorem of the alternative (see Theorem 8), we conclude that there exist $\overline{\lambda}^L \in R^p$, $\overline{\lambda}^U \in R^p$, $\overline{\mu}_j$, $j \in J\left(\overline{x}\right)$, and $\overline{\xi} \in R^q$ such that

$$\sum_{i=1}^{p} \overline{\lambda}_i^L \nabla f_i^L(\overline{x}) + \sum_{i=1}^{p} \overline{\lambda}_i^U \nabla f_i^U(\overline{x}) + \sum_{j \in J(\overline{x})} \overline{\mu}_j \nabla g_j(\overline{x}) + \sum_{k=1}^{q} \overline{\xi}_k \nabla h_k(\overline{x}) = 0. \tag{15}$$

If we set that $\overline{\mu}_j = 0$ for all $j \in J \backslash J\left(\overline{x}\right)$, then (15) gives (6). Further, note that also (7) is satisfied. Indeed, if $g_j(\overline{x}) < 0$, then $j \in J \backslash J\left(\overline{x}\right)$ and $\overline{\mu}_j = 0$. ■

In order to illustrate the above result, we present an example of such a vector optimization problem (IVP) with the multiple interval-valued objective function for which the Kuhn-Tucker constraint qualification is not satisfied.

Example 10: Consider the following differentiable vector optimization problem with the multiple interval-valued objective function:

$$\begin{gathered} f(x) = \left(\left[f_1^L(x), f_1^U(x)\right], \left[f_2^L(x), f_2^U(x)\right]\right) = \\ \left(\left[\left(x_1 - 3\right)^2 + 2x_2^2 \ , \ \left(x_1 - 3\right)^2 + 2x_2^2 + 1\right] \ , \right. \\ \left. \left[\left(x_1 - 2\right)^4 + x_2^2 - 1 \ , \ \left(x_1 - 2\right)^4 + x_2^2\right]\right) \to V\text{-}\min \quad \text{(IVP1)} \\ g_1(x) = x_2 - \left(1 - x_1\right)^3 \leqq 0, \\ g_2(x) = -x_2 = 0. \end{gathered}$$

Note that the feasible solution $\overline{x} = (1, 0)$ is a LU-Pareto solution in the considered vector optimization problem (IVP1) with the multiple interval-valued objective function. However, the Karush-Kuhn-Tucker necessary optimality conditions are not satisfied at this point. Indeed, by (6), it follows that $-4\left(\overline{\lambda}_1^L + \overline{\lambda}_1^U + \overline{\lambda}_2^L + \overline{\lambda}_2^U\right) = 0$, what is not possible. This is a consequence of the fact that the Kuhn-Tucker constraint qualification is not fulfilled at $\overline{x} = (1, 0)$. Indeed, for any function $\varphi : [0, 1] \to R^n$, which is continuously differentiable at 0, satisfying $\varphi\left(0\right) = \overline{x}$, $\varphi\left(\alpha\right) \in \Omega$ for all $\alpha \in [0, 1]$, the condition that there exists a scalar $\beta > 0$ such that $\varphi'\left(0\right) = \beta d$ is not satisfied. Indeed, if we set, for example $\varphi\left(\alpha\right) = \left(1 - \alpha\right) d$, where $d = (1, 0)$, then, in fact, the condition $\varphi'\left(0\right) = \beta d$ is not satisfied for each $\beta > 0$.

Now, we give the definition of a Karush-Kuhn-Tucker point in problem (IVP).

Definition 11: The point $\left(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}, \overline{\xi}\right) \in \Omega \times R^p \times R^p \times R^m \times R^q$ is said to be a Karush-Kuhn-Tucker point for the considered multiobjective programming problem (IVP) with the multiple interval-valued objective function, if the conditions (6)-(8) are satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\lambda}^L$, $\overline{\lambda}^U$, $\overline{\mu}$ and $\overline{\xi}$.

Now, we prove the sufficiency of the Karush-Kuhn-Tucker necessary optimality conditions for the considered differentiable vector optimization problem (IVP) with the multiple interval-valued objective function under (F,ρ)-convexity assumptions imposed on the involved functions.

Theorem 12: Let $\left(\overline{x},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\mu},\overline{\xi}\right) \in \Omega\times R^p\times R^p\times R^m\times R^q$ be a Karush-Kuhn-Tucker point in the considered differentiable vector optimization problem (IVP) with the multiple interval-valued objective function. Further, assume that f_i^L, $i\in I$, is a $\left(F,\rho_{f_i}^L\right)$-convex function at $\overline{x}$ on Ω, f_i^U, $i\in I$, is a $\left(F,\rho_{f_i}^U\right)$-convex function at $\overline{x}$ on Ω, each constraint function g_j, $j\in J(\overline{x})$, is a $\left(F,\rho_{g_j}\right)$-convex function at $\overline{x}$ on Ω, each constraint function h_k, $k\in K^+(\overline{x})=\left\{k\in K:\overline{\xi}_k>0\right\}$, is a $\left(F,\rho_{h_k}^+\right)$-convex function at $\overline{x}$ on Ω, each function $-h_k$, $k\in K^-(\overline{x})=\left\{k\in K:\overline{\xi}_k<0\right\}$, is a $\left(F,\rho_{h_k}^-\right)$-convex function at $\overline{x}$ on Ω. If $\sum_{i=1}^p\overline{\lambda}_i^L\rho_{f_i}^L+\sum_{i=1}^p\overline{\lambda}_i^U\rho_{f_i}^U+\sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j}+\sum_{k\in K^+(\overline{x})}\overline{\xi}_k\rho_{h_k}^+-\sum_{k\in K^-(\overline{x})}\overline{\xi}_k\rho_{h_k}^-\geqq 0$, then $\overline{x}$ is a weak LU-Pareto solution in problem (IVP).

Proof. Assume that $\left(\overline{x},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\mu},\overline{\xi}\right)\in\Omega\times R^p\times R^p\times R^m\times R^q$ is a Karush-Kuhn-Tucker point for the considered optimization problem with the interval-valued objective function (IVP). Hence, by Definition 11, the Karush-Kuhn-Tucker necessary optimality conditions (6)-(8) are satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\lambda}^L\in R^p$, $\overline{\lambda}^U\in R^p$, $\overline{\mu}\in R^m$ and $\overline{\xi}\in R^q$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not a weak LU-Pareto solution in the considered optimization problem with the interval-valued objective function (IVP). Hence, by Definition 5, there exists another feasible solution $\widetilde{x}$ such that the inequality

$$f_i(\widetilde{x}) <_{LU} f_i(\overline{x}),\ i\in I \tag{16}$$

holds. Hence, by the definition of the relation $<_{LU}$, (16) implies that for each $i\in I$,

$$\begin{gathered}\left(f_i^L(\widetilde{x})<f_i^L(\overline{x})\ \wedge\ f_i^U(\widetilde{x})\leqq f_i^U(\overline{x})\right)\\ \text{or}\ \left(f_i^L(\widetilde{x})\leqq f_i^L(\overline{x})\ \wedge\ f_i^U(\widetilde{x})<f_i^U(\overline{x})\right)\\ \text{or}\ \left(f_i^L(\widetilde{x})<f_I^L(\overline{x})\ \wedge\ f_i^U(\widetilde{x})<f_i^U(\overline{x})\right).\end{gathered}$$

By the Karush-Kuhn-Tucker necessary optimality condition (8), we have that $\overline{\lambda}^L\geq 0$, $\overline{\lambda}^U\geq 0$. Then, the above inequalities yield

$$\begin{gathered}\sum_{i=1}^p\overline{\lambda}_i^Lf_i^L(\widetilde{x})+\sum_{i=1}^p\overline{\lambda}_i^Lf_i^L(\widetilde{x})<\\ \sum_{i=1}^p\overline{\lambda}_i^Uf_i^U(\overline{x})+\sum_{i=1}^p\overline{\lambda}_i^Uf_i^U(\overline{x}).\end{gathered}\tag{17}$$

By assumption, f_i^L, $i\in I$, is a $\left(F,\rho_{f_i}^L\right)$-convex function at $\overline{x}$ on Ω, f_i^U, $i\in I$, is a $\left(F,\rho_{f_i}^U\right)$-convex function at $\overline{x}$ on Ω, each constraint function g_j, $j\in J(\overline{x})$, is a $\left(F,\rho_{g_j}\right)$-convex function at $\overline{x}$ on Ω, each constraint function h_k, $k\in K^+(\overline{x})=\left\{k\in K:\overline{\xi}_k>0\right\}$, is a $\left(F,\rho_{h_k}^+\right)$-convex function at $\overline{x}$ on Ω, each function $-h_k$, $k\in K^-(\overline{x})=\left\{k\in K:\overline{\xi}_k<0\right\}$, is a $\left(F,\rho_{h_k}^-\right)$-convex function at $\overline{x}$ on Ω. Thus, by Definition 3, it follows that the following inequalities

$$\begin{aligned}f_i^L(\widetilde{x})-f_i^L(\overline{x})\geqq F\left(\widetilde{x},\overline{x};\nabla f_i^L(\overline{x})\right)+\\ \rho_{f_i}^Ld^2(\widetilde{x},\overline{x}),\ i\in I,\end{aligned}\tag{18}$$

$$\begin{aligned}f_i^U(\widetilde{x})-f_i^U(\overline{x})\geqq F\left(\widetilde{x},\overline{x};\nabla f_i^U(\overline{x})\right)+\\ \rho_{f_i}^Ud^2(\widetilde{x},\overline{x}),\ i\in I,\end{aligned}\tag{19}$$

$$\begin{aligned}g_j(\widetilde{x})-g_j(\overline{x})\geqq F\left(\widetilde{x},\overline{x};\nabla g_j(\overline{x})\right)+\\ \rho_{g_j}d^2(\widetilde{x},\overline{x}),\ \ j\in J(\overline{x}),\end{aligned}\tag{20}$$

$$\begin{aligned}h_k(\widetilde{x})-h_k(\overline{x})\geqq F\left(\widetilde{x},\overline{x};\nabla h_k(\overline{x})\right)+\\ \rho_{h_k}^+d^2(\widetilde{x},\overline{x}),\ k\in K^+(\overline{x}),\end{aligned}\tag{21}$$

$$\begin{aligned}-h_k(\widetilde{x})+h_k(\overline{x})\geqq F\left(\widetilde{x},\overline{x};-\nabla h_k(\overline{x})\right)+\\ \rho_{h_k}^-d^2(\widetilde{x},\overline{x}),\ k\in K^-(\overline{x}).\end{aligned}\tag{22}$$

hold. Thus, by the Karush-Kuhn-Tucker necessary optimality condition (8), inequalities (18)-(22) give, respectively,

$$\begin{aligned}\overline{\lambda}_i^Lf_i^L(\widetilde{x})-\overline{\lambda}_i^Lf_i^L(\overline{x})\geqq\overline{\lambda}_i^LF\left(\widetilde{x},\overline{x};\nabla f_i^L(\overline{x})\right)+\\ \overline{\lambda}_i^L\rho_{f_i}^Ld^2(\widetilde{x},\overline{x}),\ i\in I,\end{aligned}\tag{23}$$

$$\begin{aligned}\overline{\lambda}_i^Uf_i^U(\widetilde{x})-\overline{\lambda}_i^Uf_i^U(\overline{x})\geqq\overline{\lambda}_i^UF\left(\widetilde{x},\overline{x};\nabla f_i^U(\overline{x})\right)+\\ \overline{\lambda}_i^U\rho_{f_i}^Ud^2(\widetilde{x},\overline{x}),\ i\in I,\end{aligned}\tag{24}$$

$$\begin{aligned}\overline{\mu}_jg_j(\widetilde{x})-\overline{\mu}_jg_j(\overline{x})\geqq\overline{\mu}_jF\left(\widetilde{x},\overline{x};\nabla g_j(\overline{x})\right)+\\ \overline{\mu}_j\rho_{g_j}d^2(\widetilde{x},\overline{x}),j\in J(\overline{x}),\end{aligned}\tag{25}$$

$$\begin{aligned}\overline{\xi}_kh_k(\widetilde{x})-\overline{\xi}_kh_k(\overline{x})\geqq\overline{\xi}_kF\left(\widetilde{x},\overline{x};\nabla h_k(\overline{x})\right)+\\ \overline{\xi}_k\rho_{h_k}^+d^2(\widetilde{x},\overline{x}),k\in K^+(\overline{x}),\end{aligned}\tag{26}$$

$$\begin{aligned}\overline{\xi}_kh_k(\widetilde{x})-\overline{\xi}_kh_k(\overline{x})\geqq-\overline{\xi}_kF\left(\widetilde{x},\overline{x};-\nabla h_k(\overline{x})\right)-\\ \overline{\xi}_k\rho_{h_k}^-d^2(\widetilde{x},\overline{x}),\ k\in K^-(\overline{x}).\end{aligned}\tag{27}$$

Adding both sides of (23) and (24), we get

$$\begin{gathered}\sum_{i=1}^p\overline{\lambda}_i^Lf_i^L(\widetilde{x})+\sum_{i=1}^p\overline{\lambda}_i^Lf_i^L(\widetilde{x})-\sum_{i=1}^p\overline{\lambda}_i^Uf_i^U(\overline{x})\\ -\sum_{i=1}^p\overline{\lambda}_i^Uf_i^U(\overline{x})\geqq\sum_{i=1}^p\overline{\lambda}_i^LF\left(\widetilde{x},\overline{x};\nabla f_i^L(\overline{x})\right)+\\ \sum_{i=1}^p\overline{\lambda}_i^UF\left(\widetilde{x},\overline{x};\nabla f_i^U(\overline{x})\right)+\\ \sum_{i=1}^p\overline{\lambda}_i^L\rho_{f_i}^Ld^2(\widetilde{x},\overline{x})+\sum_{i=1}^p\overline{\lambda}_i^U\rho_{f_i}^Ud^2(\widetilde{x},\overline{x}).\end{gathered}\tag{28}$$

Combining (17) and (28), we have

$$\begin{gathered}\sum_{i=1}^p\overline{\lambda}_i^LF\left(\widetilde{x},\overline{x};\nabla f_i^L(\overline{x})\right)+\sum_{i=1}^p\overline{\lambda}_i^UF\left(\widetilde{x},\overline{x};\nabla f_i^U(\overline{x})\right)+\\ \sum_{i=1}^p\overline{\lambda}_i^L\rho_{f_i}^Ld^2(\widetilde{x},\overline{x})+\sum_{i=1}^p\overline{\lambda}_i^U\rho_{f_i}^Ud^2(\widetilde{x},\overline{x})<0.\end{gathered}\tag{29}$$

Using $\widetilde{x}\in\Omega$, $\overline{x}\in\Omega$ together with the Karush-Kuhn-Tucker necessary optimality condition (7) in inequalities (25)-(27), and then adding both sides of the resulting inequalities, we get

$$\sum_{j\in J(\overline{x})}\overline{\mu}_j F\left(\widetilde{x},\overline{x};\nabla g_j\left(\overline{x}\right)\right)+\sum_{k\in K^+(\overline{x})}\overline{\xi}_k F\left(\widetilde{x},\overline{x};\nabla h_k\left(\overline{x}\right)\right)$$
$$+\sum_{k\in K^-(\overline{x})}\left(-\overline{\xi}_k\right)F\left(\widetilde{x},\overline{x};-\nabla h_k\left(\overline{x}\right)\right)+\left[\sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j}+\right.$$
$$\left.\sum_{k\in K^+(\overline{x})}\overline{\xi}_k\rho^+_{h_k}-\sum_{k\in K^-(\overline{x})}\overline{\xi}_k\rho^-_{h_k}\right]d^2\left(\widetilde{x},\overline{x}\right)\leqq 0. \tag{30}$$

Combining (29) and (30), we have

$$\sum_{i=1}^p\overline{\lambda}_i^L F\left(\widetilde{x},\overline{x};\nabla f_i^L\left(\overline{x}\right)\right)+\sum_{i=1}^p\overline{\lambda}_i^U F\left(\widetilde{x},\overline{x};\nabla f_i^U\left(\overline{x}\right)\right)+$$
$$\sum_{j\in J(\overline{x})}\overline{\mu}_j F\left(\widetilde{x},\overline{x};\nabla g_j\left(\overline{x}\right)\right)+\sum_{k\in K^+(\overline{x})}\overline{\xi}_k F\left(\widetilde{x},\overline{x};\nabla h_k\left(\overline{x}\right)\right)$$
$$+\sum_{k\in K^-(\overline{x})}\left(-\overline{\xi}_k\right)F\left(\widetilde{x},\overline{x};-\nabla h_k\left(\overline{x}\right)\right)+$$
$$\left[\sum_{i=1}^p\overline{\lambda}_i^L\rho^L_{f_i}d^2\left(\widetilde{x},\overline{x}\right)+\sum_{i=1}^p\overline{\lambda}_i^U\rho^U_{f_i}+\sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j}+\right.$$
$$\left.\sum_{k\in K^+(\overline{x})}\overline{\xi}_k\rho^+_{h_k}d^2\left(\widetilde{x},\overline{x}\right)-\sum_{k\in K^-(\overline{x})}\overline{\xi}_k\rho^-_{h_k}\right]d^2\left(\widetilde{x},\overline{x}\right)<0. \tag{31}$$

Using the sublinearity of the functional F (with respect to the third component) and taking into account Lagrange multipliers $\overline{\mu}_j=0$, $j\notin J\left(\overline{x}\right)$ and $\overline{\xi}_k=0$, $k\notin K^+\left(\overline{x}\right)\cup K^-\left(\overline{x}\right)$, we obtain

$$F\left(\widetilde{x},\overline{x};\left(\sum_{i=1}^p\overline{\lambda}_i^L\nabla f_i^L\left(\overline{x}\right)+\sum_{i=1}^p\overline{\lambda}_i^U\nabla f_i^U\left(\overline{x}\right)+\right.\right.$$
$$\left.\left.\sum_{j=1}^m\overline{\mu}_j\nabla g_j\left(\overline{x}\right)+\sum_{k=1}^q\overline{\xi}_k\nabla h_k\left(\overline{x}\right)\right)\right)$$
$$+\left[\sum_{i=1}^p\overline{\lambda}_i^L\rho^L_{f_i}d^2\left(\widetilde{x},\overline{x}\right)+\sum_{i=1}^p\overline{\lambda}_i^U\rho^U_{f_i}+\right. \tag{32}$$
$$\sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j}+\sum_{k\in K^+(\overline{x})}\overline{\xi}_k\rho^+_{h_k}d^2\left(\widetilde{x},\overline{x}\right)$$
$$\left.-\sum_{k\in K^-(\overline{x})}\overline{\xi}_k\rho^-_{h_k}\right]d^2\left(\widetilde{x},\overline{x}\right)<0.$$

By assumption, we have

$$\sum_{i=1}^p\overline{\lambda}_i^L\rho^L_{f_i}d^2\left(\widetilde{x},\overline{x}\right)+\sum_{i=1}^p\overline{\lambda}_i^U\rho^U_{f_i}+\sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j}+$$
$$\sum_{k\in K^+(\overline{x})}\overline{\xi}_k\rho^+_{h_k}d^2\left(\widetilde{x},\overline{x}\right)-\sum_{k\in K^-(\overline{x})}\overline{\xi}_k\rho^-_{h_k}\geqq 0.$$

Hence, (32) implies

$$F\left(\widetilde{x},\overline{x};\left(\sum_{i=1}^p\overline{\lambda}_i^L\nabla f_i^L\left(\overline{x}\right)+\sum_{i=1}^p\overline{\lambda}_i^U\nabla f_i^U\left(\overline{x}\right)\right.\right.$$
$$\left.\left.+\sum_{j=1}^m\overline{\mu}_j\nabla g_j\left(\overline{x}\right)+\sum_{k=1}^q\overline{\xi}_k\nabla h_k\left(\overline{x}\right)\right)\right)<0. \tag{33}$$

By the Karush-Kuhn-Tucker necessary optimality condition (6), (33) implies that the following inequality

$$F\left(\widetilde{x},\overline{x};0\right)<0$$

holds, contradicting (3). This completes the proof of this theorem. ■

Theorem 13: Let $\left(\overline{x},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\mu},\overline{\xi}\right)\in\Omega\times R^p\times R^p\times R^m\times R^q$ be a Karush-Kuhn-Tucker point in the considered differentiable multiobjective programming problem (IVP) with the multiple interval-valued objective function. Further, assume that f_i^L, $i\in I$, is a strictly $\left(F,\rho^L_{f_i}\right)$-convex function at $\overline{x}$ on Ω, f_i^U, $i\in I$, is a strictly $\left(F,\rho^U_{f_i}\right)$-convex function at $\overline{x}$ on Ω, each constraint function g_j, $j\in J\left(\overline{x}\right)$, is a $\left(F,\rho_{g_j}\right)$-convex function at $\overline{x}$ on Ω, each constraint function h_k, $k\in K^+\left(\overline{x}\right)=\left\{k\in K:\overline{\xi}_k>0\right\}$, is a $\left(F,\rho^+_{h_k}\right)$-convex function at $\overline{x}$ on Ω, each function $-h_k$, $k\in K^-\left(\overline{x}\right)=\left\{k\in K:\overline{\xi}_k<0\right\}$, is a $\left(F,\rho^-_{h_k}\right)$-convex function at $\overline{x}$ on Ω. If $\sum_{i=1}^p\overline{\lambda}_i^L\rho^L_{f_i}+\sum_{i=1}^p\overline{\lambda}_i^U\rho^U_{f_i}+\sum_{j\in J(\overline{x})}\overline{\mu}_j\rho_{g_j}+\sum_{k\in K^+(\overline{x})}\overline{\xi}_k\rho^+_{h_k}-\sum_{k\in K^-(\overline{x})}\overline{\xi}_k\rho^-_{h_k}\geqq 0$, then $\overline{x}$ is a LU-Pareto solution in problem (IVP).

In order to illustrate the optimality results established in the paper, we consider an example of a differentiable optimization problem with the multiple interval-valued objective function, in which the involved functions are differentiable (F,ρ)-convex.

Example 14: Consider the following differentiable vector optimization problem with the multiple interval-valued objective function:

$$\begin{gathered} f(x)=\left([1,1]\left(\ln^2\left(1-x_1\right)+x_1^3+\arctan x_1+\right.\right.\\ \left.\arctan x_2\right)+[0,1]\,,[1,1]\left(\ln^2\left(1-x_2\right)+x_2^3\right)+\\ \left.\left[\tfrac{1}{2},\tfrac{1}{2}\right]\left(x_1+x_2\right)+[0,1]\right)\to V\text{-}\min\\ g_1(x)=-x_1\leqq 0,\\ h_1(x)=x_1-x_2=0,\\ X=\left\{(x_1,x_2)\in R^2:x_1<1\wedge x_2<1\right\}. \end{gathered} \tag{IVP2}$$

We now re-write the considered differentiable vector optimization problem (IVP2) with interval-valued objective functions in the following form:

$$\begin{gathered} f(x)=\left(\left[f_1^L(x),f_1^U(x)\right],\left[f_2^L(x),f_2^U(x)\right]\right)=\\ \left(\left[x_1^3+\ln^2\left(1-x_1\right)+\arctan x_1+\arctan x_2,\right.\right.\\ \left.\ln^2\left(1-x_1\right)+x_1^3+\arctan x_1+\arctan x_2+1\right],\\ \left[\ln^2\left(1-x_2\right)+x_2^3+\tfrac{1}{2}x_1+\tfrac{1}{2}x_2\ ,\right.\\ \left.\left.\ln^2\left(1-x_2\right)+\ x_2^3+\tfrac{1}{2}x_1+\tfrac{1}{2}x_2+1\right]\right)\to V\text{-}\min\\ g_1(x)=-\arctan x_1\leqq 0,\\ h_1(x)=x_1-x_2=0,\\ X=\left\{(x_1,x_2)\in R^2:x_1<1\wedge x_2<1\right\}. \end{gathered} \tag{IVP2}$$

Note that $\Omega=\left\{(x_1,x_2)\in X:-\arctan x_1\leqq 0\ \wedge\ x_1-x_2=0\right\}$ and $\overline{x}=(0,0)$ is a feasible point in problem (IVP2). Further, it can be shown by Definition 6 that $\overline{x}=(0,0)$ is a LU-Pareto solution in problem (IVP2). Thus, the Karush-Kuhn-Tucker necessary optimality conditions (6)-(8) are satisfied at $\overline{x}=(0,0)$ with the Lagrange multipliers $\overline{\lambda}_1^L=\overline{\lambda}_2^L=\overline{\lambda}_2^L=\overline{\lambda}_2^U=\frac{1}{4}$, $\overline{\mu}_1=\frac{3}{2}$, $\overline{\xi}_1=\frac{3}{4}$. Let $d:R^2\times R^2\to R$ be defined by $d\left(x,\overline{x}\right)=\left|x_1-\overline{x}_1\right|+\left|x_2-\overline{x}_2\right|$. Let us define the sublinear functional F as follows

$$F\left(x,\overline{x};\vartheta\right)=2\left(x_1-\overline{x}_1\right)\vartheta_1+2\left(x_2-\overline{x}_1\right)\vartheta_1$$

and, moreover,

$$\rho_{f_1}=\left(\rho^L_{f_1},\rho^U_{f_1}\right)=\left(-1,-1\right),\ \rho_{f_2}=\left(\rho^L_{f_2},\rho^U_{f_2}\right)=\left(-\frac{1}{2},-\frac{1}{2}\right),$$
$$\rho_{g_1}=1,\ \rho^+_{h_1}=-1.$$

Note that $F(\overline{x},\overline{x};\cdot)=0$ and $\sum_{i=1}^{2}\overline{\lambda}_i^L\rho_{f_i}^L+\sum_{i=1}^{2}\overline{\lambda}_i^U\rho_{f_i}^U+\overline{\mu}_1\rho_{g_1}+\overline{\xi}_1\rho_{h_1}^{+}=0$. Further, it can be proved, by Definition 3, that the interval-valued objective functions f_1, f_2 are strictly (F,ρ_{f_1})-convex and (F,ρ_{f_2}) at $\overline{x}=(0,0)$ on Ω, respectively. Also the constraint function g_1 is (F,ρ_{g_1})-convex at $\overline{x}=(0,0)$ on Ω and the constraint function h_1 is $\left(F,\rho_{h_1}^{+}\right)$-convex at $\overline{x}=(0,0)$ on Ω. Since all hypotheses of Theorem 13 are satisfied, $\overline{x}=(0,0)$ is a LU-Pareto solution in problem (IVP2).

IV. Mond-Weir duality

In this section, for the considered differentiable multiobjective programming problem with the multiple interval-valued objective function (IVP), we define its vector dual problem with the interval-valued objective function in the sense of Mond-Weir [10]. Then we prove several duality results between problems (IVP) and (IVD) under assumption that the involved functions are differentiable (F,ρ)-convex, not necessarily, with respect to the same ρ.

Consider the following dual problem related to problem (IVP):

$$f(y)=\left(\left[f_1^L(y),f_1^U(y)\right],...,\left[f_p^L(y),f_p^U(y)\right]\right)\to V\text{-}\max$$

$$\begin{aligned}&\textstyle\sum_{i=1}^{p}\lambda_i^L\nabla f_i^L(y)+\sum_{i=1}^{p}\lambda_i^U\nabla f_i^U(y)+\\&\textstyle\sum_{j=1}^{m}\mu_j\nabla g_j(y)+\sum_{k=1}^{q}\xi_k\nabla h_k(y)=0,\end{aligned}\tag{34}$$

$$\sum_{j=1}^{m}\mu_j g_j(y)+\sum_{k=1}^{q}\xi_k h_k(y)\geqq 0,\quad \text{(IVD)}\tag{35}$$

$$\begin{aligned}&x\in X,\ \lambda^L\in R^p,\ \lambda^L\geq 0,\ \lambda^U\in R^p,\ \lambda^U\geq 0,\\&\mu\in R^m,\ \mu\geqq 0,\ \xi\in R^q.\end{aligned}\tag{36}$$

where the functions f^L, f^U, g, h are defined in the similar way as in the formulation of the considered differentiable multiobjective programming problem with the multiple interval-valued objective function (IVP).

Let

$$\Gamma=\left\{\begin{aligned}&\left(y,\lambda^L,\lambda^U,\mu,\xi\right)\in R^n\times R^p\times R^p\times R^m:\\&\sum_{i=1}^{p}\lambda_i^L\nabla f_i^L(y)+\sum_{i=1}^{p}\lambda_i^U\nabla f_i^U(y)+\\&\sum_{j=1}^{m}\mu_j\nabla g_j(y)+\sum_{k=1}^{q}\xi_k\nabla h_k(y)=0,\\&\sum_{j=1}^{m}\mu_j g_j(y)+\sum_{k=1}^{q}\xi_k h_k(y)\geq 0,\lambda^U\geq 0,\ \lambda^L\geq 0,\ \mu\geqq 0\end{aligned}\right\}$$

be the set of all feasible solutions in problem (IVD). Further, let us denote by Y the projection of Γ on X, that is, $Y=\left\{y\in R^n:\left(y,\lambda^L,\lambda^U,\mu,\xi\right)\in\Gamma\right\}$.

We now prove Mond-Weir weak duality under assumption that the involved functions are (F,ρ)-convex in vector optimization problems (IVP) and (IVD) with the multiple interval-valued objective functions.

Theorem 15: (Weak duality): Let x and $\left(y,\lambda^L,\lambda^U,\mu,\xi\right)$ be feasible solutions for problems (IVP) and (IVD), respectively. Furthermore, assume that f_i^L, $i\in I$, is a $\left(F,\rho_{f_i}^L\right)$-convex function at y on $\Omega\cup Y$, f_i^U, $i\in I$, is $\left(F,\rho_{f_i}^U\right)$-convex function at y on $\Omega\cup Y$, each constraint function g_j, $j\in J(y)$, is a $\left(F,\rho_{g_j}\right)$-convex function at y on $\Omega\cup Y$, each constraint function h_k, $k\in K^{+}(y)=\{k\in K:\xi_k>0\}$, is a $\left(F,\rho_{h_k}^{+}\right)$-convex function at y on $\Omega\cup Y$, each function $-h_k$, $k\in K^{-}(y)=\{k\in K:\xi_k<0\}$, is a $\left(F,\rho_{h_k}^{-}\right)$-convex function at y on $\Omega\cup Y$. If $\sum_{i=1}^{p}\lambda_i^L\rho_{f_i}^L+\sum_{i=1}^{p}\lambda_i^U\rho_{f_i}^U+\sum_{j\in J(y)}\mu_j\rho_{g_j}+\sum_{k\in K^{+}(y)}\xi_k\rho_{h_k}^{+}-\sum_{k\in K^{-}(y)}\xi_k\rho_{h_k}^{-}\geqq 0$, then the following inequalities $f_i(x)<_{LU}f_i(y)$, $i\in I$, cannot hold.

Proof. Let x and $\left(y,\lambda^L,\lambda^U,\mu,\xi\right)$ be any feasible solutions for problems (IVP) and (IVD), respectively. If $x=y$, then the weak duality trivially holds. Now, we prove the theorem in the case $x\neq y$. We proceed by contradiction. Suppose, contrary to the result, that

$$f_i(x)<_{LU}f_i(y)\tag{37}$$

holds. Hence, by the definition of the relation $<_{LU}$, (37) gives that

$$\begin{aligned}&\left(f_i^L(x)<f_i^L(y)\ \wedge\ f_i^U(x)\leqq f_i^U(y)\right)\\\text{or }&\left(f_i^L(x)\leqq f_i^L(y)\ \wedge\ f_i^U(x)<f_i^U(y)\right)\\\text{or }&\left(f_i^L(x)<f_i^L(y)\ \wedge\ f_i^U(x)<f_i^U(y)\right).\end{aligned}$$

By the feasibility of $\left(y,\lambda^L,\lambda^U,\mu,\xi\right)$ in problem (IVD), the above inequalities yield

$$\sum_{i=1}^{p}\lambda_i^L f_i^L(x)+\sum_{i=1}^{p}\lambda_i^L f_i^L(x)<\sum_{i=1}^{p}\lambda_i^U f_i^U(y)+\sum_{i=1}^{p}\lambda_i^U f_i^U(y).\tag{38}$$

By assumption, f_i^L, $i\in I$, is a $\left(F,\rho_{f_i}^L\right)$-convex function at y on $\Omega\cup Y$, f_i^U, $i\in I$, is a $\left(F,\rho_{f_i}^U\right)$-convex function at y on $\Omega\cup Y$, each constraint function g_j, $j\in J(y)$, is a $\left(F,\rho_{g_j}\right)$-convex function at y on $\Omega\cup Y$, each constraint function h_k, $k\in K^{+}(y)=\{k\in K:\xi_k>0\}$, is a $\left(F,\rho_{h_k}^{+}\right)$-convex function at y on $\Omega\cup Y$, each function $-h_k$, $k\in K^{-}(y)=\{k\in K:\xi_k<0\}$, is a $\left(F,\rho_{h_k}^{-}\right)$-convex function at y on $\Omega\cup Y$. Hence, by Definition 3, the following inequalities

$$\begin{aligned}f_i^L(z)-f_i^L(y)\geqq{}&F\left(z,y;\nabla f_i^L(y)\right)+\\&\rho_{f_i}^L d^2(z,y),\quad i\in I,\end{aligned}\tag{39}$$

$$\begin{aligned}f_i^U(z)-f_i^U(y)\geqq{}&F\left(z,y;\nabla f_i^U(y)\right)+\\&\rho_{f_i}^U d^2(z,y),\quad i\in I,\end{aligned}\tag{40}$$

$$\begin{aligned}g_j(z)-g_j(y)\geqq{}&F\left(z,y;\nabla g_j(y)\right)+\\&\rho_{g_j}d^2(z,y),\ j\in J(y),\end{aligned}\tag{41}$$

$$\begin{aligned}h_k(z)-h_k(y)\geqq{}&F\left(z,y;\nabla h_k(y)\right)+\\&\rho_{h_k}^{+}d^2(z,y),\ k\in K^{+}(y),\end{aligned}\tag{42}$$

$$\begin{aligned}-h_k(z)+h_k(y)\geqq{}&F\left(z,y;-\nabla h_k(y)\right)+\\&\rho_{h_k}^{-}d^2(z,y),\ k\in K^{-}(y)\end{aligned}\tag{43}$$

hold for all $z\in\Omega\cup Y$. Therefore, they are also satisfied for $z=x\in\Omega$. Thus, inequalities (39)-(43) yield, respectively,

$$\begin{aligned}f_i^L(x)-f_i^L(y)\geqq{}&F\left(x,y;\nabla f_i^L(y)\right)+\\&\rho_{f_i}^L d^2(x,y),\ i\in I,\end{aligned}\tag{44}$$

$$f_i^U(x) - f_i^U(y) \geqq F\left(x, y; \nabla f_i^U(y)\right) + \rho_{f_i}^U d^2(x, y),\ i \in I, \tag{45}$$

$$g_j(x) - g_j(y) \geqq F(x, y; \nabla g_j(y)) + \rho_{g_j} d^2(x, y), \quad j \in J(y), \tag{46}$$

$$h_k(x) - h_k(y) \geqq F(x, y; \nabla h_k(y)) + \rho_{h_k}^+ d^2(x, y),\ k \in K^+(y), \tag{47}$$

$$-h_k(x) + h_k(y) \geqq F(x, y; -\nabla h_k(y)) + \rho_{h_k}^- d^2(x, y),\ k \in K^-(y). \tag{48}$$

By the feasibility of $\left(y, \lambda^L, \lambda^U, \mu, \xi\right)$ in problem (IVD), it follows that

$$\lambda_i^L f_i^L(x) - \lambda_i^L f_i^L(y) \geqq \lambda_i^L F\left(x, y; \nabla f_i^L(y),\right) + \lambda_i^L \rho_{f_i}^L d^2(x, y), i \in I, \tag{49}$$

$$\lambda_i^U f_i^U(x) - \lambda_i^U f_i^U(y) \geqq \lambda_i^U F\left(x, y; \nabla f_i^U(y)\right) + \lambda_i^U \rho_{f_i}^U d^2(x, y), i \in I, \tag{50}$$

$$\mu_j g_j(x) - \mu_j g_j(y) \geqq \mu_j F(x, y; \nabla g_j(y)) + \mu_j \rho_{g_j} d^2(x, y), j \in J(y), \tag{51}$$

$$\xi_k h_k(x) - \xi_k h_k(y) \geqq \xi_k F(x, y; \nabla h_k(y)) + \xi_k \rho_{h_k}^+ d^2(x, y), k \in K^+(y), \tag{52}$$

$$\xi_k h_k(x) - \xi_k h_k(y) \geqq -\xi_k F(x, y; -\nabla h_k(y)) - \xi_k \rho_{h_k}^- d^2(x, y), k \in K^-(y). \tag{53}$$

Combining (49) and (50), we have

$$\begin{gathered}\sum_{i=1}^p \lambda_i^L f_i^L(x) + \sum_{i=1}^p \lambda_i^L f_i^L(x) - \\ \sum_{i=1}^p \lambda_i^U f_i^U(y) - \sum_{i=1}^p \lambda_i^U f_i^U(y) \geqq \\ \sum_{i=1}^p \lambda_i^L F\left(x, y; \nabla f_i^L(y)\right) + \sum_{i=1}^p \lambda_i^U F\left(x, y; \nabla f_i^U(y)\right) \\ + \sum_{i=1}^p \lambda_i^L \rho_{f_i}^L d^2(x, y) + \sum_{i=1}^p \lambda_i^U \rho_{f_i}^U d^2(x, y).\end{gathered} \tag{54}$$

Hence, (34) and (38) yield

$$\begin{gathered}\sum_{i=1}^p \lambda_i^L F\left(x, y; \nabla f_i^L(y)\right) + \sum_{i=1}^p \lambda_i^U F\left(x, y; \nabla f_i^U(y)\right) \\ + \left[\sum_{i=1}^p \lambda_i^L \rho_{f_i}^L + \sum_{i=1}^p \lambda_i^U \rho_{f_i}^U\right] d^2(x, y) < 0.\end{gathered} \tag{55}$$

Adding both sides of (51)-(53), we get

$$\begin{gathered}\sum_{j \in J(y)} \mu_j g_j(x) + \sum_{j \in J(y)} \mu_j g_j(y) - \\ \sum_{k \in K^+(y) \cup K^-(y)} \xi_k h_k(x) - \sum_{k \in K^+(y) \cup K^-(y)} \xi_k h_k(y) \\ \geqq \sum_{j \in J(y)} \mu_j F(x, y; \nabla g_j(y)) + \\ \sum_{k \in K^+(y)} \xi_k F(x, y; \nabla h_k(y)) + \\ \sum_{k \in K^-(y)} (-\xi_k) F(x, y; -\nabla h_k(y)) + \left[\sum_{j \in J(y)} \mu_j \rho_{g_j}\right. \\ \left.\sum_{k \in K^+(y)} \xi_k \rho_{h_k}^+ - \sum_{k \in K^-(y)} \xi_k \rho_{h_k}^-\right] d^2(x, y) \geqq 0.\end{gathered} \tag{56}$$

Since F is a sublinear functional with respect to the third component, therefore, (56) gives

$$\begin{gathered}\sum_{j \in J(y)} \mu_j g_j(x) + \sum_{j \in J(y)} \mu_j g_j(y) - \\ \sum_{k \in K^+(y) \cup K^-(y)} \xi_k h_k(x) - \sum_{k \in K^+(y) \cup K^-(y)} \xi_k h_k(y) \geqq \\ F\left(x, y; \sum_{j \in J(y)} \mu_j \nabla g_j(y) + \sum_{k \in K^+(y) \cup K^-(y)} \xi_k \nabla h_k(y)\right) \\ + \left[\sum_{j \in J(y)} \mu_j \rho_{g_j} + \sum_{k \in K^+(y)} \xi_k \rho_{h_k}^+ - \right. \\ \left.\sum_{k \in K^-(y)} \xi_k \rho_{h_k}^-\right] d^2(x, y).\end{gathered}$$

Taking into account Lagrange multipliers $\mu_j = 0,\ j \notin J(y)$ and $\xi_k = 0,\ k \notin K^+(y) \cup K^-(y)$, we have

$$\begin{gathered}\sum_{j=1}^m \mu_j g_j(x) - \sum_{j=1}^m \mu_j g_j(y) + \sum_{k=1}^q \xi_k h_k(x) - \\ \sum_{k=1}^q \xi_k h_k(y) \geqq F\left(x, y; \sum_{j=1}^m \mu_j \nabla g_j(y) + \right. \\ \left.\sum_{k=1}^q \xi_k \nabla h_k(y)\right) + \left[\sum_{j \in J(y)} \mu_j \rho_{g_j} + \right. \\ \left.\sum_{k \in K^+(y)} \xi_k \rho_{h_k}^+ - \sum_{k \in K^-(y)} \xi_k \rho_{h_k}^-\right] d^2(x, y).\end{gathered}$$

By $x \in \Omega$, $\left(y, \lambda^L, \lambda^U, \mu, \xi\right) \in \Gamma$, it follows that

$$\begin{gathered}F\left(x, y; \sum_{j=1}^m \mu_j \nabla g_j(y) + \sum_{k=1}^q \xi_k \nabla h_k(y)\right) + \\ \left[\sum_{j \in J(y)} \mu_j \rho_{g_j} + \sum_{k \in K^+(y)} \xi_k \rho_{h_k}^+ - \right. \\ \left.\sum_{k \in K^-(y)} \xi_k \rho_{h_k}^-\right] d^2(x, y) \leqq 0.\end{gathered} \tag{57}$$

Combining (55) and (57), we get

$$\begin{gathered}\sum_{i=1}^p \lambda_i^L F\left(x, y; \nabla f_i^L(y)\right) + \\ \sum_{i=1}^p \lambda_i^U F\left(x, y; \nabla f_i^U(y)\right) + \\ F\left(x, y; \sum_{j=1}^m \mu_j \nabla g_j(y) + \sum_{k=1}^q \xi_k \nabla h_k(y)\right) + \\ \left[\sum_{i=1}^p \lambda_i^L \rho_{f_i}^L + \sum_{i=1}^p \lambda_i^U \rho_{f_i}^U + \sum_{j \in J(y)} \mu_j \rho_{g_j} + \right. \\ \left.\sum_{k \in K^+(y)} \xi_k \rho_{h_k}^+ - \sum_{k \in K^-(y)} \xi_k \rho_{h_k}^-\right] d^2(x, y) < 0.\end{gathered} \tag{58}$$

Using the sublinearity of the functional F (with respect to the third component), we have

$$\begin{gathered}F\left(x, y; \left(\sum_{i=1}^p \lambda_i^L \nabla f_i^L(y) + \sum_{i=1}^p \lambda_i^U \nabla f_i^U(y) + \right.\right. \\ \left.\left.\sum_{j=1}^m \mu_j \nabla g_j(y) + \sum_{k=1}^q \xi_k \nabla h_k(y)\right)\right) + \\ \left[\sum_{i=1}^p \lambda_i^L \rho_{f_i}^L + \sum_{i=1}^p \lambda_i^U \rho_{f_i}^U + \sum_{j \in J(y)} \mu_j \rho_{g_j} + \right. \\ \left.\sum_{k \in K^+(y)} \xi_k \rho_{h_k}^+ - \sum_{k \in K^-(y)} \xi_k \rho_{h_k}^-\right] d^2(x, y) < 0.\end{gathered} \tag{59}$$

By assumption, $\sum_{i=1}^p \lambda_i^L \rho_{f_i}^L + \sum_{i=1}^p \lambda_i^U \rho_{f_i}^U + \sum_{j \in J(y)} \mu_j \rho_{g_j} + \sum_{k \in K^+(y)} \xi_k \rho_{h_k}^+ - \sum_{k \in K^-(y)} \xi_k \rho_{h_k}^- \geqq 0$. Hence, (59) yields

$$\begin{gathered}F\left(x, y; \left(\sum_{i=1}^p \lambda_i^L \nabla f_i^L(y) + \sum_{i=1}^p \lambda_i^U \nabla f_i^U(y) + \right.\right. \\ \left.\left.\sum_{j=1}^m \mu_j \nabla g_j(y) + \sum_{k=1}^q \xi_k \nabla h_k(y)\right)\right) < 0.\end{gathered}$$

By the feasibility of $\left(y, \lambda^L, \lambda^U, \mu, \xi\right)$ in problem (IVD), it follows that the following inequality

$$F(x, y; 0) < 0$$

holds, contradicting (3). This completes the proof of weak duality. ■

If stronger hypothesis of (F, ρ)-convexity is imposed on the objective function, then the following stronger result is true:

Theorem 16: (Weak duality): Let x and $\left(y, \lambda^L, \lambda^U, \mu, \xi\right)$ be feasible solutions for problems (IVP) and (IVD), respectively. Furthermore, assume that f_i^L, $i \in I$, is a strictly $\left(F, \rho_{f_i}^L\right)$-convex function at y on $\Omega \cup Y$, $f_i^U, i \in I$, is a strictly $\left(F, \rho_{f_i}^U\right)$-convex function at y on $\Omega \cup Y$, each constraint function g_j, $j \in J(y)$, is a $\left(F, \rho_{g_j}\right)$-convex function at y on $\Omega \cup Y$, each constraint function h_k, $k \in K^+(y) = \{k \in K : \xi_k > 0\}$, is a $\left(F, \rho_{h_k}^+\right)$-convex function at y on $\Omega \cup Y$, each function $-h_k$, $k \in K^-(y) = \{k \in K : \xi_k < 0\}$, is a $\left(F, \rho_{h_k}^-\right)$-convex function at y on $\Omega \cup Y$. If $\sum_{i=1}^p \lambda_i^L \rho_{f_i}^L + \sum_{i=1}^p \lambda_i^U \rho_{f_i}^U + \sum_{j \in J(y)} \mu_j \rho_{g_j} + \sum_{k \in K^+(y)} \xi_k \rho_{h_k}^+ - \sum_{k \in K^-(y)} \xi_k \rho_{h_k}^- \geqq 0$, then the following inequality $f(x) \leqq_{LU} f(y)$ and $f_i(x) <_{LU} f_i(y)$ for at least one $i \in I$ cannot hold.

Theorem 17: (Strong duality): Let $\overline{x}$ be a weak LU-Pareto solution (LU-Pareto solution) in problem (IVP) and the Kuhn-Tucker constraint qualification be satisfied at $\overline{x}$. Then there exist $\overline{\lambda}^L \in R^p$, $\overline{\lambda}^U \in R^p$, $\overline{\mu} \in R^m$ and $\overline{\xi} \in R^q$ such that $\left(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}, \overline{\xi}\right)$ is feasible in (IVD) and the objective functions of (IVP) and (IVD) are equal at these points. If all hypotheses of the weak duality theorem (Theorems 15 or 16, respectively) are also satisfied, then $\left(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}, \overline{\xi}\right)$ is a weak LU-Pareto solution (LU-Pareto solution) of a maximum type in problem (IVD).

Proof. By assumption, $\overline{x}$ is a LU-Pareto solution (weak LU-Pareto solution) in problem (IVP) and the Kuhn-Tucker constraint qualification is satisfied at $\overline{x}$. Then there exist $\overline{\lambda}^L \in R^p$, $\overline{\lambda}^U \in R^p$, $\overline{\mu} \in R^m$ and $\overline{\xi} \in R^q$ such that the Karush-Kuhn-Tucker necessary optimality conditions (6)-(8) are satisfied at $\overline{x}$ with these Lagrange multipliers. Therefore, the feasibility of $\left(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}, \overline{\xi}\right)$ in problem (IVD) follows directly from the conditions (6)-(8). Hence, the objective functions of problems (IVP) and (IVD) have the same values at $\overline{x}$ and $\left(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}, \overline{\xi}\right)$, respectively. From the weak duality theorem (Theorem 15 or 16, respectively), it follows that the inequality $f(\overline{x}) <_{LU} f(y)$ ($f(\overline{x}) \leqq_{LU} f(y)$ and $f_k(\overline{x}) <_{LU} f_k(y)$ for at least one $k \in K$) cannot hold for any feasible point $\left(y, \lambda^L, \lambda^U, \mu, \xi\right)$ in dual problem (IVD). Hence, we conclude that $\left(\overline{x}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}, \overline{\xi}\right)$ is weakly LU-efficient (LU-efficient) of a maximum type in problem (IVD). This completes the proof of strong duality. ■

Theorem 18: (Converse duality): Let $\left(\overline{y}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}, \overline{\xi}\right)$ be a weakly LU-efficient (LU-efficient) solution of a maximum type in Mond-Weir dual problem (IVD) such that $\overline{y} \in \Omega$. Further, assume that f_i^L, $i \in I$, is a $\left(F, \rho_{f_i}^L\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$, f_i^U, $i \in I$, is a $\left(F, \rho_{f_i}^U\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$, each constraint function g_j, $j \in J(\overline{y})$, is a $\left(F, \rho_{g_j}\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$, each constraint function h_k, $k \in K^+(\overline{y}) = \left\{k \in K : \overline{\xi}_k > 0\right\}$, is a $\left(F, \rho_{h_k}^+\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$, each function $-h_k$, $k \in K^-(\overline{y}) = \left\{k \in K : \overline{\xi}_k < 0\right\}$, is a $\left(F, \rho_{h_k}^-\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$ and $\sum_{i=1}^p \overline{\lambda}_i^L \rho_{f_i}^L + \sum_{i=1}^p \overline{\lambda}_i^U \rho_{f_i}^U + \sum_{j \in J(\overline{y})} \overline{\mu}_j \rho_{g_j} + \sum_{k \in K^+(\overline{y})} \overline{\xi}_k \rho_{h_k}^+ - \sum_{k \in K^-(\overline{y})} \overline{\xi}_k \rho_{h_k}^- \geqq 0$. Then $\overline{y}$ is a weak LU-Pareto solution (LU-Pareto solution) in problem (IVP).

Proof. The proof of this theorem follows directly from weak duality (Theorem 15 or 16, respectively). ■

A restricted version of converse duality for (IVP) and (IVD) is the following result:

Theorem 19: (Restricted converse duality): Let $\left(\overline{y}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}, \overline{\xi}\right)$ be feasible in Mond-Weir vector dual problem (IVD) with the multiple interval-valued objective function. Further, assume that f_i^L, $i \in I$, is a (strictly) $\left(F, \rho_{f_i}^L\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$, f_i^U, $i \in I$, is a (strictly) $\left(F, \rho_{f_i}^U\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$, each constraint function g_j, $j \in J(\overline{y})$, is a $\left(F, \rho_{g_j}\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$, each constraint function h_k, $k \in K^+(\overline{y}) = \left\{k \in K : \overline{\xi}_k > 0\right\}$, is a $\left(F, \rho_{h_k}^+\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$, each function $-h_k$, $k \in K^-(\overline{y}) = \left\{k \in K : \overline{\xi}_k < 0\right\}$, is a $\left(F, \rho_{h_k}^-\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$ and $\sum_{i=1}^p \overline{\lambda}_i^L \rho_{f_i}^L + \sum_{i=1}^p \overline{\lambda}_i^U \rho_{f_i}^U + \sum_{j \in J(\overline{y})} \overline{\mu}_j \rho_{g_j} + \sum_{k \in K^+(\overline{y})} \overline{\xi}_k \rho_{h_k}^+ - \sum_{k \in K^-(\overline{y})} \overline{\xi}_k \rho_{h_k}^- \geqq 0$. If there exists $\overline{x} \in \Omega$ such that $f(\overline{x}) = f(\overline{y})$, then $\overline{x}$ is a weak LU-Pareto solution (LU-Pareto solution) in the considered differentiable multiobjective programming problem with the multiple interval-valued objective function (IVP).

Proof. By means of contradiction, suppose that $\overline{x}$ is not a LU-weak Pareto solution in problem (IVP). This means, by Definition 5, that there exists $\widetilde{x} \in \Omega$ such that

$$f_i(\widetilde{x}) <_{LU} f_i(\overline{x}). \tag{60}$$

By assumption, $f(\overline{x}) = f(\overline{y})$. Hence, (60) yields

$$f_i(\widetilde{x}) <_{LU} f_i(\overline{y}). \tag{61}$$

By the definition of the relation $<_{LU}$, (61) gives that for each $i \in I$,

$$\begin{gathered}\left(f_i^L(\widetilde{x}) < f_i^L(\overline{y}) \ \wedge \ f_i^U(\widetilde{x}) \leqq f_i^U(\overline{y})\right) \\ \text{or } \left(f_i^L(\widetilde{x}) \leqq f_i^L(\overline{y}) \ \wedge \ f_i^U(\widetilde{x}) < f_i^U(\overline{y})\right) \\ \text{or } \left(f_i^L(\widetilde{x}) < f_i^L(\overline{y}) \ \wedge \ f_i^U(\widetilde{x}) < f_i^U(\overline{y})\right).\end{gathered}$$

By $\left(\overline{y}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}\right) \in \Gamma$, it follows that $\overline{\lambda}^L \geq 0$, $\overline{\lambda}^U \geq 0$. Hence, the above inequalities yield

$$\sum_{k=1}^p \overline{\lambda}_k^L f_k^L(\widetilde{x}) - \sum_{k=1}^p \overline{\lambda}_k^L f_k^L(\overline{y}) < \sum_{k=1}^p \overline{\lambda}_k^U f_k^U(\widetilde{x}) - \sum_{k=1}^p \overline{\lambda}_k^U f_k^U(\overline{y}). \tag{62}$$

By assumption, f_i^L, $i \in I$, is a $\left(F, \rho_{f_i}^L\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$, f_i^U, $i \in I$, is a $\left(F, \rho_{f_i}^U\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$, each constraint function g_j, $j \in J(\overline{y})$, is a $\left(F, \rho_{g_j}\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$, each constraint function h_k, $k \in K^+(\overline{y}) = \left\{k \in K : \overline{\xi}_k > 0\right\}$, is a $\left(F, \rho_{h_k}^+\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$, each function $-h_k$, $k \in K^-(\overline{y}) = \left\{k \in K : \overline{\xi}_k < 0\right\}$, is a $\left(F, \rho_{h_k}^-\right)$-convex function at $\overline{y}$ on $\Omega \cup Y$. Hence, by Definition 3, the following inequalities

$$\begin{aligned} f_i^L(z) - f_i^L(\overline{y}) \geqq F\left(z, \overline{y}; \nabla f_i^L(\overline{y}), \right) + \\ \rho_{f_i}^L d^2(z, \overline{y}), \ i \in I, \end{aligned} \tag{63}$$

$$f_i^U(z) - f_i^U(\overline{y}) \geqq F\left(z, \overline{y}; \nabla f_i^U(\overline{y})\right) + \rho_{f_i}^U d^2(z, \overline{y}), i \in I, \tag{64}$$

$$g_j(z) - g_j(\overline{y}) \geqq F(z, \overline{y}; \nabla g_j(\overline{y})) + \rho_{g_j} d^2(z, \overline{y}), \ j \in J(\overline{y}), \tag{65}$$

$$h_k(z) - h_k(\overline{y}) \geqq F(z, \overline{y}; \nabla h_k(\overline{y})) + \rho_{h_k}^+ d^2(z, \overline{y}), \ k \in K^+(\overline{y}), \tag{66}$$

$$-h_k(z) + h_k(\overline{y}) \geqq F(z, \overline{y}; -\nabla h_k(\overline{y})) + \rho_{h_k}^- d^2(z, \overline{y}), \ k \in K^-(y) \tag{67}$$

hold for $z \in \Omega \cup Y$. Thus, they are also fulfilled for $z = \widetilde{x} \in \Omega$. Hence, (63)-(67) yield, respectively,

$$f_i^L(\widetilde{x}) - f_i^L(\overline{y}) \geqq F\left(\widetilde{x}, \overline{y}; \nabla f_i^L(\overline{y}), \right) + \rho_{f_i}^L d^2(\widetilde{x}, \overline{y}), \ i \in I, \tag{68}$$

$$f_i^U(\widetilde{x}) - f_i^U(\overline{y}) \geqq F\left(\widetilde{x}, \overline{y}; \nabla f_i^U(\overline{y})\right) + \rho_{f_i}^U d^2(\widetilde{x}, \overline{y}), \ i \in I, \tag{69}$$

$$g_j(\widetilde{x}) - g_j(\overline{y}) \geqq F(\widetilde{x}, \overline{y}; \nabla g_j(\overline{y})) + \rho_{g_j} d^2(\widetilde{x}, \overline{y}), \quad j \in J(\overline{y}), \tag{70}$$

$$h_k(\widetilde{x}) - h_k(\overline{y}) \geqq F(\widetilde{x}, \overline{y}; \nabla h_k(\overline{y})) + \rho_{h_k}^+ d^2(\widetilde{x}, \overline{y}), \ k \in K^+(\overline{y}), \tag{71}$$

$$-h_k(\widetilde{x}) + h_k(\overline{y}) \geqq F(\widetilde{x}, \overline{y}; -\nabla h_k(\overline{y})) + \rho_{h_k}^- d^2(\widetilde{x}, \overline{y}), \ k \in K^-(\overline{y}). \tag{72}$$

By the feasibility of $\left(\overline{y}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}, \overline{\xi}\right)$ in problem (IVD), it follows that

$$\overline{\lambda}_i^L f_i^L(\widetilde{x}) - \overline{\lambda}_i^L f_i^L(\overline{y}) \geqq \overline{\lambda}_i^L F\left(\widetilde{x}, \overline{y}; \nabla f_i^L(\overline{y}), \right) + \overline{\lambda}_i^L \rho_{f_i}^L d^2(\widetilde{x}, \overline{y}), \quad i \in I, \tag{73}$$

$$\overline{\lambda}_i^U f_i^U(\widetilde{x}) - \overline{\lambda}_i^U f_i^U(\overline{y}) \geqq \overline{\lambda}_i^U F\left(\widetilde{x}, \overline{y}; \nabla f_i^U(\overline{y})\right) + \overline{\lambda}_i^U \rho_{f_i}^U d^2(\widetilde{x}, \overline{y}), \quad i \in I, \tag{74}$$

$$\overline{\mu}_j g_j(\widetilde{x}) - \overline{\mu}_j g_j(\overline{y}) \geqq \overline{\mu}_j F(\widetilde{x}, \overline{y}; \nabla g_j(\overline{y})) + \overline{\mu}_j \rho_{g_j} d^2(\widetilde{x}, \overline{y}), \quad j \in J(\overline{y}), \tag{75}$$

$$\overline{\xi}_k h_k(\widetilde{x}) - \overline{\xi}_k h_k(\overline{y}) \geqq \overline{\xi}_k F(\widetilde{x}, \overline{y}; \nabla h_k(\overline{y})) + \overline{\xi}_k \rho_{h_k}^+ d^2(\widetilde{x}, \overline{y}), \quad k \in K^+(\overline{y}), \tag{76}$$

$$\overline{\xi}_k h_k(\widetilde{x}) - \overline{\xi}_k h_k(\overline{y}) \geqq -\overline{\xi}_k F(\widetilde{x}, \overline{y}; -\nabla h_k(\overline{y})) - \overline{\xi}_k \rho_{h_k}^- d^2(\widetilde{x}, \overline{y}), \quad k \in K^-(\overline{y}). \tag{77}$$

Adding both sides of inequalities (73) and (74) and, moreover, adding both sides of inequalities (75)-(77), respectively, we get

$$\begin{gathered}\sum_{i=1}^p \overline{\lambda}_i^L f_i^L(\widetilde{x}) + \sum_{i=1}^p \overline{\lambda}_i^U f_i^U(\widetilde{x}) \\ - \sum_{i=1}^p \overline{\lambda}_i^L f_i^L(\overline{y}) - \sum_{i=1}^p \overline{\lambda}_i^U f_i^U(\overline{y}) \geqq \\ \sum_{i=1}^p \overline{\lambda}_i^L F\left(\widetilde{x}, \overline{y}; \nabla f_i^L(\overline{y})\right) + \sum_{i=1}^p \overline{\lambda}_i^U F\left(\widetilde{x}, \overline{y}; \nabla f_i^U(\overline{y})\right) \\ + \sum_{i=1}^p \overline{\lambda}_i^L \rho_{f_i}^L d^2(\widetilde{x}, \overline{y}) + \sum_{i=1}^p \overline{\lambda}_i^U \rho_{f_i}^U d^2(\widetilde{x}, \overline{y}),\end{gathered} \tag{78}$$

$$\begin{gathered}\sum_{j \in J(\overline{y})} \overline{\mu}_j g_j(\widetilde{x}) - \sum_{j \in J(\overline{y})} \overline{\mu}_j g_j(\overline{y}) + \\ \sum_{k \in K^+(\overline{y}) \cup K^-(\overline{y})} \overline{\xi}_k h_k(\widetilde{x}) - \sum_{k \in K^+(\overline{y}) \cup K^-(\overline{y})} \overline{\xi}_k h_k(\overline{y}) \\ \geqq \sum_{j \in J(\overline{y})} \overline{\mu}_j F(\widetilde{x}, \overline{y}; \nabla g_j(\overline{y})) + \\ \sum_{k \in K^+(\overline{y})} \overline{\xi}_k F(\widetilde{x}, \overline{y}; \nabla h_k(\overline{y})) + \\ \sum_{k \in K^-(\overline{y})} \left(-\overline{\xi}_k\right) F(\widetilde{x}, \overline{y}; -\nabla h_k(\overline{y})) + \Big[\sum_{j \in J(\overline{y})} \overline{\mu}_j \rho_{g_j} \\ \sum_{k \in K^+(\overline{y})} \overline{\xi}_k \rho_{h_k}^+ - \sum_{k \in K^-(\overline{y})} \overline{\xi}_k \rho_{h_k}^-\Big] d^2(\widetilde{x}, \overline{y}).\end{gathered} \tag{79}$$

Thus, (62) and (78) imply

$$\begin{gathered}\sum_{i=1}^p \overline{\lambda}_i^L F\left(\widetilde{x}, \overline{y}; \nabla f_i^L(\overline{y})\right) + \\ \sum_{i=1}^p \overline{\lambda}_i^U F\left(\widetilde{x}, \overline{y}; \nabla f_i^U(\overline{y})\right) + \\ \left[\sum_{i=1}^p \overline{\lambda}_i^L \rho_{f_i}^L + \sum_{i=1}^p \overline{\lambda}_i^U \rho_{f_i}^U\right] d^2(\widetilde{x}, \overline{y}) < 0.\end{gathered} \tag{80}$$

Since F is a sublinear functional with respect to the third component, (79) yields

$$\begin{gathered}\sum_{j \in J(\overline{y})} \overline{\mu}_j g_j(\widetilde{x}) - \sum_{j \in J(\overline{y})} \overline{\mu}_j g_j(\overline{y}) + \\ \sum_{k \in K^+(\overline{y}) \cup K^-(\overline{y})} \overline{\xi}_k h_k(\widetilde{x}) - \sum_{k \in K^+(\overline{y}) \cup K^-(\overline{y})} \overline{\xi}_k h_k(\overline{y}) \geqq \\ F\Big(\widetilde{x}, \overline{y}; \sum_{j \in J(\overline{y})} \overline{\mu}_j \nabla g_j(\overline{y}) + \\ \sum_{k \in K^+(\overline{y}) \cup K^-(\overline{y})} \overline{\xi}_k \nabla h_k(\overline{y})\Big) + \Big[\sum_{j \in J(\overline{y})} \overline{\mu}_j \rho_{g_j} + \\ \sum_{k \in K^+(\overline{y})} \overline{\xi}_k \rho_{h_k}^+ - \sum_{k \in K^-(\overline{y})} \overline{\xi}_k \rho_{h_k}^-\Big] d^2(\widetilde{x}, \overline{y}).\end{gathered}$$

Taking into account Lagrange multipliers $\overline{\mu}_j = 0$, $j \notin J(\overline{y})$ and $\overline{\xi}_k = 0$, $k \notin K^+(\overline{y}) \cup K^-(\overline{y})$, we have

$$\begin{gathered}\sum_{j=1}^m \overline{\mu}_j g_j(\widetilde{x}) - \sum_{j=1}^m \overline{\mu}_j g_j(\overline{y}) + \sum_{k=1}^q \overline{\xi}_k h_k(\widetilde{x}) - \\ \sum_{k=1}^q \overline{\xi}_k h_k(\overline{y}) \geqq F\Big(\widetilde{x}, \overline{y}; \sum_{j=1}^m \overline{\mu}_j \nabla g_j(\overline{y}) + \\ \sum_{k=1}^q \overline{\xi}_k \nabla h_k(\overline{y})\Big) + \Big[\sum_{j \in J(\overline{y})} \overline{\mu}_j \rho_{g_j} + \\ \sum_{k \in K^+(\overline{y})} \overline{\xi}_k \rho_{h_k}^+ - \sum_{k \in K^-(\overline{y})} \overline{\xi}_k \rho_{h_k}^-\Big] d^2(\widetilde{x}, \overline{y}).\end{gathered} \tag{81}$$

By $\widetilde{x} \in \Omega$, $\left(\overline{y}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\mu}, \overline{\xi}\right) \in \Gamma$, (81) gives

$$\begin{gathered}F\Big(\widetilde{x}, \overline{y}; \sum_{j=1}^m \overline{\mu}_j \nabla g_j(\overline{y}) + \sum_{k=1}^q \overline{\xi}_k \nabla h_k(\overline{y})\Big) + \\ \Big[\sum_{j \in J(\overline{y})} \overline{\mu}_j \rho_{g_j} + \sum_{k \in K^+(\overline{y})} \overline{\xi}_k \rho_{h_k}^+ - \\ \sum_{k \in K^-(\overline{y})} \overline{\xi}_k \rho_{h_k}^-\Big] d^2(\widetilde{x}, \overline{y}) \leqq 0.\end{gathered} \tag{82}$$

Combining (80) and (82), we get

$$\begin{gathered}\sum_{i=1}^p \overline{\lambda}_i^L F\left(\widetilde{x}, \overline{y}; \nabla f_i^L(\overline{y})\right) + \sum_{i=1}^p \overline{\lambda}_i^U F\left(\widetilde{x}, \overline{y}; \nabla f_i^U(\overline{y})\right) + \\ F\Big(\widetilde{x}, \overline{y}; \sum_{j=1}^m \overline{\mu}_j \nabla g_j(\overline{y}) + \sum_{k=1}^q \overline{\xi}_k \nabla h_k(\overline{y})\Big) + \\ \Big[\sum_{i=1}^p \overline{\lambda}_i^L \rho_{f_i}^L + \sum_{i=1}^p \overline{\lambda}_i^U \rho_{f_i}^U + \sum_{j \in J(\overline{y})} \overline{\mu}_j \rho_{g_j} + \\ \sum_{k \in K^+(\overline{y})} \overline{\xi}_k \rho_{h_k}^+ - \sum_{k \in K^-(\overline{y})} \overline{\xi}_k \rho_{h_k}^-\Big] d^2(\widetilde{x}, \overline{y}) < 0.\end{gathered}$$

Using the sublinearity of the functional F (with respect to the third component) again, we obtain

$$F\left(\widetilde{x},\overline{y};\left(\sum_{i=1}^{p}\overline{\lambda}_i^L\nabla f_i^L\left(\overline{y}\right)+\sum_{i=1}^{p}\overline{\lambda}_i^U\nabla f_i^U\left(\overline{y}\right)+\right.\right.$$
$$\left.\left.\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j\left(\overline{y}\right)+\sum_{k=1}^{q}\overline{\xi}_k\nabla h_k\left(\overline{y}\right)\right)\right)+$$
$$\left[\sum_{i=1}^{p}\overline{\lambda}_i^L\rho_{f_i}^L+\sum_{i=1}^{p}\overline{\lambda}_i^U\rho_{f_i}^U+\right. \tag{83}$$
$$\sum_{j\in J(\overline{y})}\overline{\mu}_j\rho_{g_j}+\sum_{k\in K^+(\overline{y})}\overline{\xi}_k\rho_{h_k}^+-$$
$$\left.\sum_{k\in K^-(\overline{y})}\overline{\xi}_k\rho_{h_k}^-\right]d^2\left(\widetilde{x},\overline{y}\right)<0.$$

By assumption, $\sum_{i=1}^{p}\overline{\lambda}_i^L\rho_{f_i}^L+\sum_{i=1}^{p}\overline{\lambda}_i^U\rho_{f_i}^U+\sum_{j\in J(\overline{y})}\overline{\mu}_j\rho_{g_j}+\sum_{k\in K^+(\overline{y})}\overline{\xi}_k\rho_{h_k}^+-\sum_{k\in K^-(\overline{y})}\overline{\xi}_k\rho_{h_k}^-\geqq 0$. Thus, (83) implies

$$F\left(\widetilde{x},\overline{y};\left(\sum_{i=1}^{p}\overline{\lambda}_i^L\nabla f_i^L\left(\overline{y}\right)+\sum_{i=1}^{p}\overline{\lambda}_i^U\nabla f_i^U\left(\overline{y}\right)+\right.\right.$$
$$\left.\left.\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j\left(\overline{y}\right)+\sum_{k=1}^{q}\overline{\xi}_k\nabla h_k\left(\overline{y}\right)\right)\right)<0.$$

By the feasibility of $\left(\overline{y},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\mu},\overline{\xi}\right)$ in problem (IVD), it follows that the following inequality

$$F\left(\widetilde{x},\overline{y};0\right)<0$$

holds, contradicting (3). This means that $\overline{x}$ is a weak LU-Pareto solution of the considered interval-valued vector optimization problem (IVP) and completes the proof of this theorem. ■

V. Conclusion

In the paper, we have considered a differentiable vector optimization problem with the multiple interval-valued objective function and with both inequality and equality constraints. The Karush-Kuhn-Tucker necessary optimality conditions for a weak LU-Pareto solution in such differentiable vector optimization problems have been derived under the Kuhn-Tucker constraint qualification. The sufficiency of the Karush-Kuhn-Tucker necessary optimality conditions for weak LU-Pareto optimality (LU-Pareto optimality) have been established under assumptions that the involved functions in the considered differentiable vector optimization problem with the multiple interval-valued objective function are (F,ρ)-convex, not necessarily with respect to the same ρ. Further, for the considered differentiable multiobjective programming problem with the multiple interval-valued objective function, its Mond-Weir vector dual problem with the multiple interval-valued objective function has been defined and several duality results have been proved also under the concept of differentiable (F,ρ)-convexity. Hence, the optimality conditions and duality results established in the paper are applicable for a larger class of nonconvex differentiable vector optimization problems with the multiple interval-objective function than similar ones existing actually in the literature.

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