Hybrid Boundary Value Methods for the Solution of First Order Stiff Systems

Grace O. Akinlabi, Raphael B. Adeniyi, and Enahoro A. Owoloko

***Abstract*—** Recently, several Boundary Value Methods (BVMs) have been developed to overcome the limitations of the popular Linear Multistep Methods (LMMs). In this work, we introduce a new class of BVMs called the Hybrid Boundary Value Methods (HBVMs), which are based on the LMMs by utilizing data at both step and off-step points. Numerical tests on both linear and nonlinear stiff systems were presented so as to illustrate the process by using the specific cases: $k = 4$ and 6. The results were of high accuracy as the Rate of Convergence (ROC) of the solutions were compared to a symmetric scheme known as Extended Trapezoidal Rule (ETR) of order 6.

***Keywords*—** Boundary value methods, hybrid formula, initial value problems, linear multistep method, stiff systems.

I. Introduction

DUE to the modelisation of real world phenomena, the numerical approximation of solutions of Differential Equations (DEs) continues to be an active field of investigation.

In this paper, our focus will be on the first order Initial Value Problem (IVP) of the form:

$$y' = f(x,y), \quad y(x_0) = y_0, \quad x \in [x_0, x_N] \tag{1.1}$$

where all $f : \mathbb{R}^m \to \mathbb{R}^m$ are continuous functions and satisfy the conditions for existence and uniqueness of solutions, which are guaranteed by the theorem of Henrici in [1].

Numerical analysts have developed several numerical methods for the solution of this type of IVP and other types of Differential problems [2] – [7]. Among such methods, we have the Linear Multistep Formula (LMF), which is of the form:

$$\sum_{r=0}^{k} \alpha_r y_{n+r} = h \sum_{r=0}^{k} \beta_r f_{n+r} \tag{1.2}$$

Several modifications of these formulas have been introduced, which include, but not limited to, the hybrid methods. These hybrid methods share the characteristic property of Runge-Kutta methods, which are more flexible than the LMMs in the way they are used as data are been utilized at off-step points [8] – [12].

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We define a $k-$step hybrid formula to be a formula of the type:

$$\sum_{r=0}^{k} \alpha_r y_{n+r} = h \sum_{r=0}^{k} \beta_r f_{n+r} + h\beta_v f_{n+v} \tag{1.3}$$

where α_r, β_r are constants, $v \notin \{0,1,\ldots,k\}$ and $f_{n+v} = f(x_{n+v}, y_{n+v})$.

Although the LMMs are used for the approximation of both IVPs and Boundary Value Problems BVPs (via the shooting method), they however have limitations: requiring starting methods or values, stability problems (Dahlquist barriers) [13], and so on.

The new Boundary Value Methods (BVMs) were introduced to overcome these limitations. For instance, the same continuous scheme used to generate the main methods is also used in generating the additional methods, which are then applied at the end points thereby avoiding some of the stability problems encountered by the LMMs [14].

Several BVMs have been developed and used for the numerical integration of ODEs, PDEs, and so on [14] – [25]. Also their convergence and stability analysis have been fully discussed. For a comprehensive work on BVMs, see [26].

Our focus in this work is to develop new BVMs that utilize data at off-step points and which will be called Hybrid Boundary Value Methods (HBVMS). In deriving these methods, we will be adopting the Adams Moulton methods, which is a LMM of the form:

$$y_{n+k} - y_{n+k-1} = h \sum_{i=0}^{k} \beta_i f_{n+i} \tag{1.4}$$

II. Overview of the Boundary Value Methods [27]- [28]

In this section, we present an overview of the BVMs.
Consider the IVP in (1.1). To approximate this problem, we consider the *k*-step LMF:

$$\sum_{r=0}^{k} \alpha_r y_{n+r} = h \sum_{r=0}^{k} \beta_r f_{n+r} \tag{2.1}$$

This discrete problem needs k independent conditions to be imposed so as to get the discrete solution $\{y_n\}$. Now, the first $k-1$ values need to be generated, since the IVP (1.1) has provided the first value y_0. Hence, we are to obtain the $k-1$ values: $y_0, \ldots, y_{k-1}$ of the discrete solution.

By this process, we say that the given continuous IVP has been approximated by means of a discrete IVP and this is what is known as IVM.

On the other hand, if we decide to fix the first k_1 values of the discrete solution, $y_0,\ldots,y_{k_1-1}$ and the last k_2 values of the discrete solution, $y_N,\ldots,y_{N+k_2-1}$ such that k_1,k_2 are integers and $k_1+k_2=k$. The discrete problem becomes.

$$\sum_{r=-k_1}^{k_2}\alpha_{r+k_1}y_{n+r}=h\sum_{r=-k_1}^{k_2}\beta_{r+k_1}f_{n+r} \qquad (2.2)$$

By this, we have succeeded in fixing the first k_1 and final k_2 values of the discrete solution.

By this process, we say that the continuous IVP has been approximated by means of a discrete BVP and this approach is what is called BVM.

III. Derivation of Methods (HBVMs)

We shall construct, via interpolation and collocation, methods of the form:

$$y_{n+v}-y_{n+v-1}=h\sum_{r=0\left(\frac{1}{2}\right)}^{k}\beta_r f_{n+r}$$

where $v=\begin{cases}\dfrac{k+1}{2}, & \text{for odd } k\\ \dfrac{k}{2}, & \text{for even } k\end{cases}$

For example for $k=1$, $v=1$ we have the formula

$$y_{n+1}-y_n=h\left[\beta_0 f_n+\beta_{\frac{1}{2}}f_{n+\frac{1}{2}}+\beta_0 f_{n+1}\right]$$

After the derivation, we implement these LMMs as BVMs while considering two specific cases: $k=4$ and 6.

A. For case k=4

The main method is as follows:

$$y_{n+3}-y_{n+1}=\frac{h}{28350}\left[\begin{array}{l}13f_n+5494f_{n+1}+10870f_{n+2}\\+5494f_{n+3}+13f_{n+4}\\-32\left(7f_{n+\frac{1}{2}}-551\left(f_{n+\frac{3}{2}}+f_{n+\frac{5}{2}}\right)+7f_{n+\frac{7}{2}}\right)\end{array}\right]$$

which is used together with the following initial methods:

$$y_{\frac{1}{2}}-y_1=\frac{h}{7257600}\left[\begin{array}{l}33953f_0-3244786f_1-1317280f_2\\-294286f_3-7297f_4-1375594f_{\frac{1}{2}}\\+1752542f_{\frac{3}{2}}+755042f_{\frac{5}{2}}+68906f_{\frac{7}{2}}\end{array}\right]$$

$$y_{\frac{3}{2}}-y_1=\frac{h}{7257600}\left[\begin{array}{l}7297f_0+1638286f_1-833120f_2\\-142094f_3-3233f_4-99626f_{\frac{1}{2}}\\+2631838f_{\frac{3}{2}}+397858f_{\frac{5}{2}}+31594f_{\frac{7}{2}}\end{array}\right]$$

$$y_{\frac{5}{2}}-y_1=\frac{h}{89600}\left[\begin{array}{l}81f_0+19118f_1+44640f_2-2862f_3-49f_4\\-1098f_{\frac{1}{2}}+50814f_{\frac{3}{2}}+23234f_{\frac{5}{2}}+552f_{\frac{7}{2}}\end{array}\right]$$

and the final methods

$$y_{N-4}-y_{N-1}=-\frac{h}{2800}\left[\begin{array}{l}-9f_N+158f_{N-1}-360f_{N-2}+18f_{N-3}\\+401f_{N-4}+8\left(\begin{array}{l}9f_{N-\frac{1}{2}}+333f_{N-\frac{3}{2}}\\+403f_{N-\frac{5}{2}}+279f_{N-\frac{7}{2}}\end{array}\right)\end{array}\right]$$

$$y_{N-\frac{7}{2}}-y_{N-1}=-\frac{5h}{290304}\left[\begin{array}{l}85f_N+13606f_{N-1}+10546f_{N-\frac{7}{2}}\\+5\left(\begin{array}{l}6560f_{N-2}+7442f_{N-3}-49f_{N-4}\\-202f_{N-\frac{1}{2}}+6014f_{N-\frac{3}{2}}+4418f_{N-\frac{5}{2}}\end{array}\right)\end{array}\right]$$

$$y_{N-2}-y_{N-1}=-\frac{h}{226800}\left[\begin{array}{l}127f_N+44446f_{N-1}+43480f_{N-2}\\-494f_{N-3}-23f_{N-4}-1976f_{N-\frac{1}{2}}\\+141928f_{N-\frac{3}{2}}-872f_{N-\frac{5}{2}}+184f_{N-\frac{7}{2}}\end{array}\right]$$

$$y_N-y_{N-1}=-\frac{h}{226800}\left\{\begin{array}{l}-32377f_N+42494f_{N-1}+116120f_{N-2}\\+31154f_{N-3}+833f_{N-4}\\-8\left(\begin{array}{l}22823f_{N-\frac{1}{2}}+15011f_{N-\frac{3}{2}}\\+9341f_{N-\frac{5}{2}}+953f_{N-\frac{7}{2}}\end{array}\right)\end{array}\right\}$$

B. For case k=6

The main method is as follows:

$$y_{n+4}-y_{n+2}=\frac{h}{2554051500}\left[\begin{array}{l}28151f_n+4721736f_{n+1}+529200405f_{n+2}\\+1047943344f_{n+3}+529200405f_{n+4}\\+4721736f_{n+5}+28151f_{n+6}\\-64\left(\begin{array}{l}7923f_{n+\frac{1}{2}}+531095f_{n+\frac{3}{2}}\\-23916042f_{n+\frac{5}{2}}-23916042f_{n+\frac{7}{2}}\\+531095f_{n+\frac{9}{2}}+7923f_{n+\frac{11}{2}}\end{array}\right)\end{array}\right]$$

which is used together with the following initial methods:

$$y_{\frac{1}{2}}-y_2=\frac{h}{21525504000}\left[\begin{array}{l}50840663f_0-15631690812f_1\\-17564506125f_2-13516516608f_3\\-4999623795f_4-510865092f_5\\-6279127f_6-3507456066f_{\frac{1}{2}}\\-3243018230f_{\frac{3}{2}}+15178447404f_{\frac{5}{2}}\\+9451486164f_{\frac{7}{2}}+1927727350f_{\frac{9}{2}}\\+83198274f_{\frac{11}{2}}\end{array}\right]$$

$$y_{\frac{3}{2}}-y_2=\frac{h}{5230697472000}\left[\begin{array}{l}456196373f_0+72649122828f_1\\-2008959454935f_2-576826591488f_3\\-171945526185f_4-15894332172f_5\\-184329877f_6-7728247206f_{\frac{1}{2}}\\-1152341705090f_{\frac{3}{2}}+826951939524f_{\frac{5}{2}}\\+353393854524f_{\frac{7}{2}}+62574497410f_{\frac{9}{2}}\\+2505840294f_{\frac{11}{2}}\end{array}\right]$$

$$y_{\frac{5}{2}}-y_2=\frac{h}{5230697472000}\left[\begin{array}{l}184329877f_0+22105977612f_1\\+1284137567145f_2-510641870592f_3\\-116161302825f_4-9856152588f_5\\-109551893f_6-2852484774f_{\frac{1}{2}}\\-125367467650f_{\frac{3}{2}}+1771726903236f_{\frac{5}{2}}\\+260516522556f_{\frac{7}{2}}+40149664130f_{\frac{9}{2}}\\+1516601766f_{\frac{11}{2}}\end{array}\right]$$

$$y_{\frac{7}{2}}-y_2=\frac{h}{21525504000}\left[\begin{array}{l}688087f_0+80355012f_1\\+4938122355f_2+10933456128f_3\\-824424435f_4-51176388f_5\\-521303f_6-10514754f_{\frac{1}{2}}\\-451692790f_{\frac{3}{2}}+11828012076f_{\frac{5}{2}}\\+5609039316f_{\frac{7}{2}}\\+229447670f_{\frac{9}{2}}+7465026f_{\frac{11}{2}}\end{array}\right]$$

$$y_{\frac{9}{2}}-y_2=\frac{5h}{41845579776}\left[\begin{array}{l}387173f_0+40903116f_1\\+2009196729f_2+4356823296f_3\\+4948419015f_4-100766412f_5\\-637669f_6-5670918f_{\frac{1}{2}}\\-211497890f_{\frac{3}{2}}+4450127364f_{\frac{5}{2}}\\+3692434428f_{\frac{7}{2}}+1732368034f_{\frac{9}{2}}\\+10703622f_{\frac{11}{2}}\end{array}\right]$$

$$y_{\frac{11}{2}}-y_2=-\frac{7h}{106748928000}\left\{\begin{array}{l}4616563f_0+390117588f_1\\+6701781375f_2+2481846474f_{\frac{11}{2}}\\-7\left(\begin{array}{l}2261849856f_3+2229061815f_4\\+1586077044f_5-5121461f_6\\-8852838f_{\frac{1}{2}}-224093890f_{\frac{3}{2}}\\+349026372f_{\frac{5}{2}}-230586948f_{\frac{7}{2}}\\+299226050f_{\frac{9}{2}}\end{array}\right)\end{array}\right\}$$

and the final methods

$$y_{N-3}-y_{N-2}=-\frac{h}{40864824000}\left[\begin{array}{l}584203f_N+83659728f_{N-1}\\+8440941375f_{N-2}+8383546752f_{N-3}\\+26265105f_{N-4}-8111952f_{N-5}\\-133787f_{N-6}-9718596f_{N-\frac{1}{2}}\\-561950380f_{N-\frac{3}{2}}+24975451224f_{N-\frac{5}{2}}\\-485424216f_{N-\frac{7}{2}}+18109100f_{N-\frac{9}{2}}\\+1605444f_{N-\frac{11}{2}}\end{array}\right]$$

$$y_{N-5}-y_{N-2}=-\frac{h}{168168000}\left[\begin{array}{l}-6887f_N-402672f_{N-1}+27874005f_{N-2}\\+51015552f_{N-3}+53970075f_{N-4}\\+30828528f_{N-5}+44983f_{N-6}+82644f_{N-\frac{1}{2}}\\+455900f_{N-\frac{3}{2}}+113824584f_{N-\frac{5}{2}}+117908664f_{N-\frac{7}{2}}\\+109885220f_{N-\frac{9}{2}}-976596f_{N-\frac{11}{2}}\end{array}\right]$$

$$y_{N-6}-y_{N-2}=\frac{2h}{638512875}\left\{\begin{array}{l}739276f_N+58489176f_{N-1}\\+491088915f_{N-2}+1267922544f_{N-3}\\+1006809120f_{N-4}+171401976f_{N-5}\\-42194069f_{N-6}\\-32\left(\begin{array}{l}302481f_{N-\frac{1}{2}}+6723935f_{N-\frac{3}{2}}\\+37948986f_{N-\frac{5}{2}}+51102126f_{N-\frac{7}{2}}\\+27054245f_{N-\frac{9}{2}}+9095811f_{N-\frac{11}{2}}\end{array}\right)\end{array}\right\}$$

$$y_{N-1}-y_{N-2}=\frac{h}{40864824000}\left\{\begin{array}{l}10480453f_N+7415784528f_{N-1}\\+4647521745f_{N-2}-4370314368f_{N-3}\\-1693823265f_{N-4}-173397072f_{N-5}\\-2123957f_{N-6}\\+4\left(\begin{array}{l}-57299919f_{N-\frac{1}{2}}+6811487435f_{N-\frac{3}{2}}\\+1040444586f_{N-\frac{5}{2}}+792336726f_{N-\frac{7}{2}}\\+163656245f_{N-\frac{9}{2}}+7048911f_{N-\frac{11}{2}}\end{array}\right)\end{array}\right\}$$

$$y_N-y_{N-2}=-\frac{h}{2554051500}\left\{\begin{array}{l}-337524401f_N+1375937544f_{N-1}\\+8583673365f_{N-2}+11191323696f_{N-3}\\+4457911725f_{N-4}+472635144f_{N-5}\\+5942359f_{N-6}\\-64\left(\begin{array}{l}36391167f_{N-\frac{1}{2}}+108748075f_{N-\frac{3}{2}}\\+180492462f_{N-\frac{5}{2}}+127879902f_{N-\frac{7}{2}}\\+27426835f_{N-\frac{9}{2}}+1217847f_{N-\frac{11}{2}}\end{array}\right)\end{array}\right\}$$

IV. Numerical Examples

In this section, we apply the HBVMs of order 4 and 6 derived in the previous section to two (2) stiff systems. For the first problem, the methods are compared with Extended Trapezoidal Rule (ETR) of order 6, which is the first symmetric scheme (BVM) introduced by Amodio and Mazzia (1995) [18].

Problem 4.1: Consider the linear first order stiff system [28]:

$$y_1'=-21y_1+19y_2-20y_3$$
$$y_2'=19y_1-21y_2+20y_3$$
$$y_3'=40y_1-40y_2-40y_3$$

for $x\in[0,0.4]$

with initial conditions:

$$y_1(0)=1,\quad y_2(0)=0,\quad y_3(0)=-1$$

and with exact solutions:

$$y_1=\frac{1}{2}e^{-2x}+\frac{1}{2}e^{-40x}(\sin 40x+\cos 40x)$$
$$y_2=\frac{1}{2}e^{-2x}-\frac{1}{2}e^{-40x}(\sin 40x+\cos 40x)$$
$$y_3=e^{-40x}(\sin 40x-\cos 40x)$$

Problem 4.2: Consider the nonlinear first order stiff system:

$$y_1'=-1002y_1+1000y_2^2$$
$$y_2'=y_1-y_2(1+y_2)$$

for $x\in[0,1]$

with initial conditions: $y_1(0)=1,\quad y_2(0)=1$

and with exact solutions: $y_1=e^{-2x},\quad y_2=e^{-x}$

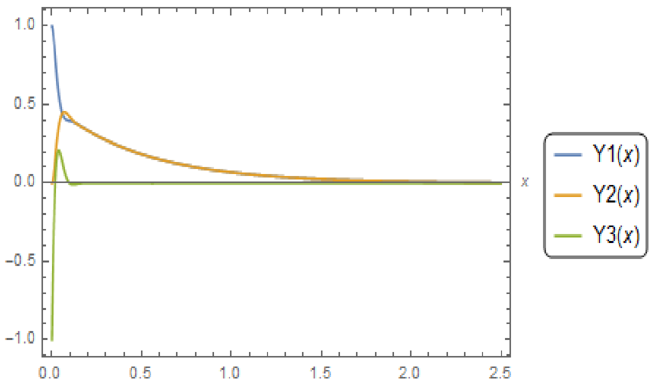

Fig. 1: Exact Solution of Problem 4.1

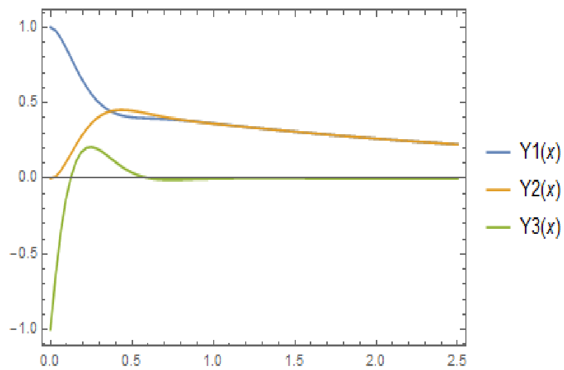

Fig. 2: Approximate Solution of Problem 4.1 computed with HBVM ($k = 4$, $h = 0.005$)

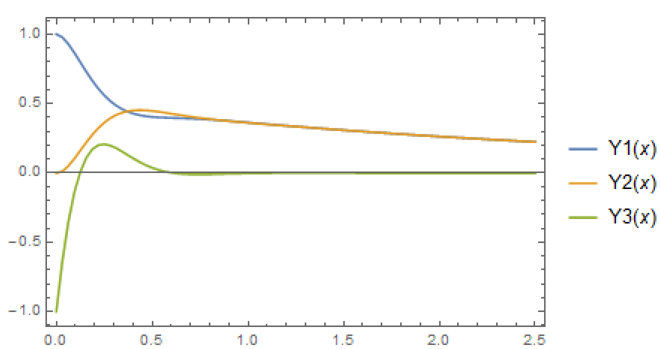

Fig. 3: Approximate Solution of Problem 4.1 computed with HBVM ($k = 6$, $h = 0.005$)

V. Conclusion

In this paper, we have proposed the hybrid of the Boundary Value Methods, which are new class of methods that are based on the Linear Multistep Methods. We implemented the Adams Moulton methods as BVMs for the numerical approximation of linear and nonlinear stiff systems. The Adams Moulton methods were derived through interpolation and collocation procedure by utilizing data at both step and off-step points. Numerical tests confirmed that these methods give high accuracy results as the results were very close to their analytical solutions.

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Table I: Maximum errors for HBVMs of order 4 and 6 with ETR of order 6 (Problem 4.1)

h	ETR of order 6		HBVM of order 4		HBVM of order 6	
	$\|e\|_\infty$	Rate	$\|e\|_\infty$	Rate	$\|e\|_\infty$	Rate
4e-2	5.4985e-02	-	1.2226e-02	-	1.8315e-02	-
2e-2	6.2297e-03	3.14	3.5013e-04	5.13	8.8319e-05	7.70
1e-2	3.23923-04	4.27	1.9979e-06	7.45	5.4155e-08	10.67
5e-3	8.0594e-06	5.33	8.8177e-09	7.82	5.4248e-09	3.32
2.5e-3	1.5402e-07	5.71	2.7689e-11	8.31	2.4902e-10	4.45

Table II: Maximum errors for HBVMs of order 4 and 6 (Problem 4.2)

h	HBVM of order 4		HBVM of order 6	
	$\|e\|_\infty$	Rate	$\|e\|_\infty$	Rate
1e-1	6.38392e-12	-	4.92964e-10	-
5e-2	4.07419e-12	0.65	4.62916e-10	0.09
2.5e-2	1.97988e-12	1.04	3.72713e-10	0.31
1.25e-2	5.33658e-12	1.43	8.62215e-10	1.21
6.25e-3	4.56693e-12	0.22	5.00906e-10	0.78