Fractional Complex Transform for the Solution of Time-Fractional Advection-Diffusion Model

S. O. Edeki, G. O. Akinlabi, and C. E. Odo

***Abstract*—** In this paper, we obtain exact solutions of time-fractional Advection-Diffusion model equations by means Fractional Complex Transform (FCT) coupled with modified differential transform method. The derivatives are defined in terms of Jumarie's sense. Two illustrative examples are considered in elucidating the effectiveness of the proposed technique. The method requires little knowledge of fractional calculus while obtaining exact solutions of fractional equations with high level of accuracy not being compromised.

***Keywords*—** Fractional calculus; modified DTM; Advection-Diffusion model.

I. Introduction

In applied science domain, most modelling problems do arise in the form of advection-diffusion models whose solutions appear to be of great importance [1-3].

Many scholars have proposed a lot of analytical, semi-analytical, and numerical solution methods for solving the advection-diffusion equations and other related linear and nonlinear differential models. These approaches include the Variational Iteration Method (VIM), Decomposition Method (ADM), Homotopy Analysis Method (HAM), Differential Transformation Method (DTM), Reduced Differential Transform Method (RDTM), Modified extended tanh-function method, Chebyshev spectral collocation method and so on [4-16].

The notion of fractional differential equation acts as a response for an expression that can be varied to describe the order of the derivative. In a generalized form, Momani [17] considered by means of ADM, the non-perturbation analytical solutions of the Burgers equation with time- and space-fractional orders.

In this present work, we will be considering *time-fractional* advection-diffusion equations whose derivative is defined in the sense of Jumarie. In general, a convection-diffusion model having no source term is traceable to stochastic dynamics with great impact in financial engineering [18].

Fractional Complex Transform (FCT) transforms fractional order differential equations to integer differential equations with the help of Riemann-Liouville derivatives [19-21]. Other solution methods for linear and nonlinear models include [22-28]. FCT as a solution method for fractional differential equations (FDEs) was first proposed by [29]. The notion of Jumarie's fractional derivative is introduced as follows before the overview of FCT.

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II. Jumarie's Fractional Derivative (JFD)

It is noted here that JFD is a modified form of the Riemann-Liouville derivatives [21]. Hence, the definition of JFD and its basic properties as follows:

Let $h(v)$ be a continuous real function of v (not necessarily differentiable), and $D_v^\alpha h = \dfrac{\partial^\alpha h}{\partial v^\alpha}$ denoting JFD of h, of order α w.r.t. v. Then,

$$D_v^\alpha h = \begin{cases} \dfrac{1}{\Gamma(-\alpha)}\dfrac{d}{dv}\displaystyle\int_0^v (v-\lambda)^{-\alpha-1}\left(h(\lambda)-h(0)\right)d\lambda, \ \alpha\in(-\infty,0) \\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dv}\displaystyle\int_0^v (v-\lambda)^{-\alpha}\left(h(\lambda)-h(0)\right)d\lambda, \ \alpha\in(0,1) \\ \left(h^{(\alpha-\eta)}(v)\right)^{(\eta)}, \ \alpha\in[\eta,\eta+1), \ \eta\geq 1 \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ denotes a gamma function. As summarized in [21], the basic properties of JFD are stated as *P1-P5*:

P1: $D_v^\alpha k = 0, \ \alpha > 0$,

P2: $D_v^\alpha \left(kh(v)\right) = kD_v^\alpha h(v), \ \alpha > 0$,

P3: $D_v^\alpha v^\beta = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} v^{\beta-\alpha}, \ \beta\geq\alpha>0,$

P4: $\begin{aligned} D_v^\alpha\left(h_1(v)h_2(v)\right) &= D_v^\alpha h_1(v)\left(h_2(v)\right) \\ &\quad + h_1(v)D_v^\alpha h_2(v) \end{aligned},$

P5: $D_v^\alpha\left(h\left(v(g)\right)\right) = D_v^1 h\cdot D_g^\alpha v$,

where k is a constant.

Note: *P1, P2, P3, P4,* and *P5* are referred to as fractional derivative of: constant function, constant multiple function, power function, product function, and function of function respectively. *P5* can be linked to Jumarie's chain rule of fractional derivative.

A. The Fractional Complex Transform and DTM

Here, we briefly introduce the concept of the FCT and the RDTM.

B. The Fractional Complex Transform

Let us consider a general fractional differential equation of the form:

$$f\left(\varpi, D_t^{\alpha}\varpi, D_x^{\beta}\varpi, D_y^{\lambda}\varpi, D_z^{\gamma}\varpi\right)=0,\ \ \varpi=\varpi\left(t,x,y,z\right). \tag{2.2}$$

Then, the Fractional Complex Transform [30] is defined as follows:

$$\begin{cases} T=\dfrac{at^{\alpha}}{\Gamma\left(1+\alpha\right)},\ \alpha\in\left(0,1\right], \\ X=\dfrac{bx^{\beta}}{\Gamma\left(1+\beta\right)},\ \beta\in\left(0,1\right], \\ Y=\dfrac{cy^{\lambda}}{\Gamma\left(1+\lambda\right)},\ \lambda\in\left(0,1\right], \\ Z=\dfrac{dz^{\gamma}}{\Gamma\left(1+\gamma\right)},\ \gamma\in\left(0,1\right], \end{cases} \tag{2.3}$$

where $a,\ b,\ c,$ and d are unknown constants.

From *P3*,

$$D_v^{\alpha}v^{\beta}=\frac{\Gamma\left(1+\beta\right)}{\Gamma\left(1+\beta-\alpha\right)}v^{\beta-\alpha},\ \beta\geq\alpha>0,$$

$$\therefore\ \ D_t^{\alpha}T=D_t^{\alpha}\left[\frac{at^{\alpha}}{\Gamma\left(1+\alpha\right)}\right]=\frac{a}{\Gamma\left(1+\alpha\right)}D_t^{\alpha}t^{\alpha}$$

$$=\frac{a}{\Gamma\left(1+\alpha\right)}\left[\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+\alpha-\alpha\right)}\right]t^{\alpha-\alpha}=a. \tag{2.4}$$

Obviously in a similar manner, using properties *P1-P5*, and the FCT in (2.3), the following are easily obtained:

$$\begin{cases} D_t^{\alpha}T=\dfrac{\partial^{\alpha}T}{\partial t^{\alpha}}=a, \\ D_x^{\beta}X=\dfrac{\partial^{\beta}X}{\partial x^{\beta}}=b, \\ D_y^{\lambda}Y=\dfrac{\partial^{\lambda}Y}{\partial y^{\lambda}}=c, \\ D_z^{\gamma}Z=\dfrac{\partial^{\gamma}Z}{\partial z^{\gamma}}=d. \end{cases} \tag{2.5}$$

Hence,

$$\begin{cases} D_t^{\alpha}\varpi\left(t,x,y,z\right)=D_t^{\alpha}\varpi\left(T\left(t\right)\right)=D_T^{1}\varpi\cdot D_t^{\alpha}T=a\dfrac{\partial\varpi}{\partial T}, \\ D_x^{\beta}\varpi\left(t,x,y,z\right)=D_x^{\beta}\varpi\left(X\left(x\right)\right)=D_X^{1}\varpi\cdot D_x^{\beta}X=b\dfrac{\partial\varpi}{\partial X}, \\ D_y^{\lambda}\varpi\left(t,x,y,z\right)=D_y^{\lambda}\varpi\left(Y\left(y\right)\right)=D_Y^{1}\varpi\cdot D_y^{\lambda}Y=c\dfrac{\partial\varpi}{\partial Y}, \\ D_z^{\gamma}\varpi\left(t,x,y,z\right)=D_z^{\gamma}\varpi\left(Z\left(z\right)\right)=D_Z^{1}\varpi\cdot D_z^{\gamma}Z=d\dfrac{\partial\varpi}{\partial Z}. \end{cases} \tag{2.6}$$

$$D_t^{\alpha}\varpi\left(t,x,y,z\right)=D_t^{\alpha}\varpi\left(T\left(t\right)\right)$$
$$=D_T^{1}\varpi\cdot D_t^{\alpha}T=a\frac{\partial\varpi}{\partial T}, \tag{2.7}$$

$$D_x^{\beta}\varpi\left(t,x,y,z\right)=D_x^{\beta}\varpi\left(X\left(x\right)\right)$$
$$=D_X^{1}\varpi\cdot D_x^{\beta}X=b\frac{\partial\varpi}{\partial X}, \tag{2.8}$$

$$D_y^{\lambda}\varpi\left(t,x,y,z\right)=D_y^{\lambda}\varpi\left(Y\left(y\right)\right)$$
$$=D_Y^{1}\varpi\cdot D_y^{\lambda}Y=c\frac{\partial\varpi}{\partial Y}, \tag{2.9}$$

$$D_z^{\gamma}\varpi\left(t,x,y,z\right)=D_z^{\gamma}\varpi\left(Z\left(z\right)\right)$$
$$=D_Z^{1}\varpi\cdot D_z^{\gamma}Z=d\frac{\partial\varpi}{\partial Z}. \tag{2.10}$$

III. The Reduced Differential Transform [1-11, 13]

Suppose $m\left(x,t\right)$ is an analytic and continuously differentiable function, defined in a domain D, then the differential transform (DF) of $m\left(x,t\right)$ is defined and denoted by:

$$M_k\left(x\right)=\frac{1}{k!}\left[\frac{\partial^k m\left(x,t\right)}{\partial t^k}\right]_{t=0} \tag{3.1}$$

where $M_k\left(x\right)$ and $m\left(x,t\right)$ are referred to as the transformed and the original functions respectively. Thus, the differential inverse transform (DIT) of $M_k\left(x\right)$ is defined and denoted as:

$$m\left(x,t\right)=\sum_{k=0}^{\infty}M_k\left(x\right)t^k. \tag{3.2}$$

A. The fundamentals properties of the DTM

D1: If $m\left(x,t\right)=\alpha p\left(x,t\right)\pm\beta q\left(x,t\right)$, then
$M_k\left(x\right)=\alpha P_k\left(x\right)\pm\beta Q_k\left(x\right)$.

D2: If $m\left(x,t\right)=\dfrac{\alpha\partial^{\eta}h\left(x,t\right)}{\partial t^{\eta}},\ \eta\in\mathbb{N}$, then

$$M_k\left(x\right)=\frac{\alpha\left(k+\eta\right)!}{k!}H_{k+\eta}\left(x\right).$$

D3: If $m(x,t)=\dfrac{g(x)\partial^{\eta}h(x,t)}{\partial x^{\eta}},\ \eta\in\mathbb{N}$, then

$$M_k(x)=\frac{g(x)\partial^{\eta}H_k(x)}{\partial x^{\eta}},\ \eta\in\mathbb{N}.$$

D4: If $m(x,t)=p(x,t)q(x,t)$, then

$$M_k(x)=\sum_{\eta=0}^{k}P_{\eta}(x)Q_{k-\eta}(x).$$

D5: If $m(x,t)=x^{n_1}t^{n_2}$, then

$$M_k=x^{n_1}\delta(k-n_2),\delta(k)=\begin{cases}1,\ k=0\\0,\ k\neq 0.\end{cases}$$

IV. Illustrative Applications

In this subsection, the proposed method is applied to some examples of Advection-Diffusion model equations of time-fractional orders as follows.

A. Example 1: Consider the time-fractional homogeneous advection diffusion equation:

$$D_t^{\alpha}u+uu_x=0,\ u=u(x,t), \tag{4.1}$$

subject to:

$$u(x,0)=-x \tag{4.2}$$

Solution procedure:
By FCT,

$$T=\frac{at^{\alpha}}{\Gamma(1+\alpha)},$$

which according to section 3 gives $D_t^{\alpha}u=\dfrac{\partial u}{\partial T}$ for $a=1.$

Hence, (4.1) and (4.2) become:

$$\begin{cases}\dfrac{\partial u}{\partial T}+uu_x=0,\\ u(x,0)=-x.\end{cases} \tag{4.3}$$

Thus, applying the RDTM to (4.3) gives the recursive relation:

$$\begin{cases}U_{k+1}=-\dfrac{1}{k+1}\displaystyle\sum_{j=0}^{k}U_j\frac{\partial}{\partial x}U_{k-j},\\ U_0=-x.\end{cases} \tag{4.4}$$

By the properties of the RDTM, we have the exact solution of (4.3) as follows:

$$u(x,T)=\frac{x}{T-1},\ T\neq 1. \tag{4.5}$$

Hence, the exact solution of (4.1) is:

$$\begin{aligned}u(x,t)&=\frac{x}{\left(\dfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right)-1}\\&=\frac{x\Gamma(1+\alpha)}{t^{\alpha}-\Gamma(1+\alpha)},\ \Gamma(1+\alpha)\neq t^{\alpha}.\end{aligned} \tag{4.6}$$

Note**:** when $\alpha=1$, we have $u(x,t)=\dfrac{x}{t-1},\ t\neq 1.$ which corresponds to the exact solution as contained [1].

B. Example 2: Consider the time-fractional diffusion equation:

$$\begin{cases}D_t^{\alpha}u=\dfrac{\partial^2 u}{\partial x^2},\ u=u(x,t),\\ u(x,0)=\sin(\pi x),\ x\in[0,1].\end{cases} \tag{4.7}$$

Solution procedure:
By FCT,

$$T=\frac{at^{\alpha}}{\Gamma(1+\alpha)},$$

which according to section 3 gives $D_t^{\alpha}u=\dfrac{\partial u}{\partial T}$ for $a=1.$

Hence, (4.7) and (4.2) become:

$$\begin{cases}\dfrac{\partial u}{\partial T}=\dfrac{\partial^2 u}{\partial x^2},\\ u(x,0)=\sin(\pi x).\end{cases} \tag{4.8}$$

Thus, applying the RDTM to (4.8) gives the recursive relation:

$$\begin{cases}U_{k+1}=\dfrac{1}{(k+1)}\dfrac{\partial^2}{\partial x^2}U_k(x),k\geq 0,\\ U_0=\sin(\pi x).\end{cases} \tag{4.9}$$

By the properties of the RDTM, we have the exact solution of (4.8) as follows:

$$u(x,T)=\exp\left(-\pi^2 T\right)\sin(\pi x). \tag{4.10}$$

Hence, the exact solution of (4.7) is:

$$u(x,t)=\exp\left(\frac{-\pi^2t^{\alpha}}{\Gamma(1+\alpha)}\right)\sin(\pi x). \tag{4.11}$$

Note**:** when $\alpha=1$, we have $u(x,t)=\exp\left(-\pi^2t\right)\sin(\pi x).$ which corresponds to the exact solution as contained [14].

Here, we present in Fig. 1 through Fig. 9, the relationship between the exact solutions of the integer cases $\alpha=1$, and the fractional cases for $\alpha\in\mathbb{Q}$ with respect to example I.

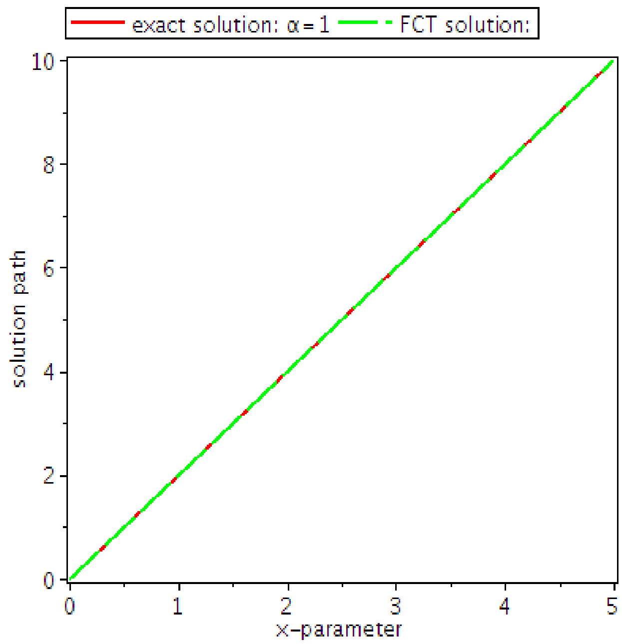

Fig. 1: The solution graph for $t = 1.5$ w.r.t. Example 1

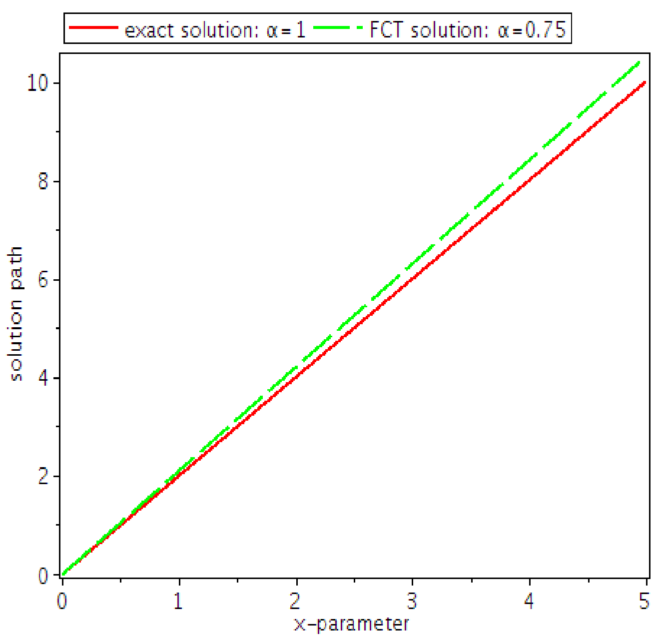

Fig. 2: The solution graph for $t = 1.5$ w.r.t. Example 1

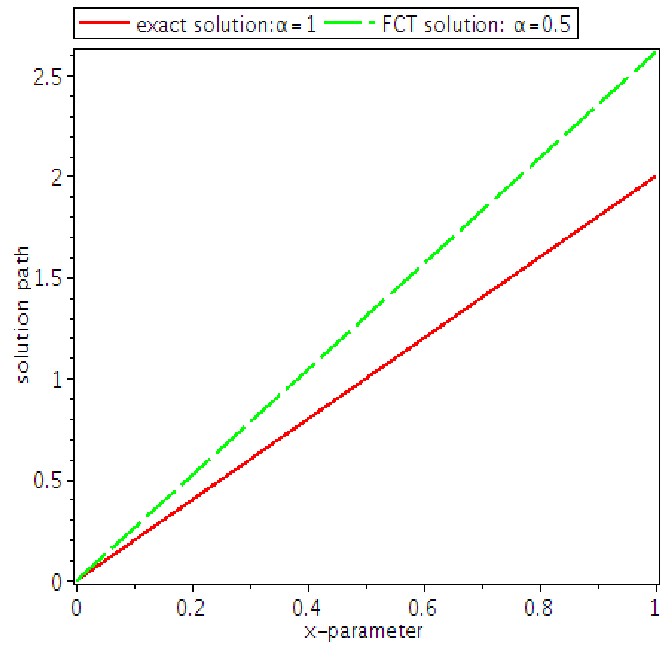

Fig. 3: The solution graph for $t = 1.5$ w.r.t. Example 1

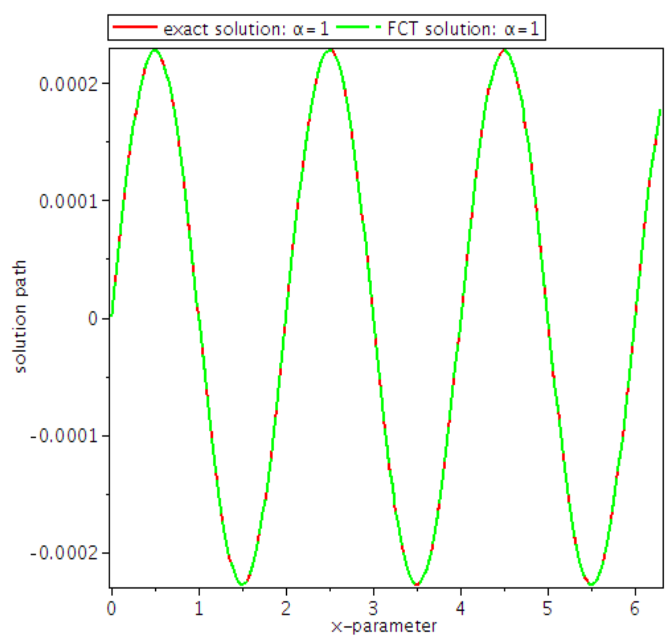

Fig. 4: The solution graph for $t = 0.8$ w.r.t. Example 2

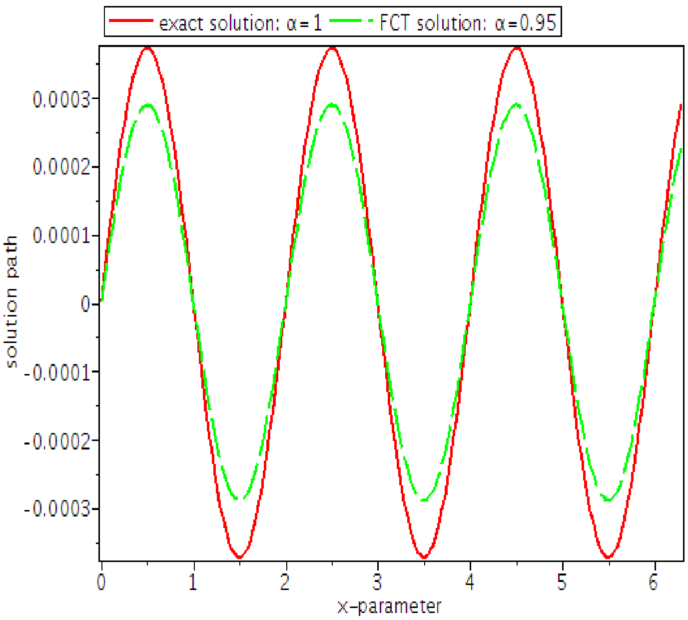

Fig. 5: The solution graphs for $t = 0.8$ w.r.t. Example 2

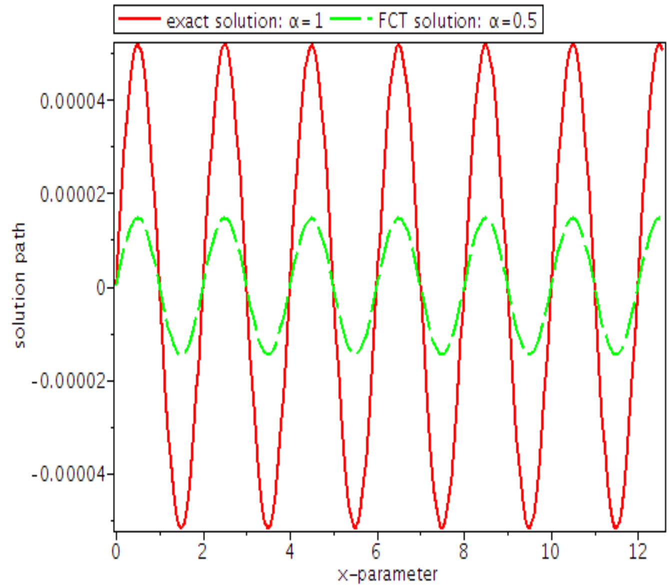

Fig. 8: The solution graphs for $t = 1$ w.r.t. Example 2

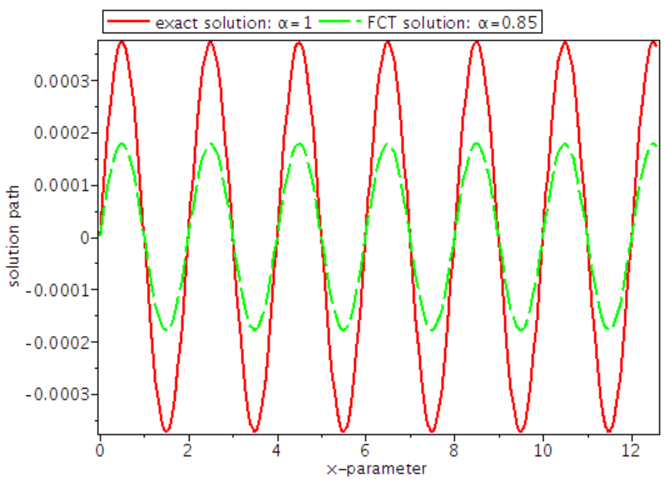

Fig. 6: The solution graph for $t = 0.8$ w.r.t. Example 2

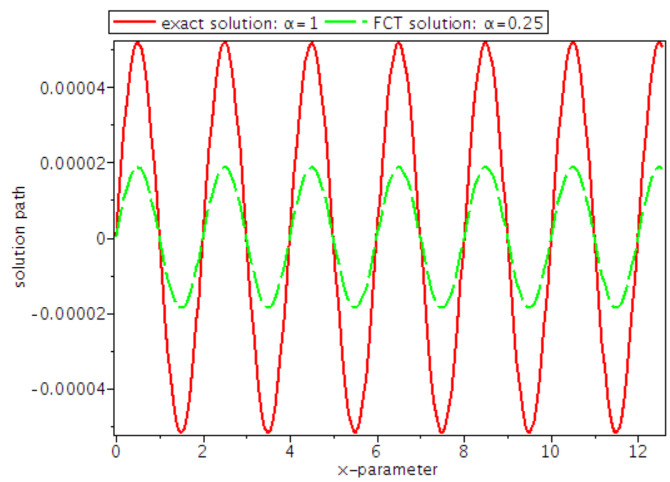

Fig. 9: The solution graph for $t = 1$ w.r.t. Example 2

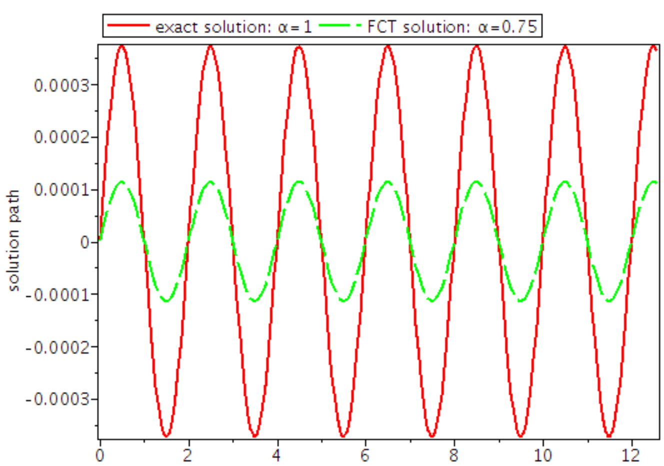

Fig. 7: The solution graphs for $t = 0.8$ w.r.t. Example 2

C. Example 3: Consider the time-fractional linear advection-diffusion equation:

$$D_t^{\alpha} u + k u_x = \nu u_{xx}, \quad x \in (-1,1)\ , t \in [0,T] \qquad (4.12)$$

where u signifies the velocity such that is the constant advection velocity and the kinetic viscosity [31].

By FCT,

$$T = \frac{a t^{\alpha}}{\Gamma(1+\alpha)},$$

which according to section 3 gives $D_t^{\alpha} u = \dfrac{\partial u}{\partial T}$ for $a = 1$.

Hence, (4.12) becomes:

$$u_T + ku_x = vu_{xx} \quad . \tag{4.13}$$

We supposed the solution of (4.13) to be:

$$u(x,T) = v(x,T)e^{\lambda x+\beta T} \tag{4.14}$$

for a change of variable. Hence, (4.13) becomes:

$$v_T + (\beta + kx - \lambda^2 v)v + (k-v)v_x = vv_{xx} \tag{4.15}$$

We choose λ and β in such a way that $k = 2\lambda v$ and $k^2 = -4\beta v$. Therefore (4.15) becomes:

$$v_T = vv_{xx} \quad . \tag{4.16}$$

Thus, applying the RDTM to (4.16) gives the recursive relation:

$$V_{k+1} = \frac{1}{k+1}\sum_{j=0}^{k} V_{k-j}\frac{\partial^2}{\partial x^2}V_k \,. \tag{4.17}$$

By the properties of the RDTM, we have the exact solution of (4.3) as follows:

Therefore, for $k \ge 0$, we have:

$$V_1 = V_0\ddot{V}_0,\; V_2 = \frac{1}{2}\left(V_1\ddot{V}_0 + V_0\ddot{V}_1\right),$$

$$V_3 = \frac{1}{3}\left(V_2\ddot{V}_0 + V_1\ddot{V}_1 + V_0\ddot{V}_2\right),$$

$$V_4 = \frac{1}{4}\left(V_3\ddot{V}_0 + V_2\ddot{V}_1 + V_1\ddot{V}_2 + V_0\ddot{V}_3\right),$$

$$V_5 = \frac{1}{5}\left(V_4\ddot{V}_0 + V_3\ddot{V}_1 + V_2\ddot{V}_2 + V_1\ddot{V}_3 + V_0\ddot{V}_4\right),$$

$$\vdots$$

$$V_{k+1} = \frac{1}{k+1}\left(\sum_{i=o}^{k} V_{k-i}\ddot{V}_i\right),\; k \ge 0\,.$$

Thus subjecting (4.16) to:

$$v(x,0) = 1 + x^2 \tag{4.18}$$

gives:

$$V(x,1) = 2(x^2+1),$$

$$V(x,2) = 4(x^2+1),$$

$$V(x,3) = 8(x^2+1),$$

$$V(x,4) = 16(x^2+1),$$

$$V(x,5) = 32(x^2+1),$$

$$\vdots$$

$$V(x,p) = 2^{p}(x^2+1),\; p \ge 1.$$

Hence,

$$\begin{aligned} v(x,T) &= \sum_{\hbar=0}^{\infty} V(\hbar)T^{\alpha\hbar} \\ &= V(0) + V(1)T + V(2)T^2 + V(3)T^3 + \cdots \\ &= \begin{cases} (x^2+1) + 2(x^2+1)T + 4(x^2+1)T^2 \\ +8(x^2+1)T^3 + 16(x^2+1)T^4 + \cdots \end{cases} \\ &= (x^2+1)\{1 + 2T + 4T^2 + 8T^3 + 16T^4 + \cdots\} \\ &= \frac{(x^2+1)}{1-2T},\; |2T| < 1. \end{aligned} \tag{4.18}$$

$$\therefore \quad u(x,T) = \frac{(x^2+1)}{1-2T}e^{\lambda x+\beta T} \tag{4.19}$$

Hence, the exact solution of (4.12) is:

$$u(x,t) = \frac{(x^2+1)\exp\left\{\lambda x + \beta\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right)\right\}}{1-\left(\frac{2t^\alpha}{\Gamma(1+\alpha)}\right)} \tag{4.20}$$

Note**:** when $\alpha = 1$, we have:

$$u(x,t) = \frac{(x^2+1)}{1-2t}e^{\lambda x+\beta t}, 1 \ne 2t \,. \tag{4.21}$$

Note: we present in Fig. 10 through Fig. 12, the graphical views of the obtained solution with respect to (w.r.t) example 3 to see the effects of the associate parameters. For Fig. 10, $\alpha = 1$, $\lambda = 0.85$, and $\beta = 0.5$. For Fig. 11, $\alpha = 0.95$, $\lambda = 0.85$, and $\beta = 1$. For Fig. 12, $\alpha = 0.95$, $\lambda = 0.65$, and $\beta = 1$.

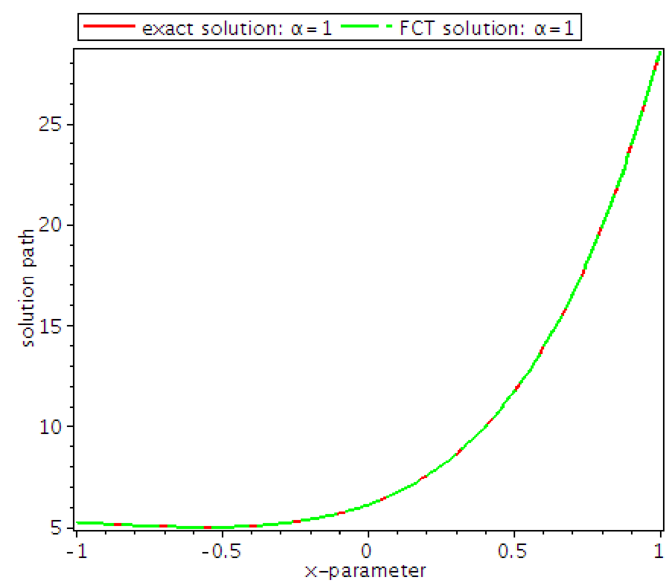

Fig. 10: The solution graph for $t = 0.4$ w.r.t. Example 3

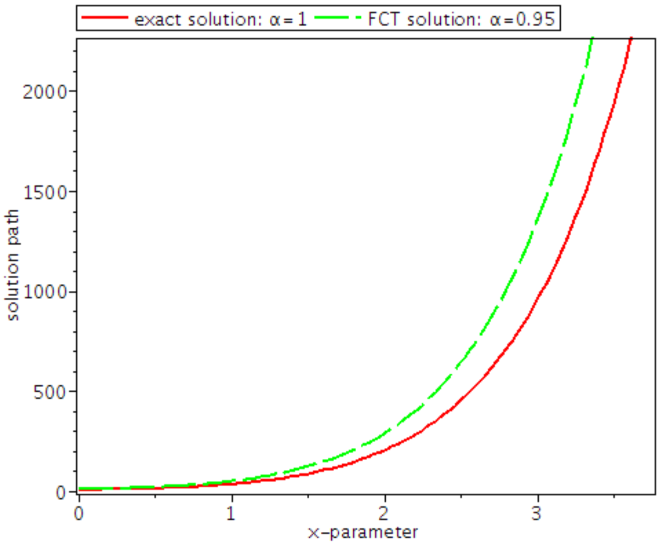

Fig. 11: The solution graph for $t = 0.4$ w.r.t. Example 3

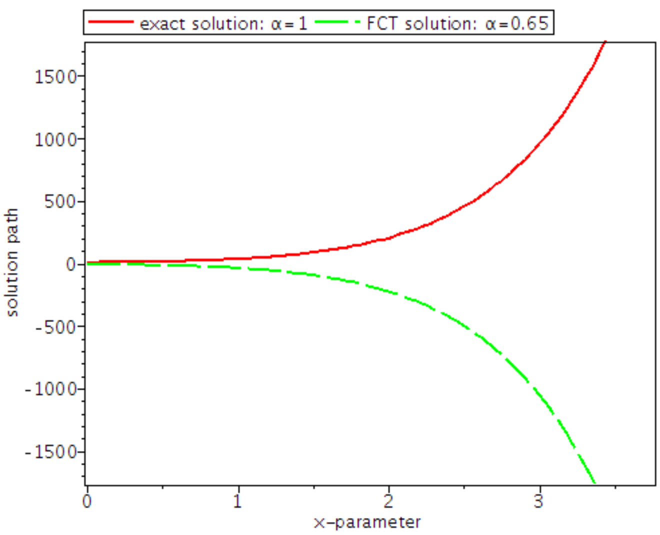

Fig. 12: The solution graph for $t = 0.4$ w.r.t. Example 3

V. Concluding Remarks

We obtained exact solutions of time-fractional Advection-Diffusion model equations by means of FCT coupled with modified differential transform method. The FCT is indeed simple but effective and accurate for the solutions of fractional differential equations. The associated derivatives were defined in terms of Jumarie's sense. Basic knowledge of advanced calculus is more required than that of fractional calculus while obtaining exact solutions of fractional equations with high level of accuracy not being compromised. This can therefore be extended to space-fractional derivatives of higher orders.

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