Numerical optimization solution of system of non-linear equations based on interval algorithm

Guobing Fan, Chen Guo and Sha Fu

***Abstract*—**Non-linear problem extensively exists in our life; however the solution algorithm for system of non-linear equations is always not perfect enough because of local optimum and low numerical calculation preciseness. This study aims to solve the numerical solution optimization of system of non-linear equations through algorithm. Based on traditional interval algorithm, this study put forward optimized interval algorithm which had improved Krawczyk iterative operator and combined it with genetic algorithm to compose genetic-optimization interval algorithm to perform optimization solution on system of non-linear equations. Then it was tested using several numerical examples. Finally the results suggested that the genetic-optimization interval solution algorithm could precisely calculate the solution of system of non-linear equations in shorter time, which achieved the research purpose of numerical optimization solution of system of non-linear equations.

***Keywords*—**genetic algorithm, interval algorithm, optimization, solution, system of non-linear equations

I. Introduction

WITH the rapid development of electronic computer technology, more and more computer mathematical models have been established and applied in solving some problems in real life. Problems in life and relevant influence factors are usually presented in a non-linear relationship. Therefore solution of various non-linear equations has been one of the key research directions in China and abroad. Yun BI [1] put forward an iterative method for non-linear equation with finitely many roots in an interval to obtain all the roots of the non-linear equation. Pang J et al. [2] solved two kinds of non-linear KdV equations with variable coefficient with (G'/G) expansion method and successively obtained some new precise solutions. It indicated that (G'/G) expansion method was also applicable to the solution of non-linear evolution equation with variable coefficient in addition to non-linear evolution equation with constant coefficient. Dutta J et al. [3] put forward a new iterative method which could find out the zero point of non-linear equations based on the concept of probability. Miodrag S et al. [4] proposed a simple iterative method to find out the roots of non-linear equations. It verified that the new method which involved no differential coefficient had quadric-form convergence order. Moreover it was found that the hybrid method in combination with non-iterative method could further improve convergence speed. In this work, the advantages and disadvantages of solving non-linear equations with interval algorithm were analyzed, and moreover the algorithm was improved to make up its defects and optimize the numerical solution ability.

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II. Overview of Interval Algorithm

In this study, interval algorithm was investigated and improved [5] to obtain the optimal solution of system of non-linear equations. Firstly interval algorithm and its algebraic properties were introduced.

A. Definition of interval computation

Definition 1: For given real number [6] $\underline{x}, \overline{x} \in R$ (R is a set of real numbers), if $\underline{x} \le \overline{x}$, then closed bounded number set X was called bounded interval.

$$X = [\underline{x}, \overline{x}] = \{\underline{x} \le x \le \overline{x}\} \tag{1}$$

where $\underline{x}$ stands for the lower limit point of interval x and $\overline{x}$ was the upper limit point of interval. If $\underline{x} = \overline{x}$, then interval X was a point interval. The set of all bounded closed interval of R was set as $I(R)$. The midpoint of interval was expressed as $m(X) = (\overline{x} + \underline{x})/2$, and the interval width was expressed as $W(X) = \overline{x} - \underline{x}$.

B. Interval vector

Interval vector was expressed as $X = (X_1, X_2, ..., X_n)$ ($X_i \in I(R)(i = 1,2,...,n)$). The midpoint of interval vector was expressed as $m(|x|) = (m(|x|_1), m(|x|_2), ..., m(|x|_n))$; the width of interval vector was as $w(|x|) = \max_i w(|x|_i)$; the radius of interval vector was as $r(|x|) = \max_i r(|x|_i) = w(|x|)/2$; the norm of interval vector was expressed as $\|x\| = \max_i \|x|_i|$.

C. Convergence of interval

For interval sequence $\{X^{(k)}\}$, if there was $X \in I(R)$ which can let $\lim_{k\to\infty} d(X^{(k)}, X) = 0$, then the sequence was considered as convergent [7] and X was its limit.

D. Extension of interval

Suppose $f : R'' \to R$. For the mapping of interval value [8] $F : I(R'') \to I(R)$, if $F([x_1, x_2],...,[x_n, x_n]) = f(x_1,..., x_n)$ could be established for any $x_i \in X_i (i = 1,2,...,n)$, then F was called the extension of function f.

Suppose $F : I(R^n) \to I(R)$. Moreover $X, Y \in I(R^n)$ and $X \subseteq Y$. If $F(X) \subseteq F(Y)$ could be established, then the mapping of interval value F had inclusion monotonicity.

III. Solution of non-linear equations with interval iteration optimization algorithm

A. Selection of interval operator

Firstly non-linear function was defined as

$$f(x) = 0, x \in D \subset R^n \tag{5}$$

Interval iteration method generally solves the included value of function interval using interval extension. Therefore an interval iteration operation was needed to analyze whether there was solution in interval value field.

Non-linear equation was transformed to fixed-point iteration mapping problem, equivalent to

$$x = p(x) = x - Hf(x) \tag{6}$$

where H stands for non-singular matrix [9], $H = \{m(F'([x]))\}^{-1}$.

For any interval vector $[x] \in IR^n$, point vector $h \in |x|$ and non-singular matrix $H \in R^{n\times n}$, $F'([x])$ was the extension of interval function $f(x)$. Interval mapping was expressed as

$$K([X], h) = h - Hf(h) + (I - HF'([x]))([x] - h) \tag{7}$$

Moreover it was also the improved Krawczyk [10] interval operator. h was the midpoint of interval vector $[x]$ and I was unit matrix.

Suppose interval non-linear function $f : D \subset R^n \to R^n$ was continuously differentiable, and x^* was expressed as any real number solution of system of non-linear equations. Then

(1) If $x^* \in [x]$, then $x^* \in K([x], h)$;

(2) If $K([x], h) \cap [x] = \varnothing$, then there was no solution of system of non-linear equations $f(x)$ in interval [x].

(3) If $K([x], h) \subseteq [x]$, then there was at least one solution for system of non-linear equations in [x].

Proof: (1) As x^* stands for a solution of system of non-linear equations, then $\exists f(x) \in f(x)$, which made $f(x^*) = 0$. Hence

$$\begin{aligned} x^* &= x^* - Hf(x^*) = h - Hf(h) + x^* - h - H(f(x^*) - f(h)) \\ &= h - Hf(h) + x^* - h - Hf'(h + \theta(x^* - h))(x^* - h) \\ &= h - Hf(h) + (I - Hf(h + \theta(x^* - h)))(x^* - h) \\ &\in h - Hf(h) + (I - HF'([x]))([x] - h) \subseteq K([x], h) \end{aligned} \tag{8}$$

(2) Suppose x^* was a solution of system of non-linear equations, then $x^* \in [x]$. Moreover it was concluded from equation (1) that $x^* \in K([x], h)$. It was not consistent with $K([x], h) \cap [x] = \varnothing$. Therefore [x] did not include the solution of system of non-linear equation $f(x)$.

(3) Suppose $\forall f(x) \in f(x)$, $q(x) = x - Hf(x)$, then

$$\begin{aligned} q(x) &= h - Hf(h) + x - h - H(f(x) - f(h)) \\ &= h - Hf(h) + x - h - Hf'(h + \theta(x - h))(x - h) \\ &= h - Hf(h) + (I - Hf'(h + \theta(x - h)))(x - h) \end{aligned} \tag{9}$$

As $x, h \in [x]$, $h + \theta(x - h) \in [x]$. Thus

$$q(x) \in h - Hf(h) + (I - HF'([X]))(x - h) \subseteq K([x], h) \tag{10}$$

Firstly [x] was a non-empty bounded closed set, and then $q(x)$ was in [x] and moreover continuous. If $K([x], h) \subseteq [x]$, then $\{q(x) : x \in [x]\} \subseteq [x]$. According to Brower fixed point theorem, there was at least one fixed point $x^* \in [x]$ of $q(x)$ which made $Hf(x^*) = 0$. H stands for non-singular matrix; hence $f(x^*) = 0$, indicating that [x] undoubtedly included a solution of non-linear equation. Then it was concluded that $[x]$ at least included a solution of system of non-linear equations.

B. Interval global optimization

The basic procedures of interval optimization were as follows.

Firstly an initial interval was set as X^0, target function that needed optimization was expressed as f, and the list of intervals waiting for processing was expressed as L.

Step 1: Initial interval $X^0 \in I(R^n)$, $F(X^0)=[F^L(X),F^R(X^0)]$.

Step 2: Suppose $L=\{(X^0,F^L(X^0))\}$. $F^L(X^0)$ was arranged from small to large; as a result, the interval vector of L had the smallest interval lower bound. One element could be obtained each time, denoted as $(X^0,F^L(X^0))$.

Step 3: The smallest interval vector was selected from L. $X^0 = X^1 \cup X^2$, then dichotomization was performed on the component which had the largest width.

Step 4: Calculate $F(X^i)=[F^L(X^i),F^R(X^i)],i=1,2$.

Step 5: $(X^0,F^L(X^0))$ was deleted; $(X^1,F^L(X^1))$ and $(X^2,F^L(X^2))$ were added into initial list L to make the interval vector of L had the smallest lower bound.

Step 6: Interval vectors which did not include the optimal solution of system of equations were deleted using tools.

① Midpoint detection tool

The first vector in initial list L was taken, and the value of f at the midpoint of the interval vector was calculated. The threshold value of optimal value f* was $\hat{f} = \min(f\_\text{mid}, F^R(X^0))$. Then the interval whose lower bound was larger than the threshold value $\hat{f}$ was deleted. The threshold value was reduced, which improved the efficiency of the algorithm.

② Detection of monotonicity [11]

For any i and f satisfying $0 \notin \frac{\partial f}{\partial x_i}(X^0)$, the smallest value of objective function was not inside interval, but might be end point; if objective function f had minimal value on point x^*, then the function in some field of point x^* was convex. If $\frac{\partial^2 f}{\partial x_i^2}(X)<0$ for some i, then interval matrix $F(x)$ certainly did not include positive definite matrix, there was no optimal solution in the interval, and some endpoint in the original interval could be deleted.

③ Lipschitz condition detection [12]

$f: R^n \to R$ was supposed as a Lipschitz function which took $H>0$ as constant, and $F(x)$ was supposed as the monotone-type interval extension of $f(x)$.

$$F(X) = f(m(x)) + H[X - m(X)], \quad (11)$$

which could simplify the calculation of upper and lower limits.

Step 7: Whether there was solution was determined. If there was, then iteration continued till satisfying the requirement on preciseness; otherwise the interval was deleted.

Step 8: Return to step 3 until list L was empty.

Interval algorithm was capable of processing relatively complex system of non-linear equations and calculating its upper limit in some interval. Moreover it could ensure the reliability of global optimization solution of system of non-linear equations. But problems such as slow computational speed and excessively conservative deletion of value interval led to the low overall efficiency of interval algorithm.

C. Optimization of solution for system of non-linear equations with genetic algorithm in combination with interval algorithm

a. Basic operations of genetic algorithm

Genetic algorithm has excellent global convergence and group search abilities [13]. The range of solution of objective function was processed by binary or decimal coding. The largest evolution number of population was set as T, and M individuals which randomly generated were taken as the initial population, $X_{(0)} = \{x_1, x_2, \ldots, x_n\}$. The fitness of each individual in population $F(x_1)$ was calculated, and fitness stands for the quality of the solution. Next generation of individuals and middle generation X_{r_t} generated based on genetic selection operator. New generation of population $X_{(t+1)}$ generated based on crossover operator and mutation operator ($t=t+1$). If the termination condition was not satisfied, the calculation of population fitness continued; if the termination condition was satisfied or $t>T$, then the individual with the highest fitness was regarded as the optimal solution and output.

b. Solution of system of non-linear equations based on genetic-optimization interval algorithm

The solution of equation (5) could be equivalent to the following optimization problem.

$$\min: fitness(x) = \sqrt{\sum_{i=1}^{n} (f_i(x))^2}, x \in D \subset R^n \quad (12)$$

The solution of non-linear equation based on genetic algorithm in combination with interval algorithm was as follows. Firstly the solution problem was regarded as the problem of optimization interval. It was mostly based on the favorable global convergent group search ability of genetic algorithm. Moreover interval had good judgment. Interval iterative algorithm has been able to initially determine intervals with solutions of equations and intervals without solutions of equations. In this way, the area which needed searching of equation solution was further narrowed, which greatly reduced workload, improved overall computational efficiency, and avoided the premature phenomena of genetic algorithm. The combination of the two algorithms could significantly improve the efficiency of solution of non-linear equation and obtain the optimal solution as far as possible.

The detailed procedures of the algorithm were as follows.

(1) Initial population was established, and the optimal solution was searched through genetic operation.

(2) Intervals which have been verified including no equation solution by interval iterative algorithm were excluded, and near-optimal solution X_1 was obtained. Interval vector B_1 was established by taking X_1 as the center and r_1 as the radius.

(3) A population was established in interval vector B_1, and near-optimal solution X_2 was obtained through genetic operations.

(4) Interval vector B_2 was established taking near-optimal solution X_2 as the center and r_2 as the radius ($r_1 \geq r_2$).

(5) Interval vector B_2 was taken as the initial interval. Whether there was the solution of system of non-linear equations in the interval was determined based on the properties of Krawczyk-Moore interval operator. If there was no solution, then interval vector M_2 was recorded, and it turned to procedure 1; if there was iterative divergence, then it turned to procedure 4, and radius r_2 was narrowed; if there was, then iteration continued, and the solution of the system of equations was converged.

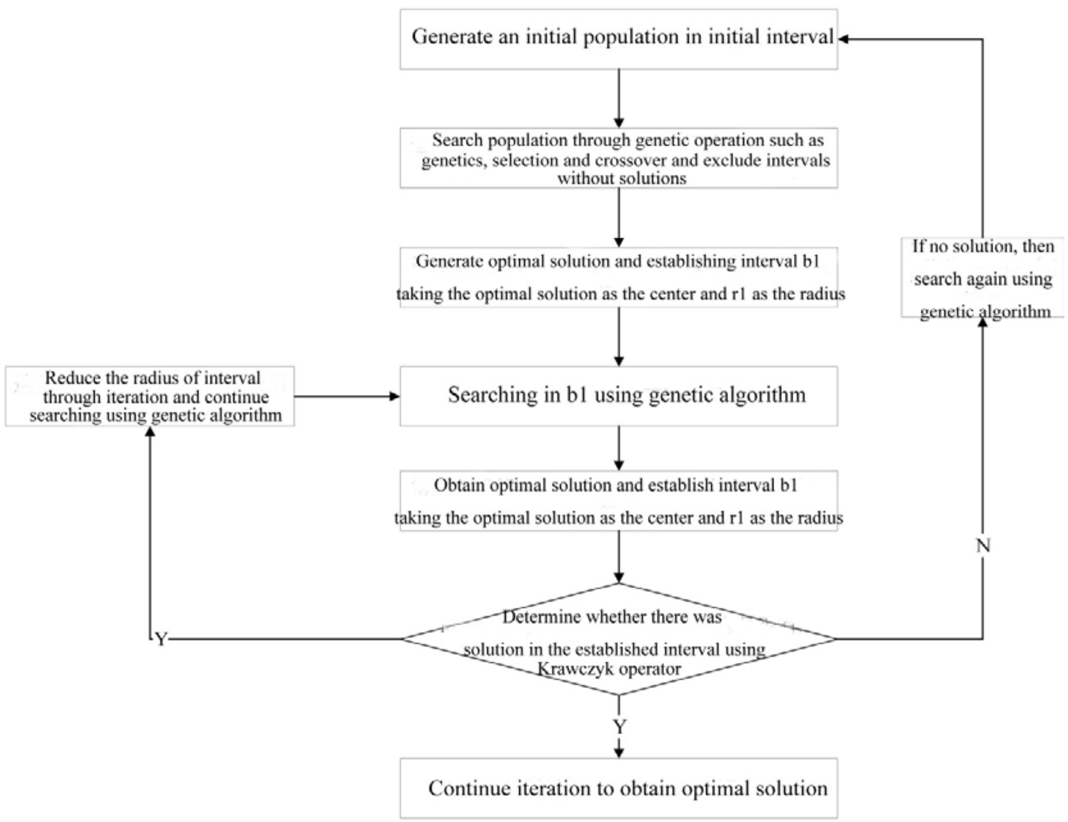

Fig. 1 The flow chart of solution of system of non-linear equations based on interval optimization-genetic search algorithm optimization

IV. NUMERICAL EXAMPLE

To test the performance of the proposed interval optimization-genetic search algorithm in the solution of system of non-linear equations, three functions were selected for numerical test. Moreover all the tested non-linear functions had multiple local optimal solutions. The numerical test was performed on Matlab2016 software which carries interval algorithm program package Intlab v5.5 [14] and genetic algorithm toolbox GUI. Some parameters of the combined algorithm were set. The population quantity of genetic algorithm was set as 60, iteration number was as 80, mutation probability was as 0.25, the radius of initially established interval vectors was $r_1 = 0.6$ and $r_2 = 0.1$ respectively, and the termination criterion was vector width $w(X) \leq 10^{-6}$. KRA stands for Krawczyk iterative algorithm, han-s stands for Hensen-Sengupta iteration method [15], and GA-KRA stands for Krawczyk interval iteration optimization – genetic search algorithm.

iter stands for number of iteration, time stands for the time consumed in calculation (unit: s), error stands for the error of iterative calculation, and [x] stands for calculation results.

Finally the results were obtained.

Numerical example 1 was a non-linear equation as follows.

$$\begin{cases} 2x_1 - x_2 - 2 = 0 \\ -x_1^2 - 4x_1 + \frac{7}{2}x_2 + 5 = 0 \end{cases} \tag{13}$$

The solution in interval vector $D = \{[0.6,1.1],[-0.2,0.5]\}^T$ was $x^* = (1,0)^T$. The numerical calculation results of the algorithms are shown in Table 1.

Numerical example 2 was a non-linear equation as follows.

$$\begin{cases} \sin x_1 + \cos x_2 - 1 = 0 \\ -2\cos x_1 - 2\cos x_2 + 3 = 0 \end{cases} \tag{14}$$

Vector interval was $D = ([0.3,0.5],[0.8,1.5])^t$, then the solution of the equation in the interval was $x^* = (0.42403103849073,0.94151614381642)^T$. The numerical calculation results of the algorithms are shown in Table 2.

Numerical example 3 was a system of non-linear equations as follows.

$$\begin{cases} x_1^3 - x_2^{x_3} - x_3^{x_2} - 60 = 0 \\ x_1^{x_2} + x_2^{x_1} - 5x_1x_2x_3 - 85 = 0 \\ x_1^{x_3} - x_2 - x_3^{x_1} - 2 = 0 \end{cases} \tag{15}$$

Interval vector was $D = ([3.7,4.2],[2.8,3.3],[0.7,1.2])^t$, and the solution in the interval was $x^* = (4,3,1)^t$. The numerical calculation results of the algorithms are shown in Table 3.

Table 1 Comparison of the numerical calculation results of numerical example 1 using the three algorithms

Algorithm	iter	[x]	error	time
KRA	6	$\begin{pmatrix} [0.99999999999999,1.00000000000000] \\ [-0.00000000000001,0.00000000000002] \end{pmatrix}$	9.4074e-15	0.4018
han-s	4	$\begin{pmatrix} [0.99999999999999,1.00000000000001] \\ [-0.00000000000002,0.00000000000001] \end{pmatrix}$	9.4074e-15	0.3443
GA-KRA	3	$\begin{pmatrix} [0.99999999999999,1.00000000000000] \\ [-0.00000000000001,0.00000000000000] \end{pmatrix}$	1.1102e-16	0.3185

Table 2 Comparison of the numerical calculation results of numerical example 2 using the three algorithms

Algorithm	iter	[x]	error	time
KRA	6	$\begin{pmatrix} [0.42403103949073,0.42403103949075] \\ [0.941517614381640,0.941516143481642] \end{pmatrix}$	5.7175e-15	0.0543
han-s	4	$\begin{pmatrix} [0.42403103949064,0.42403103949084] \\ [-0.94151614381640,0.94151134481642] \end{pmatrix}$	9.7586e-14	0.0517
GA-KRA	3	$\begin{pmatrix} [0.42403103849074,0.42403103949075] \\ [0.94151614381641,0.94151614381642] \end{pmatrix}$	1.1102e-16	0.0482

Table 3 Comparison of the numerical calculation results of numerical example 3 using the three algorithms

Algorithm	iter	[x]	error	time
KRA	8	$\begin{pmatrix} [3.99999999999999,4.00000000000001] \\ [2.99999999999991,3.00000000000009] \\ [0.99999999999998,1.00000000000002] \end{pmatrix}$	8.4821e-14	0.2187
han-s	6	$\begin{pmatrix} [4.00000000000000,4.00000000000001] \\ [2.99999999999997,3.00000000000000] \\ [0.99999999999999,1.00000000000001] \end{pmatrix}$	8.7586e-16	0.1786
GA-KRA	5	$\begin{pmatrix} [4.00000000000000,4.00000000000001] \\ [2.99999999999998,3.00000000000000] \\ [1.00000000000000,1.00000000000001] \end{pmatrix}$	8.8816e-16	0.1739

It was concluded from the numerical calculation results of the above three numerical examples that the proposed Krawczyk interval iteration optimization-genetic search algorithm has smaller error and shorter calculation time compared to Krawczyk interval iteration and Hensen-Sengupta iteration. Moreover the intervention of improved Krawczyk operator and genetic operation reduced the iteration number and further enhanced the calculation efficiency.

V. Application Example

The solution of nonlinear functions has been widely used in the fields such as finance and science. By using the genetic-optimization interval algorithm proposed in this paper, the practical problems can be solved well. The following is a case.

An investor planned a long-term portfolio investment using stock X, Y and Z. The related data of the three stocks are shown in Table 4. He expected to get the maximum benefit when the standard deviation was not larger than 12%. Please figure out the investment percentages of the three stocks.

Table 4 Related data of stock X, Y and Z

Name of stock		X	Y	Z
Five-year expected return rate (%)		94	64	42
Five-year covariance	X	180	36	110
	Y	36	120	-30
	Z	110	-30	140

Suppose x_1, x_2 and x_3 as the investment percentages of stock X, Y and Z, and then a mathematical model was established.

$$\begin{cases} \max & R = 0.94x_1 + 0.64x_2 + 0.42x_3, \\ s.t. & x_1 + x_2 + x_3 = 1, \\ & \left[180x_1^2 + 120x_2^2 + 140x_3^2 + 72x_1x_2 + 220x_1x_3 - 60x_2x_3\right]^{\frac{1}{2}} \le 12, \\ & x_1, x_2, x_3 \ge 0, \end{cases} \quad (16),$$

which is equivalent to

$$\begin{cases} \min & R = -0.94x_1 - 0.64x_2 - 0.42x_3, \\ s.t. & x_1 + x_2 + x_3 = 1, \\ & -\left[180x_1^2 + 120x_2^2 + 140x_3^2 + 72x_1x_2 + 220x_1x_3 - 60x_2x_3\right]^{\frac{1}{2}} \ge -12, \\ & x_1 + x_2 + x_3 \ge 0, \end{cases} \quad (17).$$

It was transformed to the problem of non-linear function, and then it was solved using genetic-optimization interval algorithm. Finally $x_1 = 0.8694$, $x_2 = 0.1306$ and $x_3 = 0.0000$ were obtained.

The optimal value of the target function was 0.8796. When the standard deviation was not larger than 12%, the investment percentages of stock X, Y and Z was 86.94%, 13.06% and 0% respectively, and the maximum return rate was 87.96%.

VI. Conclusion

To optimize the solution of non-linear function, interval algorithm based calculation method was proposed. In this study, the definition of interval algorithm and its attributes and characteristics were introduced, and then some improvement and adjustment were made based on it. The proposed improved Krawczyk interval algorithm could effectively help determine whether the solution of system of non-linear equation was in some interval and obtain the position of solution the interval. Moreover Lipschitz condition detection, midpoint detection and monotonicity detection tool were used to coordinate the determination on the solution domain of system of non-linear equation, which enhanced the global optimization performance of interval iteration algorithm. Then based on the problems of interval iteration algorithm such as slow iterative calculation and excessively conservative deletion of interval, genetic algorithm with strong global convergence and group search capacities was introduced. Interval algorithm was combined with genetic algorithm to compose genetic-optimization interval algorithm. Finally Matlab software was used to test the performance of the algorithm. The results suggested that genetic-optimization interval algorithm could optimize the solution of system of non-linear equations and obtain globally optimal solution, and moreover its iteration number and calculation time were superior to those of interval algorithm and genetic algorithm. The algorithm proposed in this study provides an orientation for the solution of system of non-linear equations and is effective in solving non-linear problems in reality.

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