Classes of Ordinary Differential Equations Obtained for the Probability Functions of Linear Failure Rate and Generalized Linear Failure Rate Distributions

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***Abstract* —** The linear failure rate (hazard) and generalized linear failure rate (hazard) distributions are uniquely identified by their linear hazard functions. In this paper, homogenous ordinary differential equations (ODES) of different orders were obtained for the probability functions of linear failure rate and generalized linear failure rate distributions. This is possible since the aforementioned probability functions of the distributions are differentiable and the former distribution is a particular case of the later. Differentiation and modified product rule were used to derive the required ODEs, whose solutions are the respective probability functions. The different conditions necessary for the existence of the ODEs were obtained and it is in consistent with the support that defined the various probability functions considered. The parameters that defined each distribution greatly affect the nature of the ODEs obtained. This method provides new ways of classifying and approximating other probability distributions apart from one considered in this research. Algorithms for implementation can be helpful in improving the results.

***Keywords* —** Differentiation, product rule, quantile function, failure rate, approximation, hazard function, inverse survival function.

I. Introduction

DIFFERENT mathematical techniques are viable tools in statistics. In mathematical statistics, different mathematical areas are used heavily in better understanding of probability distributions. Some of these are calculus, differential equations, algebra, measure theory, fixed point and topology and so on. Hitherto most of the use of ordinary differential equation (ODE) is often in mode and parameter estimation and approximation. Approximation of quantile function features prominently in the use of ODE in approximation [1-6].

Few available literatures have considered the study of the ODE of different probability functions of the studied distribution in particular and probability distributions in general. The available ones contain previous works done on the ODE of the following distributions: beta distribution [7], Lomax distribution [8], beta prime distribution [9], Laplace distribution [10] and raised cosine distribution [11].

Derivation of homogenous ordinary differential equations for the probability density function (PDF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of linear failure rate and generalized linear failure rate distributions was considered in this paper. This will also help to provide the answers as to whether there are discrepancies between the support of the distribution and the conditions necessary for the nature and existence of the ODEs. Similar results for other distributions have been proposed and can be seen in [12-24]

The linear failure rate (hazard) and generalized linear failure rate (hazard) distributions are uniquely identified by their linear hazard function and the former is generalized to obtain the later distribution.

The details of the linear failure distribution can be found in [25]. Kantam et al. [26] gave the detailed comparison between the distribution and the Rayleigh distribution while Block et al. [27] reviewed some mixture of distributions with linear failure rates. Estimation of parameters of the distribution has been explored intensively such as: Bayes estimate [28], detailed inference procedures [29], Bayesian estimation based on records [30], use of simulation in the Bayesian estimates of the parameters [31], parameter estimation by the use of masked data [32], Bayesian inference for randomly progressive random censored samples [33].

The variants or generalizations and modifications of the distribution include: Generalized Linear Failure rate distribution was proposed by Sarhan and Kundu [34].

Others are: bivariate linear failure rate distribution [35-36], bivariate and multivariate generalized linear failure rate distribution [37], McDonald generalized linear failure rate distribution [38], modified generalized linear failure rate distribution [39], new five parameter modified generalized linear failure rate distribution [40], beta-linear failure rate distribution [41] and extended linear failure rate distribution [42].

Others are: bivariate generalized linear failure rate-power series class of distributions [43], Kumaraswamy generalized linear failure rate distribution [44], extension of the generalized linear failure rate distribution [45], generalized

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linear failure rate power series distribution [46], Poisson generalized linear failure rate distribution [47] and beta linear failure rate geometric distribution [48]. The distribution was applied by Bain [49] in the analysis of lifetime data.

The ordinary differential calculus was used to obtain the results presented in different sections.

II. Linear Failure Rate Distribution

A Probability Density Function

The PDF of the Linear failure rate distribution is given as;

$$f(x)=(a+bx)\mathrm{e}^{-(ax+\frac{bx^2}{2})} \quad (1)$$

When $b=0$ & $a\neq 0$, the distribution reduces to the exponential distribution.

When $a=0$ & $\mathrm{b}\neq 0$, the distribution reduces to the Rayleigh distribution.

Differentiate equation (1), to obtain;

$$f'(x)=\left\{\frac{b}{a+bx}-\frac{(a+bx)\mathrm{e}^{-(ax+\frac{bx^2}{2})}}{\mathrm{e}^{-(ax+\frac{bx^2}{2})}}\right\}f(x) \quad (2)$$

$$f'(x)=\left\{\frac{b}{a+bx}-(a+bx)\right\}f(x) \quad (3)$$

The equation can only exists for $a-b\neq 0, x>0.$

The first order ODE for the PDF of the Linear failure rate distribution is given by;

$$(a+bx)f'(x)-(b-(a+bx)^2)f(x)=0 \quad (4)$$

$$f(1)=(a+b)\mathrm{e}^{-(a+\frac{b}{2})} \quad (5)$$

Special cases are considered;

When $b=0$ & $a\neq 0$, equation (4) becomes;

$$f'(x)+af(x)=0 \quad (6)$$

When $a=0$ & $\mathrm{b}\neq 0$, equation (4) becomes;

$$xf'(x)-(1-bx^2)f(x)=0 \quad (7)$$

B Quantile Function

The QF of the Linear failure rate distribution is given as;

$$aQ(p)+\frac{b}{2}Q^2(p)=-\ln(1-p) \quad (8)$$

Differentiate equation (8) to obtain;

$$aQ'(p)+bQ(p)Q'(p)=\frac{1}{1-p} \quad (9)$$

The equation can only exists for $a-b\neq 0, 0<p<1.$

The first order ODE for the QF of the Linear failure rate distribution is given by;

$$(1-p)(a+bQ(p))Q'(p)-1=0 \quad (10)$$

$$aQ(0.1)+\frac{b}{2}Q^2(0.1)=0.1054 \quad (11)$$

$$Q(0.1)=\frac{-2a\pm 2\sqrt{a^2+0.21b}}{2b} \quad (12)$$

Special cases are considered;

When $b=0$ & $a\neq 0$, equation (10) becomes;

$$a(1-p)Q'(p)-1=0 \quad (13)$$

When $a=0$ & $\mathrm{b}\neq 0$, equation (10) becomes;

$$b(1-p)Q(p)Q'(p)-1=0 \quad (14)$$

C Survival Function

The SF of the Linear failure rate distribution is given as;

$$S(t)=\mathrm{e}^{-(at+\frac{b}{2}t^2)} \quad (15)$$

Differentiate equation (15) to obtain;

$$S'(t)=-(a+bt)\mathrm{e}^{-(at+\frac{b}{2}t^2)} \quad (16)$$

The equation can only exists for $a-b\neq 0, t>0$.

The first order ODE for the SF of the Linear failure rate distribution is given by;

$$S'(t)+(a+bt)S(t)=0 \quad (17)$$

$$S(1)=\mathrm{e}^{-(a+\frac{b}{2})} \quad (18)$$

Special cases are considered;

When $b=0$ & $a\neq 0$, equation (17) becomes;

$$S'(t)+aS(t)=0 \quad (19)$$

When $a=0$ & $\mathrm{b}\neq 0$, equation (17) becomes;

$$S'(t)+btS(t)=0 \quad (20)$$

D Inverse Survival Function

The ISF of the Linear failure rate distribution is given as;

$$aQ(p)+\frac{b}{2}Q^2(p)=-\ln p \quad (21)$$

Differentiate equation (21) to obtain;

$$aQ'(p)+bQ(p)Q'(p)=-\frac{1}{p} \quad (22)$$

The equation can only exists for $a-b\neq 0, 0<p<1.$

The first order ODE for the ISF of the Linear failure rate distribution is given by;

$$p(a+bQ(p))Q'(p)+1=0 \quad (23)$$

$$aQ(0.1)+\frac{b}{2}Q^2(0.1)=2.3025 \quad (24)$$

$$Q(0.1)=-\frac{a}{b}\pm 2\sqrt{2a^2-4.605b} \quad (25)$$

E Hazard Function

The HF of the Linear failure rate distribution is given as;

$$h(t)=a+bt \quad (26)$$

Differentiate equation (26) to obtain;

$$h'(t)=b \quad (27)$$

From equation (26), it can be obtained that;

$$b=\frac{h(t)-a}{t} \quad (28)$$

Substitute equation (28) into equation (27);

$$h'(t)=\frac{h(t)-a}{t} \tag{29}$$

The first order ODE for the HF of the Linear failure rate distribution is given by;

$$th'(t)-h(t)+a=0 \tag{30}$$

$$h(1)=a+b \tag{31}$$

When $b=0$ & $a\neq 0$, equation (27) becomes;

$$h'(t)=0 \tag{32}$$

When $a=0$ & $\mathrm{b}\neq 0$, equation (30) becomes;

$$th'(t)-h(t)=0 \tag{33}$$

The nature of the ODEs point to the linearity of the hazard function.

F Reversed Hazard Function

The RHF of the Linear failure rate distribution is given as;

$$j(t)=\frac{(a+bt)\,\mathrm{e}^{-(at+\frac{bt^2}{2})}}{1-\mathrm{e}^{-(at+\frac{bt^2}{2})}} \tag{34}$$

Differentiate equation (34) to obtain;

$$j'(t)=\left\{\begin{array}{l}\dfrac{(a+bt)}{b}-\dfrac{(a+bt)\,\mathrm{e}^{-(at+\frac{bt^2}{2})}}{\mathrm{e}^{-(at+\frac{bt^2}{2})}}\\ -\dfrac{(a+bt)\,\mathrm{e}^{-(at+\frac{bt^2}{2})}(1-\mathrm{e}^{-(at+\frac{bt^2}{2})})^{-2}}{(1-\mathrm{e}^{-(at+\frac{bt^2}{2})})^{-1}}\end{array}\right\}j(t) \tag{35}$$

$$j'(t)=\left\{\frac{(a+bt)}{b}-(a+bt)-\frac{(a+bt)\,\mathrm{e}^{-(at+\frac{bt^2}{2})}}{(1-\mathrm{e}^{-(at+\frac{bt^2}{2})})}\right\}j(t) \tag{36}$$

The equation can only exists for $a-b\neq 0, t>0$.

$$bj'(t)=\big((a+bt)-b(a+bt)-bj(t)\big)\,j(t) \tag{37}$$

The first order ODE for the RHF of the Linear failure rate distribution is given by;

$$bj'(t)+bj^2(t)+(b-1)(a+bt)\,j(t)=0 \tag{38}$$

$$j(1)=\frac{(a+b)\,\mathrm{e}^{-(a+\frac{b}{2})}}{1-\mathrm{e}^{-(a+\frac{b}{2})}}=\frac{a+b}{\mathrm{e}^{(a+\frac{b}{2})}-1} \tag{39}$$

III. Generalized Linear Failure Rate Distribution

A Probability Density Function

The PDF of the Generalized Linear failure rate distribution is given as;

$$f(x)=\theta(a+bx)\left[1-\mathrm{e}^{-(ax+\frac{bx^2}{2})}\right]^{\theta-1}\mathrm{e}^{-(ax+\frac{bx^2}{2})} \tag{40}$$

When $\theta=1$, equation (40) reduces to the Linear failure rate distribution. When $b=0$ & $a>0$, equation (40) reduces to the Generalized exponential distribution.

When $a=0$ & $\mathrm{b}>0$, equation (40) reduces to the Generalized Rayleigh distribution.

Differentiate equation (40), to obtain;

$$f'(x)=\left[\begin{array}{l}\dfrac{b}{a+bx}+\dfrac{(\theta-1)(a+bx)\,\mathrm{e}^{-(ax+\frac{bx^2}{2})}}{1-\mathrm{e}^{-(ax+\frac{bx^2}{2})}}\\ -(a+bx)\end{array}\right]f(x) \tag{41}$$

The equation can only exists for $a-b\neq 0, x, \theta>0$.

The ODEs can only be derived for any given values of a, b and θ.

When $\theta=1$, equation (41) becomes;

$$f'(x)=\left[\frac{b}{a+bx}-(a+bx)\right]f(x) \tag{42}$$

$$(a+bx)\,f'(x)-(b-(a+bx)^2)\,f(x)=0 \tag{43}$$

To obtain a simpler ODE, differentiate equation (41);

$$f''(x)=\left\{\begin{array}{l}\dfrac{b}{a+bx}+\dfrac{(\theta-1)(a+bx)\,\mathrm{e}^{-(ax+\frac{bx^2}{2})}}{\left(1-\mathrm{e}^{-(ax+\frac{bx^2}{2})}\right)}\\ -(a+bx)\end{array}\right\}f'(x)$$

$$-\left\{\frac{b^2}{(a+bx)^2}+b\right\}f(x)$$

$$+\left\{\begin{array}{l}\dfrac{(\theta-1)b\,\mathrm{e}^{-(ax+\frac{bx^2}{2})}}{\left(1-\mathrm{e}^{-(ax+\frac{bx^2}{2})}\right)}-\dfrac{(\theta-1)(a+bx)^2\,\mathrm{e}^{-(ax+\frac{bx^2}{2})}}{\left(1-\mathrm{e}^{-(ax+\frac{bx^2}{2})}\right)}\\ -\dfrac{((\theta-1)(a+bx)\,\mathrm{e}^{-(ax+\frac{bx^2}{2})})^2}{\left(1-\mathrm{e}^{-(ax+\frac{bx^2}{2})}\right)^2}\end{array}\right\}f(x) \tag{44}$$

The equation can only exists for $a-b\neq 0, x, \theta>0$.

These presented equations derived from equation (41) are required in the evaluation of equation (44);

$$\frac{f'(x)}{f(x)}=\frac{b}{a+bx}+\frac{(\theta-1)(a+bx)\mathrm{e}^{-(ax+\frac{bx^2}{2})}}{\left(1-\mathrm{e}^{-(ax+\frac{bx^2}{2})}\right)}-(a+bx) \tag{45}$$

$$\frac{(\theta-1)(a+bx)\mathrm{e}^{-(ax+\frac{bx^2}{2})}}{\left(1-\mathrm{e}^{-(ax+\frac{bx^2}{2})}\right)}=\frac{f'(x)}{f(x)}+(a+bx)-\frac{b}{a+bx} \tag{46}$$

$$\frac{(\theta-1)b\,\mathrm{e}^{-(ax+\frac{bx^2}{2})}}{\left(1-\mathrm{e}^{-(ax+\frac{bx^2}{2})}\right)}=\frac{b}{a+bx}\left(\frac{f'(x)}{f(x)}+(a+bx)-\frac{b}{a+bx}\right) \tag{47}$$

$$\frac{(\theta-1)(a+bx)^2\,\mathrm{e}^{-(ax+\frac{bx^2}{2})}}{\left(1-\mathrm{e}^{-(ax+\frac{bx^2}{2})}\right)} \tag{48}$$

$$=(a+bx)\left(\frac{f'(x)}{f(x)}+(a+bx)-\frac{b}{a+bx}\right)$$

$$\frac{((\theta-1)(a+bx)\mathrm{e}^{-(ax+\frac{bx^2}{2})})^2}{\left(1-\mathrm{e}^{-(ax+\frac{bx^2}{2})}\right)^2}=\left(\frac{f'(x)}{f(x)}+(a+bx)-\frac{b}{a+bx}\right)^2 \tag{49}$$

$$\frac{(\theta-1)((a+bx)\mathrm{e}^{-(ax+\frac{bx^2}{2})})^2}{\left(1-\mathrm{e}^{-(ax+\frac{bx^2}{2})}\right)^2} \tag{50}$$

$$=\frac{1}{\theta-1}\left(\frac{f'(x)}{f(x)}+(a+bx)-\frac{b}{a+bx}\right)^2$$

Substitute equations (45), (47), (48) and (50) into equation (44) to obtain;

$$f''(x)=\frac{f'^2(x)}{f(x)}+\left[\begin{array}{l}\frac{b}{a+bx}\left(\frac{f'(x)}{f(x)}+a+bx-\frac{b}{a+bx}\right)\\ -\frac{1}{\theta-1}\left(\frac{f'(x)}{f(x)}+a+bx-\frac{b}{a+bx}\right)^2\end{array}\right.$$

$$\left.-a+bx\left(\frac{f'(x)}{f(x)}+a+bx-\frac{b}{a+bx}\right)-\frac{b^2}{(a+bx)^2}-b\right]f(x) \tag{51}$$

$$f(1)=\theta(a+b)\left[1-\mathrm{e}^{-(a+\frac{b}{2})}\right]^{\theta-1}\mathrm{e}^{-(a+\frac{b}{2})} \tag{52}$$

$$f'(1)=\theta(a+b)\left[\begin{array}{l}\frac{b-(a+b)^2}{a+b}\\ +\frac{(\theta-1)(a+b)\mathrm{e}^{-(a+\frac{b}{2})}}{\left(1-\mathrm{e}^{-(a+\frac{b}{2})}\right)}\end{array}\right] \tag{53}$$

$$\mathrm{e}^{-(a+\frac{b}{2})}\left[1-\mathrm{e}^{-(a+\frac{b}{2})}\right]^{\theta-1}$$

Special cases of the second order differential equation for the PDF of the generalized linear failure rate distribution are considered;

When $b=0$ & $a>0$, equation (51) becomes;

$$f''(x)=\frac{f'^2(x)}{f(x)}-\left[\begin{array}{l}\frac{1}{\theta-1}\left(\frac{f'(x)}{f(x)}+a\right)^2\\ +a\left(\frac{f'(x)}{f(x)}+a\right)\end{array}\right]f(x) \tag{54}$$

When $a=0$ & $b>0$, equation (51) becomes;

$$f''(x)=\left[\frac{1}{x}\left(\frac{f'(x)}{f(x)}+bx-\frac{1}{x}\right)-\frac{1}{\theta-1}\left(\frac{f'(x)}{f(x)}+bx-\frac{1}{x}\right)^2\right.$$

$$\left.+\frac{f'^2(x)}{f(x)}-bx\left(\frac{f'(x)}{f(x)}+bx-\frac{1}{x}\right)-\frac{1}{x^2}-b\right]f(x) \tag{55}$$

B Quantile Function

The QF of the generalized Linear failure rate distribution is given as;

$$aQ(p)+\frac{b}{2}Q^2(p)=-\ln(1-p^{\frac{1}{\theta}}) \tag{56}$$

Differentiate equation (56) to obtain;

$$aQ'(p)+bQ(p)Q'(p)=\frac{p^{\frac{1}{\theta}-1}}{\theta(1-p^{\frac{1}{\theta}})} \tag{57}$$

The equation can only exists for

$a-b\neq 0,\theta>0,0<p<1$.

$$(a+bQ(p))Q'(p)=\frac{p^{\frac{1}{\theta}}}{\theta p(1-p^{\frac{1}{\theta}})} \tag{58}$$

$$\theta p(1-p^{\frac{1}{\theta}})(a+bQ(p))Q'(p)-p^{\frac{1}{\theta}}=0 \tag{59}$$

The ODE can be derived for any given values of a, p and θ.

When $\theta=1$, equation (58) becomes;

$$(a+bQ(p))Q'(p)=\frac{p}{\theta p(1-p)} \quad (60)$$

$$\theta p(1-p)(a+bQ(p))Q'(p)-p=0 \quad (61)$$

To obtain a much simpler ODE, differentiate equation (57);

$$aQ''(p)+bQ'^2(p)+bQ(p)Q''(p) =\frac{(p^{\frac{1}{\theta}-1})^2}{\theta^2(1-p^{\frac{1}{\theta}})^2}+\frac{(1-\theta)p^{\frac{1}{\theta}-1}}{\theta^2(1-p^{\frac{1}{\theta}})} \quad (62)$$

The equation can only exists for $a-b\neq 0, \theta>0, 0<p<1$.

Substitute equation (57) into equation (62);

$$(a+bQ(p))Q''(p)+bQ'^2(p) =(a+bQ(p))^2Q'^2(p)+\frac{1-\theta}{\theta p}(a+bQ(p))Q'(p) \quad (63)$$

$$(a+bQ(0))Q'(0)=0\Rightarrow Q'(0) \quad (64)$$

Different cases are considered;

When $\theta=1$, equation (63) becomes;

$$(a+bQ(p))Q''(p)-(a-b+bQ(p))^2Q'^2(p)=0 \quad (65)$$

When $b=0$ & $a>0$, equation (60) becomes;

$$aQ''(p)=a^2Q'^2(p)+\frac{1-\theta}{\theta p}aQ'(p) \quad (66)$$

$$\theta pQ''(p)-a\theta pQ'^2(p)+(\theta-1)Q'(p)=0 \quad (67)$$

When $a=0$ & $\mathrm{b}>0$, equation (63) becomes;

$$(bQ(p))Q''(p)+bQ'^2(p) =(bQ(p))^2Q'^2(p)+\frac{1-\theta}{\theta p}(bQ(p))Q'(p) \quad (68)$$

$$\theta pQ(p)Q''(p)+\theta p(1-bQ^2(p))Q'^2(p) +(\theta-1)Q(p)Q'(p)=0 \quad (69)$$

C Survival Function

The SF of the generalized Linear failure rate distribution is given as;

$$S(t)=1-(1-\mathrm{e}^{-(at+\frac{b}{2}t^2)})^{\theta} \quad (70)$$

Differentiate equation (70) to obtain;

$$S'(t)=-\theta(a+bt)(1-\mathrm{e}^{-(at+\frac{b}{2}t^2)})^{\theta-1}\mathrm{e}^{-(at+\frac{b}{2}t^2)} \quad (71)$$

The equation can only exists for $a-b\neq 0, t, \theta>0$.

These equations derived from equation (70) are required in further simplification of equation (71);

$$(1-\mathrm{e}^{-(at+\frac{b}{2}t^2)})^{\theta}=1-S(t) \quad (72)$$

$$1-\mathrm{e}^{-(at+\frac{b}{2}t^2)}=(1-S(t))^{\frac{1}{\theta}} \quad (73)$$

$$\mathrm{e}^{-(at+\frac{b}{2}t^2)}=1-(1-S(t))^{\frac{1}{\theta}} \quad (74)$$

However, equation (71) can be written as;

$$S'(t)=-\frac{\theta(a+bt)(1-\mathrm{e}^{-(at+\frac{b}{2}t^2)})^{\theta}\mathrm{e}^{-(at+\frac{b}{2}t^2)}}{1-\mathrm{e}^{-(at+\frac{b}{2}t^2)}} \quad (75)$$

Substitute equations (72)-(74) into equation (75);

$$S'(t)=-\frac{\theta(a+bt)(1-S(t))(1-(1-S(t))^{\frac{1}{\theta}})}{(1-S(t))^{\frac{1}{\theta}}} \quad (76)$$

$$S(1)=1-(1-\mathrm{e}^{-(a+\frac{b}{2})})^{\theta} \quad (77)$$

The ODEs can be derived for any given values of a, p and θ.

When $\theta=1$, equation (73) becomes;

$$S'(t)+(a+bt)S(t)=0 \quad (78)$$

D Inverse Survival Function

The ISF of the generalized Linear failure rate distribution is given as;

$$aQ(p)+\frac{b}{2}Q^2(p)=-\ln(1-(1-p)^{\frac{1}{\theta}}) \quad (79)$$

Differentiate equation (79) to obtain;

$$aQ'(p)+bQ(p)Q'(p)=-\frac{(1-p)^{\frac{1}{\theta}-1}}{\theta(1-(1-p)^{\frac{1}{\theta}})} \quad (80)$$

$$\theta(1-p)(a+bQ(p))Q'(p)=-\frac{(1-p)^{\frac{1}{\theta}}}{(1-(1-p)^{\frac{1}{\theta}})} \quad (81)$$

The equation can only exists for $a-b\neq 0, \theta>0, 0<p<1$.

The ODEs can be derived for any given values of a, p and θ.

When $\theta=1$, equation (81) becomes;

$$p(a+bQ(p))Q'(p)+1=0 \quad (82)$$

When $\theta=2$, equation (81) becomes;

$$2(1-\sqrt{1-p})(1-p)(a+bQ(p))Q'(p)+\sqrt{1-p}=0 \quad (83)$$

E Hazard Function

The HF of the generalized Linear failure rate distribution is given as;

$$h(t)=\frac{\theta(a+bt)(1-\mathrm{e}^{-(at+\frac{b}{2}t^2)})^{\theta-1}\mathrm{e}^{-(at+\frac{b}{2}t^2)}}{1-(1-\mathrm{e}^{-(at+\frac{b}{2}t^2)})^{\theta}} \quad (84)$$

Differentiate equation (84) to obtain;

$$h'(t)=\left\{\begin{array}{l}\dfrac{b}{a+bt}+\dfrac{(\theta-1)(a+bt)\mathrm{e}^{-(at+\frac{bt^2}{2})}}{1-\mathrm{e}^{-(at+\frac{bt^2}{2})}} \\ -(a+bt)+h(t)\end{array}\right\}h(t) \quad (85)$$

The equation can only exists for $a-b\neq 0, t, \theta>0$.

Differentiate equation (85) and using the results obtained from the PDF. This is easily done by the modification of equation (51);

$$h''(t)=\frac{h'^2(t)}{h(t)}+\left[\begin{array}{l}\frac{b}{a+bt}\left(\frac{h'(t)}{h(t)}+a+bt-\frac{b}{a+bt}-h(t)\right)\\-\frac{1}{\theta-1}\left(\frac{h'(t)}{h(t)}+a+bt-\frac{b}{a+bt}-h(t)\right)^2\\-\begin{array}{l}a+bt\left(\frac{h'(t)}{h(t)}+a+bt-\frac{b}{a+bt}-h(t)\right)\\-\frac{b^2}{(a+bt)^2}-b+h'(t)\end{array}\end{array}\right]h(t) \tag{86}$$

The equation can only exists for $a-b\neq 0,t,\theta>0$.

$$h(1)=\frac{\theta(a+b)(1-\mathrm{e}^{-(a+\frac{b}{2})})^{\theta-1}\mathrm{e}^{-(a+\frac{b}{2})}}{1-(1-\mathrm{e}^{-(a+\frac{b}{2})})^{\theta}} \tag{87}$$

$$h'(1)=\left\{\begin{array}{l}\frac{b}{a+b}+\frac{(\theta-1)(a+b)\mathrm{e}^{-(a+\frac{b}{2})}}{1-\mathrm{e}^{-(a+\frac{b}{2})}}\\-(a+bt)+h(1)\end{array}\right\}h(1) \tag{88}$$

Different cases are considered;

When $b=0$ & $a>0$, equation (86) becomes;

$$h''(t)=\frac{h'^2(t)}{h(t)}-\left[\frac{1}{\theta-1}\left(\frac{h'(t)}{h(t)}+a-h(t)\right)^2 +a\left(\frac{h'(t)}{h(t)}+a-h(t)\right)-h'(t)\right]h(t) \tag{89}$$

When $a=0$ & $\mathrm{b}>0$, equation (86) becomes;

$$h''(t)=\frac{h'^2(t)}{h(t)}+\left[\begin{array}{l}\frac{1}{t}\left(\frac{h'(t)}{h(t)}+bt-\frac{1}{t}-h(t)\right)\\-\frac{1}{\theta-1}\left(\frac{h'(t)}{h(t)}+bt-\frac{1}{t}-h(t)\right)^2\\-bt\left(\frac{h'(t)}{h(t)}+bt-\frac{1}{t}-h(t)\right)-\frac{1}{t^2}-b+h'(t)\end{array}\right]h(t) \tag{90}$$

F Reversed Hazard Function

The RHF of the generalized Linear failure rate distribution is given as;

$$j(t)=\frac{\theta(a+bt)\mathrm{e}^{-(at+\frac{b}{2}t^2)}}{1-\mathrm{e}^{-(at+\frac{b}{2}t^2)}} \tag{91}$$

Differentiate equation (91) to obtain;

$$j'(t)=\left\{\begin{array}{l}\frac{b}{a+bt}-\frac{(a+bt)\mathrm{e}^{-(at+\frac{bt^2}{2})}}{\mathrm{e}^{-(at+\frac{bt^2}{2})}}\\-\frac{(a+bt)\mathrm{e}^{-(at+\frac{bt^2}{2})}(1-\mathrm{e}^{-(at+\frac{bt^2}{2})})^{-2}}{(1-\mathrm{e}^{-(at+\frac{bt^2}{2})})^{-1}}\end{array}\right\}j(t) \tag{92}$$

$$j'(t)=\left\{\frac{b}{a+bt}-(a+bt)-\frac{(a+bt)\mathrm{e}^{-(at+\frac{bt^2}{2})}}{(1-\mathrm{e}^{-(at+\frac{bt^2}{2})})}\right\}j(t) \tag{93}$$

The equation can only exists for $a-b\neq 0,t,\theta>0$.

$$j'(t)=\left\{\frac{b}{a+bt}-(a+bt)-\frac{j(t)}{\theta}\right\}j(t) \tag{94}$$

The first order ODE for the RHF of the generalized Linear failure rate distribution is given by;

$$\begin{array}{l}\theta(a+bt)\,j'(t)+\theta((a+bt)^2-b)\,j(t)\\-(a+bt)\,j^2(t)=0\end{array} \tag{95}$$

$$j(1)=\frac{\theta(a+b)\mathrm{e}^{-(a+\frac{b}{2})}}{1-\mathrm{e}^{-(a+\frac{b}{2})}}=\frac{\theta(a+b)}{\mathrm{e}^{(a+\frac{b}{2})}-1} \tag{96}$$

IV. Concluding Remarks

Differentiation and modified product rule were used to obtain the ordinary differential equations (ODES) of different orders for the probability functions of linear failure rate and generalized linear failure rate distributions. This was largely due to differentiability of the probability functions. Every changes in the parameters result to a unique ODE. Overall, the ODEs are in consistent with the support and parameter domains that characterize the distributions. In addition, several research methods can be used to derive the solutions of the ODEs [50-67]. This method of characterizing distributions cannot be applied to distributions whose PDF or CDF are either not differentiable or the domain of the support of the distribution contains singular points.

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