A Perishable Production Inventory System with Service Time and Its Performance Evaluation

Yaling Qin and Dequan Yue

***Abstract*—**This paper studied a perishable inventory system with a service facility, which is a kind of production inventory system that require a certain service time before customers receive the specified goods. The continuous review (*s*, *S*) production inventory policy was adopted for the system. It means that productive facilities start to produce goods, while the inventory level decreases to s; otherwise, the productive facilities will be stopped to produce goods, while the inventory level achieves to S. The customers arrive according to a Poisson process. All arriving customers during stockout are lost. The life time of the item, production time and service time are assumed to have independent exponential distributions. The stationary joint distribution of the queue length and the on-hand inventory is obtained. Various system performance measures are derived and the total expected cost is calculated. The impact of different parameters to the system performance measures and the total expected cost are illustrated numerically.

***Keywords*—**production inventory system, (*s*, *S*) policy, positive service time, perishable products

I. Introduction

FOR a classical model of inventory management, the arriving demand will be immediately satisfied while the storehouse exists sufficient stock; otherwise, the customers have to wait or leave dependent on themselves willing while the storehouse is out of stock. In this case, a single item inventory will be actually between the two processes of supply and demand. However, a certain time is necessary before the customers receive the inventory items in a real production inventory system, and the certain time can be called as a service time. That is in such system, satisfying each demand needs not only an on-hand inventory item but also some service time. Following the definition, assignment, packing, and delivery times can be regarded as a service time [1].

Sigman and Simchi-Levi [2] investigated an M/G/1 queueing-inventory system. They developed a light traffic systems with positive service time. Berman and Kim [3] and heuristic approximation procedure for finding performance measures of the system. This is the first work on inventory Berman and Kim [4] addressed the optimal inventory control in a supply chain with a service facility, and found the optimal replenishment policy to maximize profit by using a Markov decision process.

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Krishnamoorthy and his colleagues firstly studied a production inventory system considering positive service time [5]. A continuous review (*s*, *S*) production inventory policy was adopted, to decide the starting and stopping time points of producing facilities through assessment of inventory levels. Their study deduced the steady state probability vector in an explicit product form by application of matrix analysis method. A cost function in s and S was constructed to numerically investigate their optimal values. Baek and Moon [6] studied a production inventory system with service time, where the stocks were replenished either by an external order under a (*r*, *Q*) policy, or by an internal production. They derived the stationary joint distribution of the queue length and the on-hand inventory in product form. By comparison of the results in the reference [7], it was concluded that the inventory process of the system with internal production is more stable than the model without the internal production. Afterwards, Baek and Moon [8] applied same method to extend its study from single server to multiple servers. They proved the independence of the inventory level process and the queue length process and derived the explicit stationary joint probability in product form. The cost function was developed and investigated numerical examples.

However, the above studies lose their sights into perishable character of the items of inventory. Krishnamoorthy et al. [9] studied a perishable inventory system with service time by application of N service policy. It assumed that the server begin to provide services only while the amount of the waiting customers achieves N numbers and the demands will lost during the stock out periods. The items of inventory have exponential life time. They obtained the joint probability distribution of the number of customers and the inventory level in the steady state case. Some performance measures of the system were derived. Manuel et al. [10] analyzed a perishable inventory system with a service facility. They assumed that the arrival processes of positive and negative customers form Markovian arrival processes, the service time has phase-type distribution, the lead time and the life time of the item have exponential distribution. Manuel et al [11] extended the model with retrial customer at service facility. Shophia Lawrence et al. [12] considered a case in which the demands are generated by a finite homogeneous population. The service time and the lead time are followed phase-type distribution, the life time of the item has exponential distribution. A numerical example was used to investigate the changing situations of mean waiting times, while the service time and the lead time are followed the three special cases of exponential, Erlang and hyper exponential distributions, respectively. Hamadi et al. [13] studied a perishable inventory control problem at a service facility with impatient customers where customers arrive at service facility according to a Poisson process. The lead time, the service time, the life time of item and the reneging time have exponential distribution. The optimal service rate that minimizes the long run expected cost rate has been derived.

To the best of our knowledge, no research has been found on the production- inventory system considering both positive service time and perishable products. In this paper we have considered a perishable production-inventory system with positive service time.

Related to the paragraph structure, the second part is organized to describe research problem and establish corresponding systematic model. The third part is for theoretical analyses and its solution to deduce stationary joint probability distribution for the inventory level and the number of customers, and further calculate system performance measures to establish system cost function. In the fourth part, numerical analyses are applied to evaluate influences of different system parameters on the performance measures; moreover, a proposed algorithm was demonstrated to obtain optimal inventory control variables. Additionally, a comprehensive conclusion was given in the final part.

II. Model Description

Considering a production inventory system while the inventory goods is perishable, the following hypotheses are defined for the studying topic.

Customers arrive in the system according to a Poisson process with parameter λ, and the customers come to the system as a queue by a coming order.

The server serves the arrived customers one by one under a First-Come, First-Served (FCFS) discipline. The service times are exponentially distributed with parameter μ. Each customer leaves the system with one item from the inventory at his service completion epoch.

The system has a single production facility and the production times are exponentially distributed with parameter η. The life time of each item has exponential distribution with parameter τ.

An (s, S) production inventory policy is adopted. When the inventory level becomes S, the production facility is turned off; and when the inventory level depletes to s, the production facility is immediately turned on, and is kept in the on model until the inventory level becomes S.

During the period when the inventory level is zero, all new arriving customers are rejected and lost.

III. Model Analysis

Let $N(t)$ be the number of customers in the system at time t, $I(t)$ the on-hand inventory level at time t, and $J(t)$ the status of the production process at time t,which is defined as,

$$J(t)=\begin{cases}1, & \text{if the production facility is in on-state at time } t,\\ 0, & \text{if the production facility is in off-state at time } t.\end{cases}$$

Then the process $\{X(t), t\geq 0\}=\{N(t), I(t), J(t), t\geq 0\}$ is a continuous-time Markov process with the state space

$$\Omega=\{(i,j,1), i\geq 0, 0\leq j\leq S-1\}\cup\{(i,j,0), i\geq 0, s+1\leq j\leq S\}$$

The infinitesimal generator of the process $\{X(t),t\geq 0\}$ is as follows

$$\mathbf{Q}=\begin{pmatrix}\mathbf{Q}_0 & \mathbf{Q}_3 & & & \\ \mathbf{Q}_1 & \mathbf{Q}_2 & \mathbf{Q}_3 & & \\ & \mathbf{Q}_1 & \mathbf{Q}_2 & \mathbf{Q}_3 & \\ & & \ddots & \ddots & \ddots\end{pmatrix} \quad (1)$$

where $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$ are all $2S-s$ matrices which are given as follows

$$\mathbf{Q_0}=\begin{pmatrix}-\eta & \eta & & & & & & & & & & \\ \mathbf{C}_1 & \mathbf{A}_1 & \eta & & & & & & & & & \\ & \mathbf{C}_2 & \mathbf{A}_2 & \eta & & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & & \\ & & & \mathbf{C}_{s-1} & \mathbf{A}_{s-1} & \eta & & & & & & \\ & & & & \mathbf{C}_s & \mathbf{A}_s & \mathbf{B}_1 & & & & & \\ & & & & & \mathbf{C}_{s+1} & \mathbf{A}_{s+1} & \mathbf{B}_2 & & & & \\ & & & & & & \mathbf{C}_{s+2} & \mathbf{A}_{s+2} & \mathbf{B}_2 & & & \\ & & & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & & & \mathbf{C}_{S-2} & \mathbf{A}_{S-2} & \mathbf{B}_2 & \\ & & & & & & & & & \mathbf{C}_{S-1} & \mathbf{A}_{S-1} & \mathbf{B}_3 \\ & & & & & & & & & & \mathbf{C}_S & \mathbf{A}_S\end{pmatrix} \quad (2)$$

where

$$\mathbf{A}_i=\left(-(\lambda+\eta+i\tau)\right), i=1,\cdots,s,$$

$$\mathbf{A}_i = \begin{pmatrix} -(\lambda + i\tau) & 0 \\ 0 & -(\lambda+\eta+i\tau) \end{pmatrix}, i = s+1, \cdots, S-1$$

$$\mathbf{A}_S = (-(\lambda + S\tau)),$$

$$\mathbf{B}_1 = (0 \quad \eta), \ \mathbf{B}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \eta \end{pmatrix}, \ \mathbf{B}_3 = \begin{pmatrix} 0 \\ \eta \end{pmatrix},$$

$$\mathbf{C}_S = (S\tau \quad 0),$$

$$\mathbf{C}_{s+1} = \begin{pmatrix} (s+1)\tau \\ (s+1)\tau \end{pmatrix},$$

$$\mathbf{C}_i = \begin{pmatrix} i\tau & 0 \\ 0 & i\tau \end{pmatrix}, i = s+2, \cdots, S-1$$

$$\mathbf{Q}_1 = \begin{pmatrix} 0 & & & & & & & & \\ \mu & 0 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & \mu & 0 & & & & & \\ & & & \mathbf{F}_1 & 0 & & & & \\ & & & & \mathbf{F}_2 & 0 & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & \mathbf{F}_2 & 0 & \\ & & & & & & & \mathbf{F}_3 & 0 \end{pmatrix}, \quad (3)$$

where

$$\mathbf{F}_1 = \begin{pmatrix} \mu \\ \mu \end{pmatrix}, \ \mathbf{F}_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}, \ \mathbf{F}_3 = (\mu \quad 0),$$

$$\mathbf{Q}_3 = \begin{pmatrix} 0 & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix}, \quad (4)$$

$$\mathbf{Q}_2 = \mathbf{Q}_0 - \frac{\mu}{\lambda}\mathbf{Q}_3 \cdot \quad (5)$$

A. Stationary Condition

Theorem 1: The process $\{X(t), t \geq 0\}$ with the infinitesimal generator $\mathbf{Q}$ is positive recurrent if and only if

$$\lambda < \mu \quad (6)$$

Proof: Let a matrix $\mathbf{H} = \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3$, which is given by

$$\mathbf{H} = \begin{pmatrix} -\eta & \eta & & & & & & & & & & \\ \mathbf{L}_1 & \mathbf{H}_1 & \eta & & & & & & & & & \\ & \mathbf{L}_2 & \mathbf{H}_2 & \eta & & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & & \\ & & & \mathbf{L}_{s-1} & \mathbf{H}_{s-1} & \eta & & & & & & \\ & & & & \mathbf{L}_s & \mathbf{H}_s & \mathbf{B}_1 & & & & & \\ & & & & & \mathbf{L}_{s+1} & \mathbf{H}_{s+1} & \mathbf{B}_2 & & & & \\ & & & & & & \mathbf{L}_{s+2} & \mathbf{H}_{s+2} & \mathbf{B}_2 & & & \\ & & & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & & & \mathbf{L}_{S-2} & \mathbf{H}_{S-2} & \mathbf{B}_2 & \\ & & & & & & & & & \mathbf{L}_{S-1} & \mathbf{H}_{S-1} & \mathbf{B}_3 \\ & & & & & & & & & & \mathbf{L}_S & \mathbf{H}_S \end{pmatrix} \quad (7)$$

where

$$\mathbf{H}_i = (-(\eta + \mu + i\tau)), i = 1, \cdots, s,$$

$$\mathbf{H}_i = \begin{pmatrix} -(\mu + i\tau) & 0 \\ 0 & -(\eta+\mu+i\tau) \end{pmatrix}, i = s+1, \cdots, S-1,$$

$$\mathbf{H}_S = (-(\mu + S\tau)),$$

$$\mathbf{L}_i = (\mu + i\tau), i = 1, \cdots, s,$$

$$\mathbf{L}_{s+1} = \mathbf{F}_1 + \mathbf{C}_{s+1},$$

$$\mathbf{L}_i = \mathbf{F}_2 + \mathbf{C}_i, i = s+2, \cdots, S-1,$$

$$\mathbf{L}_S = \mathbf{F}_3 + \mathbf{C}_S.$$

Let $\boldsymbol{\alpha} = (\alpha(0,1), \cdots, \alpha(s,1), \alpha(s+1,0), \alpha(s+1,1), \cdots, \alpha(S-1,0), \alpha(S-1,1), \alpha(S,0))$ be the steady state probability vector of the generator $\mathbf{H}$. Then the vectors $\boldsymbol{\alpha}$ should satisfy equations:

$$\boldsymbol{\alpha}\mathbf{H} = 0, \ \boldsymbol{\alpha}\mathbf{e} = 1 \quad (8)$$

where $\mathbf{e}$ is a column vector of 1's of appropriate dimension. Based on Neuts [14], the stochastic process $\{X(t), t \geq 0\}$ with the infinitesimal generator $\mathbf{Q}$ is positive recurrent if and only if

$$\boldsymbol{\alpha}\mathbf{Q}_3\mathbf{e} < \boldsymbol{\alpha}\mathbf{Q}_1\mathbf{e}, \quad (9)$$

the following equations can be derived by calculation of $\boldsymbol{\alpha}\mathbf{Q}_3\mathbf{e}$ and $\boldsymbol{\alpha}\mathbf{Q}_1\mathbf{e}$,

$$\boldsymbol{\alpha}\mathbf{Q}_3\mathbf{e} = \lambda\left(\sum_{i=1}^{s}\alpha(i,1) + \sum_{i=s+1}^{S-1}(\alpha(i,0) + \alpha(i,1)) + \alpha(S,0)\right) = \lambda\alpha(0,1) \quad (10)$$

$$\boldsymbol{\alpha}\mathbf{Q}_1\mathbf{e} = \mu\left(\sum_{i=1}^{s}\alpha(i,1) + \sum_{i=s+1}^{S-1}(\alpha(i,0) + \alpha(i,1)) + \alpha(S,0)\right) = \mu\alpha(0,1) \quad (11)$$

Thereby, if and only if $\lambda < \mu$, the process $\{X(t), t \geq 0\}$ is positive recurrence, and the system is stable. The proof is completed.

B. Steady State Distribution

A perishable production inventory system neglecting service time is established. Before solving the steady probability distribution of process $\{X(t), t \geq 0\}$, it is necessary in the beginning to consider a perishable production inventory system where the service of customers is instantaneous and no backlogs are allowed when the inventory level depletes to zero. Thereby the waiting customers do not exist in this production inventory system, and the coresponding Markov process can be defined as $\{\hat{X}(t), t \geq 0\} = \{I(t), J(t), t \geq 0\}$,where $I(t)$ and $J(t)$ have the same definition as process $\{X(t), t \geq 0\}$. The state space of the process $\{\hat{X}(t), t \geq 0\}$ is given as,

$$\hat{\Omega} = \{(j,1) : 0 \leq j \leq s\} \cup \{(j,k) : s+1 \leq j \leq S-1; k = 0,1\} \cup \{S,0\}.$$

and its infinitesimal generator is given by

$$\hat{\mathbf{Q}} = \begin{pmatrix} -\eta & \eta & & & & & & & & & \\ \hat{\mathbf{C}}_1 & \mathbf{A}_1 & \eta & & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & & \\ & & \hat{\mathbf{C}}_{s-1} & \mathbf{A}_{s-1} & \eta & & & & & & \\ & & & \hat{\mathbf{C}}_s & \mathbf{A}_s & \mathbf{B}_1 & & & & & \\ & & & & \hat{\mathbf{C}}_{s+1} & \mathbf{A}_{s+1} & \mathbf{B}_2 & & & & \\ & & & & & \hat{\mathbf{C}}_{s+2} & \mathbf{A}_{s+2} & \mathbf{B}_2 & & & \\ & & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & & \hat{\mathbf{C}}_{S-2} & \mathbf{A}_{S-2} & \mathbf{B}_2 & \\ & & & & & & & & \hat{\mathbf{C}}_{S-1} & \mathbf{A}_{S-1} & \mathbf{B}_3 \\ & & & & & & & & & \hat{\mathbf{C}}_S & \mathbf{A}_S \end{pmatrix} \quad (12)$$

where

$$\hat{\mathbf{C}}_i = \frac{\lambda + i\tau}{\mu}\mathbf{F}_1, i = 1, \cdots, s,$$

$$\hat{\mathbf{C}}_i = \frac{\lambda + i\tau}{\mu}\mathbf{F}_2, i = s+1, \cdots, S-1,$$

$$\hat{\mathbf{C}}_S = \frac{\lambda + S\tau}{\mu}\mathbf{F}_3,$$

and all other submatrices are as defined previously in matrix $\mathbf{Q}_0$, and $\hat{\mathbf{Q}}$ is a $(2S-s) \times (2S-s)$ matrix.

Let $\boldsymbol{\pi} = (\pi(0,1), \pi(1,1), \ldots, \pi(s,1), \pi(s+1,0), \pi(s+1,1), \ldots, \pi(S-1,0), \pi(S-1,1), \pi(S,0))$ be the steady probability vector of the process $\{\hat{X}(t), t \geq 0\}$, and then the vector $\boldsymbol{\pi}$ satisfies the following equations

$$\begin{cases} \boldsymbol{\pi}\hat{\mathbf{Q}} = \mathbf{0} \\ \boldsymbol{\pi}\mathbf{e} = 1 \end{cases} \quad (13)$$

where $\mathbf{e}$ is a column vector of 1's of appropriate dimension.

The equation (13) can be rewritten as follows

$$\eta\pi(0,1) - (\lambda + \tau)\pi(1,1) = 0, \quad (14)$$

$$\eta\pi(i-1,1) - (\lambda + \eta + i\tau)\pi(i,1) + [\lambda + (i+1)\tau]\pi(i+1,1) = 0, 1 \leq i \leq S-2 \quad (15)$$

$$\eta\pi(s-1,1) - (\lambda + \eta + s\tau)\pi(s,1) + [\lambda + (s+1)\tau][\pi(s+1,0) + \pi(s+1,1)] = 0 \quad (16)$$

$$\eta\pi(S-2,1) - [\lambda + \eta + (S-1)\tau]\pi(S-1,1) = 0 \quad (17)$$

$$(\lambda + i\tau)\pi(i,0) - [\lambda + (i+1)\tau]\pi(i+1,0) = 0, s+1 \leq i \leq S-1 \quad (18)$$

$$\eta\pi(S-1,1) - (\lambda + S\tau)\pi(S,0) = 0 \quad (19)$$

$$\sum_{i=0}^{S-1}\pi(i,1) + \sum_{i=s+1}^{S}\pi(i,0) = 1 \quad (20)$$

To solve the equation set (14) - (19), and obtain the solutions as follows

$$\pi(i,0) = \frac{\lambda + S\tau}{\lambda + i\tau}\pi(S,0), s+1 \leq i \leq S-1 \quad (21)$$

$$\pi(i,1) = \frac{\lambda + S\tau}{\eta}(1 + v_i)\pi(S,0), s \leq i \leq S-1 \quad (22)$$

$$\pi(i,1) = \frac{\lambda + S\tau}{\eta}(1 + v_s)w_i\pi(S,0), 0 \leq i \leq s-1 \quad (23)$$

where

$$v_i = \sum_{j=1}^{S-i-1}\prod_{k=j}^{S-i-1}\frac{\lambda + (S-k)\tau}{\eta}, s \leq i \leq S-1 \quad (24)$$

$$w_i = \prod_{k=i+1}^{s}\frac{\lambda + k\tau}{\eta}, 0 \leq i \leq s-1 \quad (25)$$

By the normalizing equation (20), the following result can be deduced as,

$$\pi(S,0) = \frac{\dfrac{\eta}{\lambda + S\tau}}{1 + S - s + \sum_{i=s}^{S-1}v_i + (1 + v_s)\sum_{i=0}^{s-1}w_i + \sum_{i=s+1}^{S-1}\dfrac{\eta}{\lambda + i\tau}} \quad (26)$$

A perishable production inventory system considering service time is established. Let $\mathbf{P} = (\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \cdots)$ be the steady state probability vector of the process $\{X(t), t \geq 0\}$, where $\mathbf{P}_i$ is a row vector of order $2S-s$, and $\mathbf{P}$ satisfies the following equations

$$\mathbf{PQ} = \mathbf{0}, \mathbf{Pe} = 1 \quad (27)$$

where $\mathbf{e}$ is a column vector of 1's of appropriate dimension.

Theorem 2: If $\lambda < \mu$, the steady state probability distribution of the perishable production inventory system with positive service time is given by

$$\mathbf{P}_i = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^i \boldsymbol{\pi},\ i \geq 0, \tag{28}$$

where vector $\boldsymbol{\pi}$ is derived from the equations (21) - (26).

Proof: When $\mathbf{PQ} = \mathbf{0}$ from the equation (27), the following equations can be derived as,

$$\mathbf{P}_0\mathbf{Q}_0 + \mathbf{P}_1\mathbf{Q}_1 = \mathbf{0}, \tag{29}$$

$$\mathbf{P}_i\mathbf{Q}_3 + \mathbf{P}_{i+1}\mathbf{Q}_2 + \mathbf{P}_{i+2}\mathbf{Q}_1 = \mathbf{0},\ i \geq 0. \tag{30}$$

Let

$$\mathbf{P}_i = \xi\left(\frac{\lambda}{\mu}\right)^i \boldsymbol{\pi},\ i \geq 0, \tag{31}$$

where ξ is a constant. Now it is to validate the equation (28) satisfying the equations (29) and (30).

Substituting equation (28) into the left sides of equations (29) and (30), we have

$$\mathbf{P}_0\mathbf{Q}_0 + \mathbf{P}_1\mathbf{Q}_1 = \xi\boldsymbol{\pi}\left(\mathbf{Q}_0 + \frac{\lambda}{\mu}\mathbf{Q}_1\right), \tag{32}$$

$$\begin{aligned}\mathbf{P}_i\mathbf{Q}_3 + \mathbf{P}_{i+1}\mathbf{Q}_2 + \mathbf{P}_{i+2}\mathbf{Q}_1 &= \xi\left(\frac{\lambda}{\mu}\right)^i \boldsymbol{\pi}\left(\mathbf{Q}_3 + \left(\frac{\lambda}{\mu}\right)\mathbf{Q}_2 + \left(\frac{\lambda}{\mu}\right)^2\mathbf{Q}_1\right) \\ &= \xi\left(\frac{\lambda}{\mu}\right)^i \boldsymbol{\pi}\left(\mathbf{Q}_3 + \left(\frac{\lambda}{\mu}\right)\left(\mathbf{Q}_0 - \frac{\mu}{\lambda}\mathbf{Q}_3\right) + \left(\frac{\lambda}{\mu}\right)^2\mathbf{Q}_1\right) \\ &= \xi\left(\frac{\lambda}{\mu}\right)^{i+1} \boldsymbol{\pi}\left(\mathbf{Q}_0 + \frac{\lambda}{\mu}\mathbf{Q}_1\right)\end{aligned} \tag{33}$$

From the structure of the matrices $\mathbf{Q}_0$, $\mathbf{Q}_1$, and $\hat{\mathbf{Q}}$, it is easy to verify that

$$\mathbf{Q}_0 + \frac{\lambda}{\mu}\mathbf{Q}_1 = \hat{\mathbf{Q}}$$

Moreover, due to $\boldsymbol{\pi}\hat{\mathbf{Q}} = 0$ from equation (13), the left sides of equations (29) and (30) are equal to zero. Thereby, $\mathbf{P}_i = \xi\left(\frac{\lambda}{\mu}\right)^i \boldsymbol{\pi}, i \geq 0$ can satisfy the equations (29) and (30). Afterwards, further considering normalizing conditions $\mathbf{Pe} = 1$, the following results can be derived as,

$$\xi\left(1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + \cdots\right) = 1 \tag{34}$$

Therefore, under the condition of $\lambda < \mu$, $\xi = 1 - \frac{\lambda}{\mu}$. The proof is finished.

Remark: Theorem 2 shows that the steady state probability distribution of the system can be decomposed as two independent parts: one part is the distribution of the queue length in the classical M/M/1 queue system, and another part is the distribution of the inventory level in the perishable production inventory system without considering the service time.

C. System Performance Measures

Expected number of customers in the system is

$$E_N = \sum_{i=0}^{\infty} i\mathbf{P}_i\,\mathbf{e} = \frac{\lambda}{\mu - \lambda}\cdot \tag{35}$$

Expected inventory level in the system is

$$\begin{aligned}E_{inv} &= \sum_{i=0}^{S-1} i\pi(i,1) + \sum_{i=s+1}^{S-1} i\pi(i,0) + S\pi(S,0) \\ &= \frac{\lambda + S\tau}{\eta}\pi(S,0)\left(\frac{S\eta}{\lambda + S\tau} + \frac{(S+s-1)(S-s)}{2}\right. \\ &\quad \left. + (1+v_s)\sum_{i=0}^{s-1} iw_i + \sum_{i=s}^{S-1} iv_i + \sum_{i=s+1}^{S-1}\frac{i\eta}{\lambda + i\tau}\right)\end{aligned} \tag{36}$$

Expected production rate is

$$\begin{aligned}E_{pr} &= \eta\sum_{i=0}^{S-1}\pi(i,1) \\ &= (\lambda + S\tau)\pi(S,0)\left(S - s + (1+v_s)\sum_{i=0}^{s-1} w_i + \sum_{i=s}^{S-1} v_i\right)\cdot\end{aligned} \tag{37}$$

Expected customer loss rate while the inventory level is zero

$$E_{loss} = \lambda\pi(0,1) = \frac{\lambda + S\tau}{\eta}(1+v_s)w_0\pi(S,0) \tag{38}$$

Expected rate at which production process is switched on

$$E_{on} = \mu\left(\sum_{i=1}^{\infty}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^i \pi(s+1,0)\right) = \frac{\lambda(\lambda+S\tau)}{\lambda+(s+1)\tau}\pi(S,0) \tag{39}$$

Expected perishable rate is

$$E_{de} = \sum_{i=1}^{S-1} i\tau\pi(i,1) + \sum_{i=s+1}^{S} i\tau\pi(i,0) = \tau E_{inv}\,. \tag{40}$$

IV. Optimal Control Strategy

A. Cost Function

On the basis of the derived system steady state performance measures, the expected total cost per unit time for this model is defined to be

$$F(s,S) = C_h E_{inv} + C_p E_{pr} + C_l E_{loss} + C_o E_{on} + C_w E_N + C_d E_{de} \tag{41}$$

where C_h is the inventory holding cost per unit item per unit time, C_p is the cost of production per unit item per unit time, C_l is the cost caused by a loss of a customer, C_o is the cost for starting the production, C_w is the waiting cost of a customer per unit time, C_d is the perishable cost per unit item per unit time.

Afterwards, as an objective of minimum the expected total system cost $F(s,S)$, it is to obtain optimal value of the production switching on inventory level s and the maximum inventory level S. Normally, it is always difficult to obtain the optimal by application of analytic method, because the expression complexity of system performance measures. However, in a real situation of inventory management, the maximum inventory level is always bounded, and it is impossible as a infinite value. Therefore, a searching program is developed to calculate optimal *s* and *S* values for the minimum expected total cost $F(s,S)$. The searching steps can be simply demonstrated as,

Step 1. To set the maximum *S* value as S_{max}, and maximum s value as S_{max}-1;

Step 2. To calculate corresponding *F* values of the cost function for each *S* and *s*, and all of the *F* values is preserved in the Smax*Smax-1 dimensional matrix.

Step 3. To search *s* and *S* values corresponding to the minimum factor in matrix *F*, and it is namely optimal inventory (s^*,S^*), by application of the developed searching program.

B. Numerical Analysis

The effect of system parameters on performance measures and cost function is analysed. Now, we conduct a numerical analysis to consider the effect of parameters λ, η and τ on some performance measures and the expected total coat. The cost parameters are given as: $C_h = 2$, $C_p = 15$, $C_l = 200$, $C_o = 400$, $C_s = 1$, and $C_d = 1.5$. The numerical results are shown in Table 1, Table 2 and Table 3 respectively.

As presented in Table 1, the expected production rate E_{pr}, the expected customer loss rate E_{loss} and the expected total cost $F(s,S)$ increased with the increase of the customer arrival rate λ. Inversely, the expected inventory level E_{inv}, the expected switched on rate of production process E_{on}, the expected perishable rate E_{de} decreased with the increase of the customer arrival rate λ.

As presented in Table 2, the expected inventory level E_{inv}, the expected production rate E_{pr}, the expected switched on rate of production process E_{on}, the expected perishable rate E_{de}, and the expected total cost $F(s,S)$ increased with the increase of the production rate η; inversely, only the expected customer loss rate E_{loss} decreased with the increase of the production rate η.

TABLE 1. The effect of the arrival rate λ on some performance measures and expected total cost. The other parameters are give as: $\eta = 5.0, \mu = 3.0, \tau = 0.6, s = 4$ and *S*=16.

λ	E_{inv}	E_{pr}	E_{loss}	E_{on}	E_{de}	$F(s,S)$
0.8	6.828	4.893	0.0036	0.0023	4.097	95.20
0.9	6.683	4.905	0.0045	0.0023	4.010	95.20
1.0	6.537	4.916	0.0057	0.0022	3.922	95.21
1.1	6.390	4.925	0.0070	0.0021	3.834	95.25
1.2	6.242	4.934	0.0086	0.0020	3.745	95.31
1.3	6.093	4.942	0.0104	0.0019	3.656	95.41
1.4	5.945	4.949	0.0125	0.0018	3.567	95.56
1.5	5.796	4.955	0.0148	0.0017	3.478	95.77
1.6	5.648	4.960	0.0174	0.0016	3.389	96.05
1.7	5.501	4.965	0.0204	0.0015	3.301	96.40

TABLE 2. The effect of the production rate η on some performance measures and expected total cost. The other parameters are give as: $\lambda = 1.5, \mu = 3.0, \tau = 0.6, s = 4$ and S=16.

η	E_{inv}	E_{pr}	E_{loss}	E_{on}	E_{de}	$F(s,S)$
5.0	5.796	4.955	0.0148	0.0017	3.478	95.77
5.2	6.088	5.134	0.0118	0.0024	3.653	98.98
5.4	6.370	5.307	0.0094	0.0033	3.822	102.26
5.6	6.641	5.471	0.0075	0.0044	3.984	105.57
5.8	6.898	5.628	0.0060	0.0057	4.139	108.88
6.0	7.140	5.774	0.0048	0.0072	4.284	112.16
6.2	7.366	5.911	0.0039	0.0090	4.420	115.39
6.4	7.576	6.038	0.0031	0.0109	4.546	118.53
6.6	7.769	6.154	0.0026	0.0131	4.661	121.58
6.8	7.945	6.260	0.0021	0.0154	4.767	124.52

As presented in Table 3, the expected production rate E_{pr}, the expected customer loss rate E_{loss}, the expected perishable rate E_{de}, and the expected total cost $F(s,S)$ increased with the increase of the perishable rate τ; inversely, the expected inventory level E_{inv}, and the expected switched on rate of production process E_{on} decreased with the increase of the perishable rate τ.

TABLE 3. The effect of the perishable rate τ on some performance measures and expected total cost. The other parameters are give as: $\lambda = 1.5, \eta = 5.0, \mu = 3.0, s = 4$ and S=16.

τ	E_{inv}	E_{pr}	E_{loss}	E_{on}	E_{de}	$F(s,S)$
0.1	9.668	2.486	0.0005	0.0807	0.967	91.47
0.2	9.207	3.342	0.0011	0.0577	1.841	95.58
0.3	8.650	4.088	0.0020	0.0330	2.595	97.11
0.4	7.784	4.604	0.0040	0.0146	3.114	96.95
0.5	6.747	4.859	0.0081	0.0053	3.374	96.17
0.6	5.796	4.955	0.0148	0.0017	3.478	95.77
0.7	5.031	4.986	0.0236	0.0005	3.521	96.06
0.8	4.433	4.995	0.0340	0.0001	3.546	96.98
0.9	3.963	4.998	0.0455	6.02×10^{-5}	3.567	98.37
1.0	3.586	4.999	0.0577	2.17×10^{-5}	3.586	100.10

The Effect of Perishable rate on Optimal Inventory Policy and Cost Function is analysed. By given other system parameters: $\lambda = 1.5$, $\eta = 5.0$, and $\mu = 3.0$, the cost parameters of the each system performance measures can be set as: $C_h = 2$, $C_p = 15$, $C_l = 200$, $C_o = 400$, $C_s = 1$, and $C_d = 1.5$.

Thereby, in Table 4, the values of optimal production inventory policy (s^*, S^*) and cost function $F(s^*, S^*)$ are estimated for the different values of perishable rate τ, under the precondition of the value setting of the maximum system inventory level as 50.

As observed in Table 4, the value s^* in the optimal production inventory policy decreases with the increase of the perishable rate τ; meanwhile, the value S^* decreases in the beginning, and then increases with the increase of the perishable rate τ. However, there is no significantly monotonic relation between the optimal cost function $F(s^*, S^*)$ and perishable rate τ.

TABLE 4. The effect of perishable rate τ on optimal policy and cost function.

τ	(s^*, S^*)	$F(s^*, S^*)$
0.1	(2,18)	85.59
0.2	(2,16)	92.63
0.3	(2,15)	95.51
0.4	(2,14)	96.14
0.5	(3,14)	95.86
0.6	(3,14)	95.68
0.7	(4,14)	96.04

V. CONCLUSIONS

This paper mainly discussed production inventory system with positive service time where the inventory items are perishable. The stationary joint probability distribution of the number of customers in the system and the inventory level was obtained by using the matrix technique. The important stable performance measures were deduced to establish system cost function, and a simple calculation algorithm was designed to estimate optimal inventory policy. The effect of system parameters on some performance measures was investigated using numerical examples. The monotonic behavior of the optimal policy and the optimal cost on the perishable rate was illustrated numerically.

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