Multi-criteria Decision Making Method With Interval Neutrosophic Setting Based On Minimum and Maximum Operators

Limin Su, Huishuang He and Hongwen Lu

Abstract—The interval neutrosophic set (INS) is a subclass of the neutrosophic set (NS) and a generalization of the interval valued intuitionistic fuzzy set (IVIFS), which can be used in engineering and scientific practice. This paper aims to present a new multi-criteria decision making method under interval neutrosophic setting. For this purpose, a comparative method between two interval numbers is firstly given. Then, some new similarity measures based on minimum and maximum operators with INSs are proposed. Thirdly, a multi-criteria decision making method with INSs is established. Finally, an example is used to illustrate the practicality and validity of the proposed decision making method.

Keywords—Interval neutrosophic set, minimum and maximum operator, similarity measure, multi-criteria decision making method.

I. Introduction

THE neutrosophic set introduced by Smarandache [1] generalizes fuzzy set [2], interval valued fuzzy set [3]-[4], intuitionistic fuzzy set [5], interval valued intuitionistic fuzzy set [6], vague set [7], grey set [8], etc. Subsequently, to solve realistic problems, Wang et al. [9]-[10] proposed single valued neutrosophic set (SVNS) and interval neutrosophic set (INS). At present, studies on the SVNSs and INSs have triumphantly penetrated into different fields [11]-[21]. Whereafter, several similarity measures have been proposed. Said and Smarandache [22] presented several similarity measures based on Hausdorff distance. Ye [23] proposed three vector similarity measure for SVNs. Ye and Ye [24] proposed the Dice similarity measure and the weighted Dice similarity measure for single valued neutrosophic multisets. Biswas et al. [25] introduced cosine similarity measure under neutrosophic environment. Ye and Zhang [26] further proposed the similarity measures based on minimum and maximum operators of SVNSs for decision making problems.

However, to the best of our knowledge, the existing literature does not deal with similarity measures between INSs and the decision-making problem under interval neutrosophic setting. One of the mainly difficulties is to compare two intervals. Therefore, we firstly give another representation of interval number, and a comparative approach between two interval numbers is further proposed. Then three new similarity measures for INSs, extended based on the similarity measure in Ye and Zhang [26], is presented. They are more suitable in real scientific and engineering applications. Thirdly, a multi-criteria decision making is established based on the proposed similarity measures. Rest of the paper is structured as follows. Section 2 introduces preliminaries about neutrosophic sets. Section 3 is devoted to present similarity measures based on minimum and maximum operators for interval neutrosophic sets and some of their properties. Section 4 describes decision making method based on the proposed similarity measure. Section 5 presents an application of the similarity measures to the problem namely, interval neutrosophic set decision making through an example. Finally, section 6 presents the conclusions of this paper.

This research has been supported by National Social Science Foundation of China. Project No: 16BJY026.

Limin Su is a Ph.D. student at School of Management and Economics in North China University of Water Resources and Electric Power, 450046, Zhengzhou, China. (Corresponding author's E-mail: suliminlove2010@163.com).

Huishuang He is a vice professor. She works at School of Management and Economics in North China University of Water Resources and Electric Power, 450046, Zhengzhou, China.

Hongwen Lu is a professor of School of Mathematical Science at Tongji University and visiting professor of College of Mathematics and Information Sciences at North China University of Water Resources and Electric Power in China.

II. Preliminaries

In this section, we give some basic concepts and definitions, interval numbers and INSs, and their operational laws are included. They will be utilized in the later analysis.

A. Interval Number

Interval number and their operations are of utmost significance when developing the operations of INSs. Some definitions and operational laws of interval numbers are introduced below.

Definition 1[27] Let $\tilde{x} = [x^L, x^U] = \{x | x^L \leq x \leq x^U\}$, and then $\tilde{x}$ is called an interval number. In particular, if $x^L = x^U$, then $\tilde{x} = [x^L, x^U]$ is a real number.

Interval number $\tilde{x}$ is alternatively represented as $\tilde{x} = \langle m(\tilde{x}), w(\tilde{x})\rangle$, where $m(\tilde{x}) = \frac{1}{2}(x^L + x^U)$, $w(\tilde{x}) = \frac{1}{2}(x^U - x^L)$.

Based on this, we following give a representation for interval number and a comparison between two interval numbers.

Definition 2 Let $\tilde{x} = [x^L, x^U] = \{x | x^L \leq x \leq x^U\}$ be an interval number, and then

$$\tilde{x} = m(\tilde{x}) + w(\tilde{x})i, \tag{1}$$

where $i \in [-1, 1]$, $m(\tilde{x}) = \frac{1}{2}(x^L + x^U)$, $w(\tilde{x}) = \frac{1}{2}(x^U - x^L)$.

Considering two non-negative interval numbers $\tilde{x} = [x^L, x^U]$ and $\tilde{y} = [y^L, y^U]$, where $0 \leq x^L \leq x \leq x^U$ and

$0 \leq y^L \leq y \leq y^U$, we have their comparison defined as follows:

(C1) If $m(\tilde{x}) \geq m(\tilde{y})$ and $w(\tilde{x}) \leq w(\tilde{y})$, then $\tilde{x}$ is greater than $\tilde{y}$, i.e., $\tilde{x} \geq \tilde{y}$;

(C2) If $m(\tilde{x}) \geq m(\tilde{y})$, then $\tilde{x}$ is quasi-greater than $\tilde{y}$, i.e., $\tilde{x} \succ \tilde{y}$.

Definition 3[28] Let $\tilde{x} = [x^L, x^U]$ and $\tilde{y} = [y^L, y^U]$ be two interval numbers, then

$$\tilde{x} + \tilde{y} = [\min(x^L + y^U, x^U + y^L), \max(x^L + y^U, x^U + y^L)];$$

$$\lambda\tilde{x} = [\lambda x^L, \lambda x^U];$$

$$\tilde{x} \times \tilde{y} = [\min(x^L \cdot y^U, x^U \cdot y^L), \max(x^L \cdot y^U, x^U \cdot y^L)];$$

$$1/\tilde{x} = [1/x^U, 1/x^L].$$

B. Neutrosophic Sets

Definition 4[1] Let X be a space of points (objects), a neutrosophic set A is defined as

$$A = \{\langle x : T_A(x), I_A(x), F_A(x)\rangle, x \in X\},$$

where the functions $T_A(x), I_A(x), F_A(x) : X \to]^-0, 1^+[$ are the degree of membership, the degree of indeterminacy and the degree of non-membership, respectively, and satisfied with the condition $^-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

By comparison with a neutrosophic set, an INS has a wide ranging application. Following is the definition of an INS.

Definition 5[9] Let X be a space of points (objects), with a generic element $x \in X$, an interval neutrosophic set A is defined as

$$A = \{\langle x : T_A(x), I_A(x), F_A(x)\rangle, x \in X\},$$

where the functions

$$T_A(x) = [\inf T_A(x), \sup T_A(x)] \subseteq [0, 1],$$

$$I_A(x) = [\inf I_A(x), \sup I_A(x)] \subseteq [0, 1]$$

and

$$F_A(x) = [\inf F_A(x), \sup F_A(x)] \subseteq [0, 1]$$

are the degree of membership, the degree of indeterminacy and the degree of non-membership, respectively, and satisfied with the condition $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3$.

Two INSs and have the following relations:

Definition 6[9] An INS A is contained in another INS B, i.e., $A \subseteq B$, if and only if

$$\inf T_A(x) \leq \inf T_B(x), \sup T_A(x) \leq \sup T_B(x);$$

$$\inf I_A(x) \geq \inf I_B(x), \sup I_A(x) \geq \sup I_B(x);$$

$$\inf F_A(x) \geq \inf F_B(x), \sup F_A(x) \geq \sup F_B(x).$$

Definition 7[9] Two INSs A and B are equal, i.e., $A = B$, if and only if $A \supseteq B$ and $A \subseteq B$.

III. Similarity measures between INSs based on minimum and maximum operators

This section presents new similarity measures of INSs and their properties based on the minimum and maximum operators.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse, and A and B be two INSs, which are defined as

$$A = \{\langle x : T_A(x), I_A(x), F_A(x)\rangle, x \in X\},$$

and

$$B = \{\langle x : T_B(x), I_B(x), F_B(x)\rangle, x \in X\},$$

where

$$T_l(x_i) \in [\inf T_l(x_i), \sup T_l(x_i)];$$

$$I_l(x_i) \in [\inf I_l(x_i), \sup I_l(x_i)];$$

$$F_l(x_i) \in [\inf F_l(x_i), \sup F_l(x_i)];$$

for $x_i \in X$, $l = A, B$.

In general, a similarity measure between two INSs A and B is a function defined as

$$Y : N(X) \times N(X) \to [0, 1],$$

which satisfies the following properties:

1) $0 \leq Y(A, B) \leq 1$;
2) $Y(A, B) = 1$ if $A = B$;
3) $Y(A, B) = Y(B, A)$;
4) $Y(A, C) \leq Y(A, B)$ and $Y(A, C) \leq Y(B, C)$ if $A \subseteq B \subseteq C$ for an INSs C.

Based on the minimum and maximum operators, we propose the following similarity measures between two INSs A and B.

Proposition 1 Let A and B be two INSs in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, the INS similarity measure

$$Y_1(A, B) = \frac{1}{3n} \sum_{i=1}^{n} \left(\frac{\min(T_A(x_i), T_B(x_i))}{\max(T_A(x_i), T_B(x_i))} + \frac{\min(I_A(x_i), I_B(x_i))}{\max(I_A(x_i), I_B(x_i))} + \frac{\min(F_A(x_i), F_B(x_i))}{\max(F_A(x_i), F_B(x_i))} \right) \quad (2)$$

should satisfy the following properties:

1) $0 \leq Y_1(A, B) \leq 1$;
2) $Y_1(A, B) = 1$ if $A = B$;
3) $Y_1(A, B) = Y_1(B, A)$;
4) $Y_1(A, C) \leq Y_1(A, B)$ and $Y_1(A, C) \leq Y_1(B, C)$ if $A \subseteq B \subseteq C$ for an INSs C.

PROOF. It is easy to verifying that $Y_1(A, B)$ satisfies the properties 1) - 3). Therefore, we only prove the property 4). Let $A \subseteq B \subseteq C$, then, by Definition 6,

$$\inf T_A \leq \inf T_B \leq \inf T_C, \ \sup T_A \leq \sup T_B \leq \sup T_C,$$

$$\inf I_A \geq \inf I_B \geq \inf I_C, \ \sup I_A \geq \sup I_B \geq \sup I_C,$$

$$\inf F_A \geq \inf F_B \geq \inf F_C, \ \sup F_A \geq \sup F_B \geq \sup F_C$$

for every $x_i \in X$.

According to Definition 2 and the comparative rules (C1) and (C2), we obtain that

$$\max\{[\inf T_A, \sup T_A], [\inf T_C, \sup T_C]\} = [\inf T_C, \sup T_C],$$
$$\min\{[\inf T_A, \sup T_A], [\inf T_C, \sup T_C]\} = [\inf T_A, \sup T_A],$$
$$\max\{[\inf I_A, \sup I_A], [\inf I_C, \sup I_C]\} = [\inf T_A, \sup T_A],$$
$$\min\{[\inf T_A, \sup T_A], [\inf T_C, \sup T_C]\} = [\inf T_C, \sup T_C],$$
$$\max\{[\inf F_A, \sup F_A], [\inf F_C, \sup F_C]\} = [\inf F_A, \sup F_A],$$
$$\min\{[\inf F_A, \sup F_A], [\inf F_C, \sup F_C]\} = [\inf F_C, \sup F_C],$$

thus

$$\begin{aligned} Y_1(A,C) &= \frac{1}{3n}\sum_{i=1}^{n}\left\{\frac{\min\left(T_A\left(x_i\right), T_C\left(x_i\right)\right)}{\max\left(T_A\left(x_i\right), T_C\left(x_i\right)\right)}\right. \\ &\quad + \frac{\min\left(I_A\left(x_i\right), I_C\left(x_i\right)\right)}{\max\left(I_A\left(x_i\right), I_C\left(x_i\right)\right)} \\ &\quad \left. + \frac{\min\left(F_A\left(x_i\right), F_C\left(x_i\right)\right)}{\max\left(F_A\left(x_i\right), F_C\left(x_i\right)\right)}\right\} \\ &= \frac{1}{3n}\sum_{i=1}^{n}\left\{\frac{\left[\inf T_A\left(x_i\right), \sup T_A\left(x_i\right)\right]}{\left[\inf T_C\left(x_i\right), \sup T_C\left(x_i\right)\right]}\right. \\ &\quad + \frac{\left[\inf I_C\left(x_i\right), \sup I_C\left(x_i\right)\right]}{\left[\inf I_A\left(x_i\right), \sup I_A\left(x_i\right)\right]} \\ &\quad \left. + \frac{\left[\inf F_C\left(x_i\right), \sup F_C\left(x_i\right)\right]}{\left[\inf F_A\left(x_i\right), \sup F_A\left(x_i\right)\right]}\right\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} Y_1(A,B) &= \frac{1}{3n}\sum_{i=1}^{n}\left\{\frac{\min\left(T_A\left(x_i\right), T_B\left(x_i\right)\right)}{\max\left(T_A\left(x_i\right), T_B\left(x_i\right)\right)}\right. \\ &\quad + \frac{\min\left(I_A\left(x_i\right), I_B\left(x_i\right)\right)}{\max\left(I_A\left(x_i\right), I_B\left(x_i\right)\right)} \\ &\quad \left. + \frac{\min\left(F_A\left(x_i\right), F_B\left(x_i\right)\right)}{\max\left(F_A\left(x_i\right), F_B\left(x_i\right)\right)}\right\} \\ &= \frac{1}{3n}\sum_{i=1}^{n}\left\{\frac{\left[\inf T_A\left(x_i\right), \sup T_A\left(x_i\right)\right]}{\left[\inf T_B\left(x_i\right), \sup T_B\left(x_i\right)\right]}\right. \\ &\quad + \frac{\left[\inf I_B\left(x_i\right), \sup I_B\left(x_i\right)\right]}{\left[\inf I_A\left(x_i\right), \sup I_A\left(x_i\right)\right]} \\ &\quad \left. + \frac{\left[\inf F_B\left(x_i\right), \sup F_B\left(x_i\right)\right]}{\left[\inf F_A\left(x_i\right), \sup F_A\left(x_i\right)\right]}\right\}; \end{aligned}$$

$$\begin{aligned} Y_1(B,C) &= \frac{1}{3n}\sum_{i=1}^{n}\left\{\frac{\min\left(T_B\left(x_i\right), T_C\left(x_i\right)\right)}{\max\left(T_B\left(x_i\right), T_C\left(x_i\right)\right)}\right. \\ &\quad + \frac{\min\left(I_B\left(x_i\right), I_C\left(x_i\right)\right)}{\max\left(I_B\left(x_i\right), I_C\left(x_i\right)\right)} \\ &\quad \left. + \frac{\min\left(F_B\left(x_i\right), F_C\left(x_i\right)\right)}{\max\left(F_B\left(x_i\right), F_C\left(x_i\right)\right)}\right\} \\ &= \frac{1}{3n}\sum_{i=1}^{n}\left\{\frac{\left[\inf T_B\left(x_i\right), \sup T_B\left(x_i\right)\right]}{\left[\inf T_C\left(x_i\right), \sup T_C\left(x_i\right)\right]}\right. \\ &\quad + \frac{\left[\inf I_C\left(x_i\right), \sup I_C\left(x_i\right)\right]}{\left[\inf I_B\left(x_i\right), \sup I_B\left(x_i\right)\right]} \\ &\quad \left. + \frac{\left[\inf F_C\left(x_i\right), \sup F_C\left(x_i\right)\right]}{\left[\inf F_B\left(x_i\right), \sup F_B\left(x_i\right)\right]}\right\}. \end{aligned}$$

Then $Y_1(A,C) \le Y_1(A,B)$ satisfies the property 4).

If we consider the importance in the three independent elements, i.e., truth-membership, indeterminacy-membership, and falsity-membership, in an INS, then we should take the weights of the three independent terms in Equation (2) into account. Therefore, we develop another similarity measure between two INSs.

Proposition 2 Let A and B be two INSs in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, the INS similarity measure

$$\begin{aligned} Y_2\left(A,B\right) = \frac{1}{n}\sum_{i=1}^{n}&\left(\alpha\frac{\min\left(T_A\left(x_i\right), T_B\left(x_i\right)\right)}{\max\left(T_A\left(x_i\right), T_B\left(x_i\right)\right)}\right. \\ &+ \beta\frac{\min\left(I_A\left(x_i\right), I_B\left(x_i\right)\right)}{\max\left(I_A\left(x_i\right), I_B\left(x_i\right)\right)} \\ &\left. + \gamma\frac{\min\left(F_A\left(x_i\right), F_B\left(x_i\right)\right)}{\max\left(F_A\left(x_i\right), F_B\left(x_i\right)\right)}\right) \end{aligned} \tag{3}$$

should satisfy the following properties:

1) $0 \le Y_2(A,B) \le 1$;
2) $Y_2(A,B) = 1$ if $A = B$;
3) $Y_2(A,B) = Y_2(B,A)$;
4) $Y_2(A,C) \le Y_2(A,B)$ and $Y_2(A,C) \le Y_2(B,C)$ if $A \subseteq B \subseteq C$ for an INSs C,

where α, β, γ are the weights of the three independent elements (i.e., truth-membership, indeterminacy-membership, and falsity-membership) in an INS and $\alpha + \beta + \gamma = 1$. Especially, when $\alpha = \beta = \gamma = 1/3$, Equation (3) reduces to Equation (2).

By virtue of the proof of Proposition 1, Proposition 2 can be proved.

Furthermore, if the important differences are considered in the elements in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, the weight of each element $x_i (i = 1, 2, \ldots, n)$ needs to be taken into account. Let w_i be the weight for each element $x_i (i = 1, 2, \ldots, n)$, $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, and then the weighted similarity measure is defined as follows.

Proposition 3 Let A and B be two INSs in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, the INS similarity measure

$$\begin{aligned} Y_3\left(A,B\right) = \sum_{i=1}^{n} w_i &\left(\alpha\frac{\min\left(T_A\left(x_i\right), T_B\left(x_i\right)\right)}{\max\left(T_A\left(x_i\right), T_B\left(x_i\right)\right)}\right. \\ &+ \beta\frac{\min\left(I_A\left(x_i\right), I_B\left(x_i\right)\right)}{\max\left(I_A\left(x_i\right), I_B\left(x_i\right)\right)} \\ &\left. + \gamma\frac{\min\left(F_A\left(x_i\right), F_B\left(x_i\right)\right)}{\max\left(F_A\left(x_i\right), F_B\left(x_i\right)\right)}\right) \end{aligned} \tag{4}$$

should satisfy the following properties:

1) $0 \le Y_3(A,B) \le 1$;
2) $Y_3(A,B) = 1$ if $A = B$;
3) $Y_3(A,B) = Y_3(B,A)$;
4) $Y_3(A,C) \le Y_3(A,B)$ and $Y_3(A,C) \le Y_3(B,C)$ if $A \subseteq B \subseteq C$ for an INS C.

Especially, when $w_1 = \cdots = w_n = 1/n$, Equation (4) reduces to Equation (3).

Proof of Proposition 3 can be obtained from the proof method of Proposition 1.

IV. Decision-making method using the proposed similarity measures between INSs

In this section, we propose a multi-criteria decision making method under interval neutrosophic setting by means of the proposed similarity measures between INSs.

Let $A = \{A_1, A_2, \ldots, A_m\}$ and $C = \{C_1, C_2, \ldots, C_n\}$ be sets of alternatives and criteria, respectively. Assume that w_j is the weight of the criterion $C_j (j = 1, 2, \ldots, n)$, where $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and the weights of the three independent elements, i.e., truth-membership, indeterminacy-membership and falsity-membership, in an INS, are α, β and γ, which are entered by the decision maker. In this case, the characteristic of the alternative $A_i (i = 1, 2, \ldots, m)$ is represented by the following form:

$$\begin{aligned} A_i &= \{\langle C_j, T_{A_i}(C_j), I_{A_i}(C_j), F_{A_i}(C_j)\rangle | C_j \in C\} \\ &= \{\langle C_j, [\inf T_{A_i}(C_j), \sup T_{A_i}(C_j)], \\ &\quad [\inf I_{A_i}(C_j), \sup I_{A_i}(C_j)], \\ &\quad [\inf F_{A_i}(C_j), \sup F_{A_i}(C_j)]\rangle | C_j \in C\} \end{aligned}$$

where $P_{A_i} = [\inf P_{A_i}(C_j), \sup P_{A_i}(C_j)] \subseteq [0, 1]$, $P = T, I$ and F, respectively, and $0 \leq \sup T_{A_i}(C_j) + \sup I_{A_i}(C_j) + \sup F_{A_i}(C_j) \leq 3$ for $C_j \in C$, $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, m$.

For convenience, an interval neutrosophic value are denoted by $d_{ij} = \langle [a_{ij}^L, a_{ij}^U], [b_{ij}^L, b_{ij}^U], [c_{ij}^L, c_{ij}^U]\rangle$, which is usually derived from the evaluation of an alternative A_i with respect to the criterion C_j. Therefore, we can establish an interval neutrosophic decision matrix $D = (d_{ij})_{m\times n}$.

In multi-criteria decision making environments, the concept of ideal point has been used to help identify the best alternative in the decision set [12]. Although the ideal alternative does not exist in real world, it does provide a useful theoretical construct against which to evaluate alternatives.

Generally, the evaluation criteria can be categorized into two kinds: benefit criteria and cost criteria. Let H be a collection of benefit criteria and K be a collection of cost criteria. In the presented decision making method, an ideal alternative can be identified by using a maximum operator for the benefit criteria and a minimum operator for the cost criteria to determine the best value of each criteria among all alternatives. Therefore, we define an ideal INS for a benefit criteria in the ideal alternative A^* as

$$\begin{aligned} d_j^* &= \left\langle \left[a_j^{L*}, a_j^{U*}\right], \left[b_j^{L*}, b_j^{U*}\right], \left[c_j^{L*}, c_j^{U*}\right]\right\rangle \\ &= \left\langle \left[\max_i \left(a_{ij}^L\right), \max_i \left(a_{ij}^U\right)\right], \right. \\ &\quad \left[\min_i \left(b_{ij}^L\right), \min_i \left(b_{ij}^U\right)\right], \\ &\quad \left. \left[\min_i \left(c_{ij}^L\right), \min_i \left(c_{ij}^U\right)\right]\right\rangle \end{aligned}$$

for $j \in H$; while for a cost criterion, we define an ideal INS in the ideal alternative A^* as

$$\begin{aligned} d_j^* &= \left\langle \left[a_j^{L*}, a_j^{U*}\right], \left[b_j^{L*}, b_j^{U*}\right], \left[c_j^{L*}, c_j^{U*}\right]\right\rangle \\ &= \left\langle \left[\min_i \left(a_{ij}^L\right), \min_i \left(a_{ij}^U\right)\right], \right. \\ &\quad \left[\max_i \left(b_{ij}^L\right), \max_i \left(b_{ij}^U\right)\right], \\ &\quad \left. \left[\max_i \left(c_{ij}^L\right), \max_i \left(c_{ij}^U\right)\right]\right\rangle \end{aligned}$$

for $j \in K$.

Using the similarity measure defined in Equation (2), we have

$$\begin{aligned} Y_1(A^*, A_i) = \frac{1}{3n}\sum_{i=1}^{n} &\left\{ \frac{\min\left(\left[a_j^{L*}, a_j^{U*}\right], \left[a_{ij}^L, a_{ij}^U\right]\right)}{\max\left(\left[a_j^{L*}, a_j^{U*}\right], \left[a_{ij}^L, a_{ij}^U\right]\right)} \right. \\ &+ \frac{\min\left(\left[b_j^{L*}, b_j^{U*}\right], \left[b_{ij}^L, b_{ij}^U\right]\right)}{\max\left(\left[b_j^{L*}, b_j^{U*}\right], \left[b_{ij}^L, b_{ij}^U\right]\right)} \\ &\left. + \frac{\min\left(\left[c_j^{L*}, c_j^{U*}\right], \left[c_{ij}^L, c_{ij}^U\right]\right)}{\max\left(\left[c_j^{L*}, c_j^{U*}\right], \left[c_{ij}^L, c_{ij}^U\right]\right)} \right\}. \end{aligned}$$

By Equation (1) in Definition 2, another representation of ideal alternative A^* and the value of criteria d_{ij} should be obtained:

$$\begin{aligned} d_j^{**} &= \left\langle a_j^{**}, b_j^{**}, c_j^{**}\right\rangle \\ &= \left\langle \left(\max_i \left(a_{ij}^L\right) + \max_i \left(a_{ij}^U\right)\right)\Big/2 \right. \\ &+ \left(\max_i \left(a_{ij}^L\right) - \max_i \left(a_{ij}^U\right)\right) i\Big/2, \\ &\quad \left(\min_i \left(b_{ij}^L\right) + \min_i \left(b_{ij}^U\right)\right)\Big/2 \\ &+ \left(\min_i \left(b_{ij}^L\right) - \min_i \left(b_{ij}^U\right)\right) i\Big/2, \\ &\quad \left(\min_i \left(c_{ij}^L\right) + \min_i \left(c_{ij}^U\right)\right)\Big/2 \\ &+ \left. \left(\min_i \left(c_{ij}^L\right) - \min_i \left(c_{ij}^U\right)\right) i\Big/2\right\rangle \end{aligned} \tag{5}$$

for $j \in H$;

$$\begin{aligned} d_j^{**} &= \left\langle a_j^{**}, b_j^{**}, c_j^{**}\right\rangle \\ &= \left\langle \left(\min_i \left(a_{ij}^L\right) + \min_i \left(a_{ij}^U\right)\right)\Big/2 \right. \\ &+ \left(\min_i \left(a_{ij}^L\right) - \min_i \left(a_{ij}^U\right)\right) i\Big/2, \\ &\quad \left(\max_i \left(b_{ij}^L\right) + \max_i \left(b_{ij}^U\right)\right)\Big/2 \\ &+ \left(\max_i \left(b_{ij}^L\right) - \max_i \left(b_{ij}^U\right)\right) i\Big/2, \\ &\quad \left(\max_i \left(c_{ij}^L\right) + \max_i \left(c_{ij}^U\right)\right)\Big/2 \\ &+ \left. \left(\max_i \left(c_{ij}^L\right) - \max_i \left(c_{ij}^U\right)\right) i\Big/2\right\rangle \end{aligned}$$

for $j \in K$;

$$\begin{aligned} d_{ij} &= \langle a_{ij}, b_{ij}, c_{ij}\rangle \\ &= \left\langle \left(a_{ij}^L + a_{ij}^U\right)/2 + \left(a_{ij}^L - a_{ij}^U\right) i/2, \right. \\ &\quad \left(b_{ij}^L + b_{ij}^U\right)/2 + \left(b_{ij}^L - b_{ij}^U\right) i/2 \end{aligned}$$

$$\left(c_{ij}^L + c_{ij}^U\right)/2 + \left(c_{ij}^L - c_{ij}^U\right) i/2\rangle .$$

Comparing the three terms corresponding in d_j^{**} and d_{ij}, i.e., a_j^{**} and a_{ij}, b_j^{**} and b_{ij}, c_j^{**} and c_{ij}, respectively. The minimum and maximum interval numbers in numerator or denominator can be found, and then calculating the terms in the braces by the rules of interval number division and addition appeared in Definition 3.

Similarly, by applying equations (3) and (4), two other measures $Y_2(A^*, A_i)$ and $Y_3(A^*, A_i)$ can be obtained.

Through the similarity measure $Y_1(A^*, A_i)$, $Y_2(A^*, A_i)$, or $Y_3(A^*, A_i)(i = 1, 2, \ldots, m)$ between each alternative and the ideal alternative, the ranking order of all alternatives can be determined and the best one can be easily identified as well.

V. Illustrative example

In this section, an example for the multi-criteria decision making problem is used as the demonstration of the proposed decision making method, as well as the effectiveness of the proposed method.

Let us consider the decision making problem adapted from [12]. There are four different companies: A_1, A_2, A_3, A_4, and three criteria: C_1, C_2, C_3, where C_1 and C_2 are benefit criteria, and C_3 is a cost criteria. The weight vector of criteria is $w = (0.35, 0.25, 0.4)$, and the interval neutrosophic decision matrix $D = (A_1, A_2, A_3, A_4)^T$, where

$$\begin{aligned} A_1 = \{ &\langle [0.4, 0.5], [0.2, 0.3], [0.3, 0.4] \rangle, \\ &\langle [0.4, 0.6], [0.1, 0.3], [0.2, 0.4] \rangle, \\ &\langle [0.7, 0.9], [0.2, 0.3], [0.4, 0.5] \rangle \}; \\ A_2 = \{ &\langle [0.6, 0.7], [0.1, 0.2], [0.2, 0.3] \rangle, \\ &\langle [0.6, 0.7], [0.1, 0.2], [0.2, 0.3] \rangle, \\ &\langle [0.3, 0.6], [0.3, 0.5], [0.8, 0.9] \rangle \}; \\ A_3 = \{ &\langle [0.3, 0.6], [0.2, 0.3], [0.3, 0.4] \rangle, \\ &\langle [0.5, 0.6], [0.2, 0.3], [0.3, 0.4] \rangle, \\ &\langle [0.4, 0.5], [0.2, 0.4], [0.7, 0.9] \rangle \}; \\ A_4 = \{ &\langle [0.7, 0.8], [0.0, 0.1], [0.1, 0.2] \rangle, \\ &\langle [0.6, 0.7], [0.1, 0.2], [0.1, 0.3] \rangle, \\ &\langle [0.6, 0.7], [0.3, 0.4], [0.8, 0.9] \rangle \}. \end{aligned}$$

From the decision matrix D, we get the following ideal alternative:

$$\begin{aligned} A^* = \{ &\langle [0.7, 0.8], [0.0, 0.1], [0.1, 0.2] \rangle, \\ &\langle [0.6, 0.7], [0.1, 0.2], [0.1, 0.3] \rangle, \\ &\langle [0.3, 0.5], [0.3, 0.5], [0.8, 0.9] \rangle \} \end{aligned}$$

Subsequently, from (1), we can obtain that

$$\begin{aligned} A^{**} = \{ &\langle 0.75 + 0.05i, 0.05 + 0.05i, 0.15 + 0.05i \rangle, \\ &\langle 0.65 + 0.05i, 0.15 + 0.05i, 0.2 + 0.1i \rangle, \\ &\langle 0.4 + 0.1i, 0.4 + 0.1i, 0.85 + 0.05i \rangle \}, \end{aligned}$$

and decision matrix $D = (A_1, A_2, A_3, A_4)^T$ can be converted into another representation as $D_1^1 = (A_1^1, A_2^1, A_3^1, A_4^1)^T$, where

$$\begin{aligned} A_1^1 = \{ &\langle 0.45 + 0.05i, 0.25 + 0.05i, 0.35 + 0.05i \rangle, \\ &\langle 0.5 + 0.1i, 0.2 + 0.1i, 0.3 + 0.1i \rangle, \\ &\langle 0.8 + 0.1i, 0.25 + 0.05i, 0.45 + 0.05i \rangle \}; \\ A_2^1 = \{ &\langle 0.65 + 0.05i, 0.15 + 0.05i, 0.25 + 0.05i \rangle, \\ &\langle 0.65 + 0.05i, 0.15 + 0.05i, 0.25 + 0.05i \rangle, \\ &\langle 0.45 + 0.15i, 0.4 + 0.1i, 0.85 + 0.05i \rangle \}; \\ A_3^1 = \{ &\langle 0.45 + 0.15i, 0.25 + 0.05i, 0.35 + 0.05i \rangle, \\ &\langle 0.55 + 0.05i, 0.25 + 0.05i, 0.35 + 0.05i \rangle, \\ &\langle 0.45 + 0.05i, 0.3 + 0.1i, 0.8 + 0.1i \rangle \}; \\ A_4^1 = \{ &\langle 0.75 + 0.05i, 0.05 + 0.05i, 0.15 + 0.05i \rangle, \\ &\langle 0.65 + 0.05i, 0.15 + 0.05i, 0.2 + 0.1i \rangle, \\ &\langle 0.65 + 0.05i, 0.35 + 0.05i, 0.85 + 0.05i \rangle \}. \end{aligned}$$

Using Equation (4), we have the following similarity measures of $Y_1^*(A^*, A_i)(i = 1, 2, 3, 4)$:

$$Y_1^*(A^*, A_1) = 0.72 + 0.01i;\ Y_1^*(A^*, A_2) = 0.86 + 0.02i;$$
$$Y_1^*(A^*, A_3) = 0.63 + 0.04i;\ Y_1^*(A^*, A_4) = 0.95 + 0.02i.$$

Therefore, the ranking order of the four alternatives is $A_4 \succ A_2 \succ A_1 \succ A_3$. Obviously, among them A_4 is the best alternative.

Without loss of generality, let the weight values of the three independent elements be $\alpha = \beta = \gamma = 1/3$, then we can give the similarity measures of $Y_2^*(A^*, A_i)(i = 1, 2, 3, 4)$ as follows:

$$Y_2^*(A^*, A_1) = 0.72 + 0.01i;\ Y_2^*(A^*, A_2) = 0.86 + 0.02i;$$
$$Y_2^*(A^*, A_3) = 0.63 + 0.04i;\ Y_2^*(A^*, A_4) = 0.95 + 0.02i.$$

Thus, the ranking of the four alternatives is $A_4 \succ A_2 \succ A_1 \succ A_3$. Obviously, among them A_4 is the best alternative.

And the similarity measures of $Y_3^*(A^*, A_i)(i = 1, 2, 3, 4)$ as follows:

$$Y_3^*(A^*, A_1) = 0.92 + 0.05i;\ Y_3^*(A^*, A_2) = 0.87 + 0.05i;$$
$$Y_3^*(A^*, A_3) = 0.71 + 0.05i;\ Y_3^*(A^*, A_4) = 0.94 + 0.02i.$$

Therefore, the ranking of the four alternatives is $A_4 \succ A_1 \succ A_2 \succ A_3$. Obviously, among them A_4 is the best alternative.

In [12], two methods were utilized the similarity measures based on the relationship with distances, where the similarity measure of Method 1 is on the basis of the Euclidean distance and the similarity measure of Method 2 is in view of the Hamming distance. The comparison results can be found in Table I.

The comparison methods in [12] were conducted using the similarity measures that only considered the relationship between each alternative and the PIS. However, the PIS is closely related to the number of alternatives as well as the evaluation values of alternatives. If only PIS is considered and NIS is ignored, the ranking of alternatives may be incorrectly reversed. Furthermore, compared to the methods in [12], the proposed approach can handle MCDM problems more flexible and reliable, and the decision making process is clear and simple.

TABLE I
COMPARISON WITH DIFFERENT METHODS

method 1 of Ref. [12]	$A_4 \succ A_2 \succ A_3 \succ A_1$
method 2 of Ref. [12]	$A_2 \succ A_4 \succ A_3 \succ A_1$
method 1 of proposed method	$A_4 \succ A_2 \succ A_1 \succ A_3$
method 2 of proposed method	$A_4 \succ A_2 \succ A_1 \succ A_3$

VI. CONCLUSIONS

In scientific and engineering situations, there widely exist uncertain, imprecise, incomplete and inconsistent information, which can be flexibly expressed using INSs. Therefore, it is of great important significance to study MCDM methods with INSs. This study develops a multi-criteria decision making method under interval neutrosophic setting. There followed a list of contributions and innovation.

1) We propose a comparative method between two interval numbers. Comparing interval numbers is one of the difficult points in dealing with INSs, a comparative method is given through using another representation of interval number.

2) Three new similarity measures based on the minimum and maximum operators between INSs are presented, and their properties are also proved. We present three similarity measures via taking the weights of three elements in an INS.

3) A multi-criteria decision making method has been established under interval neutrosophic setting by means of the proposal similarity measure, and using the similarity measures, the ranking order of all alternatives can be determined and the best alternative can be easily identified as well.

The techniques proposed in this paper extend existing decision making methods and can provide a useful way for decision makers. It is showed that the decision making method with INSs is more suitable for engineering and scientific practice. In the future work, we shall continue to study in the application of similarity measures between INSs to other domains.

ACKNOWLEDGMENT

This research is supported by the National Social Science Foundation of China (Project No: 16BJY026).

REFERENCES

[1] F. Smarandache, "A unifying field in logics: neutrosophic logic", *Multiple Valued Logic*, vol. 8, no. 3, pp. 489-503, 2002.
[2] L. A. Zadeh, "Fuzzy sets", *Information and Control*, vol. 8, no. 3, pp. 338-353, 1965.
[3] K. U. Jahn, "Intervall-wertige mengen", *Mathematische Nachrichten*, vol. 68, no. 1, pp. 115-132, 1975.
[4] K. Atanassov and P. Rangasamy, "Intuitionistic fuzzy sets", *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87-96, 1986.
[5] K. Atanassov and G. Gargov, "Interval valued intuitionistic fuzzy sets", *Fuzzy Sets and Systems*, vol. 31, no. 3, pp. 343-349, 1989.
[6] W. L. Gau and D. J. Buehrer, "Vague sets", *IEEE Transactions on Systems Man and Cybernetics*, vol. 23, no. 2, pp. 610-614, 1993.
[7] J. L. Deng, "Control problems of grey system", *System and Control Letters*, vol. 1, no. 5, pp. 288-294, 1982.
[8] H. Wang, F. Smarandache, Y. Q. Zhang and R. Sunderraman, "Interval neutrosophic sets and logic: Theory and applications in computing", *Computer Science*, vol. 65, no. 4, vi 87, 2005.
[9] H. Wang, F. Smarandache, Y. Q. Zhang and R. Sunderraman, "Single valued neutrosophic sets", *Multispace and Multistructure*, vol. 4, no. 4, pp. 410-413, 2010.
[10] H. Wang, F. Smarandache, Y. Q. Zhang and R. Sunderraman, *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*, Hexis, Phoenix, Ariz, USA, 2005.
[11] A. Kharal, "A neutrosophic multicriteria decision making method", *New Mathematics and Natural Computation*, vol. 10, no. 4, pp:143-162, 2014.
[12] J. Ye, "Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making", *Journal of Intelligent and Fuzzy Systems*, vol. 26, no. 1, pp. 165-172, 2014.
[13] Y. X. Ma, J. Q. Wang, J. Wang, et al. "An interval neutrosophic linguistic multi-criteria group decision making method and its application in selecting medical treatment options", *Neural Computing and Applications*, vol. 28, no. 9, pp. 2745-2765, 2017.
[14] J. Ye, "Single valued neutrosophic cross entropy for multicriteria decision making problems", *Applied Mathematical Modeling*, vol. 38, no. 3, pp. 1170-1175, 2014.
[15] K. Monda and S. Pramanik, "A study on problems of Hijras in West Bengal based on neutrosophic cognitive maps", *Neutrosophic Sets and Systems*, vol. 5, pp. 21-26, 2014.
[16] K. Monda and S. Pramanik, "Multi-criteria group decision making approach for teacher recruitment in higher education under simplified neutrosophic environment", *Neutrosophic Sets and Systems*, vol. 6, pp. 28-34, 2014.
[17] P. Biswas, S. Pramanik, and B. C, Giri, "Entropy based grey relational analysis method for multi-attribute decision-making under single valued neutrosophic assessments", *Neutrosophic Sets and Systems*, vol. 2, pp. 102-110, 2014.
[18] P. Biswas, S. Pramanik, and B. C, Giri, "A new methodology for neutrosophic multi-attribute decision making with unknown weight information", *Neutrosophic Sets and Systems*, vol. 3, no. 3, pp. 42-50, 2014.
[19] J. Ye, "Single-valued neutrosophic similarity measures based on cotangent function and their application in the fault diagnosis of steam turbine", *Softing Computing*, vol. 21, no. 3, pp. 817-825, 2017.
[20] K. Mondal and S. Pramanik, "Neutrosophic decision making model of school choice", *Neutrosophic Sets and Systems*, vol. 7, pp. 62-68, 2015.
[21] K. Mondal and S. Pramanik, "Rough neutrosophic multi-attribute based on grey relational analysis", *Neutrosophic Sets and Systems*, vol. 7, pp. 8-17, 2015.
[22] B. Said and F. Smarandache, "Several similarity measures of neutrosophic sets", *Neutrosophic Sets and Systems*, vol. 1, pp. 54-62, 2013.
[23] J. Ye, "Vector similarity measures of simplified neutrosophic sets and their application in multi-criteria decision making", *International Journal of Fuzzy Systems*, vol. 16, no. 2, pp. 204-215, 2014.
[24] S. Ye and J. Ye "Dice Similarity Measure between Single Valued Neutrosophic Multisets and Its Application in Medical Diagnosis", *Neutrosophic Sets and Systems*, vol. 6, pp. 48-53, 2015.
[25] P. Biswas, S. Pramanik, B. C. Giri, "Cosine Similarity Measure Based Multi-attribute Decision-making with Trapezoidal Fuzzy Neutrosophic Numbers", *Neutrosophic Sets and Systems*, vol. 8, pp. 46-56, 2015.
[26] J. Ye and Q. S. Zhang, "Single Valued Neutrosophic Similarity Measures for Multiple Attribute Decision-Making", *Neutrosophic Sets and Systems*, vol. 2, pp. 48-54, 2014.
[27] A. Sengupta and T. K. Pal, "On comparing interval numbers", *European Journal of Operational Research*, vol. 127, no. 1, pp. 28-43, 2000.
[28] D. X. Chen and F. W. Zhang, "The supplement and improvement of a interval numbers algorithm", *Journal of Liaocheng University*(Nat. Sci), vol. 22, no. 2, pp. 20-21, 2009.

Limin Su was born on Aug. 22, 1981. She is a doctor candidate and majors in Management Science and Engineering. She has published dozens of papers in related journals.

Huishuang He was born Jun. 17, 1982. She received her doctor's degree in Industrial Economics from Zhongnan University of Economics and Law of China. Currently, She is a vice professor at North China University of Water Resources and Electric Power, China. She has published many papers in related journals.

Hongwen Lu was born on Oct. 22, 1939. His major research interest are Arithmetic in algebraic number field and module form, cryptography. He is one of the leading academic leaders in the study of algebraic number theory in China.