A fractional wavelet and its implementation using single switched-current integrators

Mu Li, Wenxin Yu, Xiaofeng Wu, and Zaifang Xi

***Abstract*—**A fractional wavelet based on fractional order system and its analog current-mode switched-current (SI) circuit implementation with few components and simply structure is presented. Firstly, it is shown that the impulse response of a fractional order band-pass filter satisfies the admissibility condition to be considered a wavelet base. Then the wavelet filter circuit is designed using single SI integrator and the different scale wavelet functions for implementing wavelet transform (WT) are obtained by only changing the clock frequency with the same circuit architecture. The time and frequency domain responses of the fractional wavelet filter circuit are given. Meanwhile, the sensitivity and imperfection of the designed circuit is analyzed. Finally, Simulations verify the correctness and feasibility of the proposed method.

***Keywords*—**Fractional wavelet, fractional order systems, wavelet transform, fractional wavelet filter, switched-current circuit.

I. Introduction

THE wavelet transform (WT) is a very promising mathematical tool that gives good estimation of time and frequency localization for analysis of non-stationary and fast transient signals [1-3]. So in the last few decades the WT was widely applied to signal processing in various fields. Wavelet analysis is performed using a prototype function referred to the wavelet base (or mother wavelet), which decomposes a signal into components appearing at different scales. Traditionally, the WT is implemented using digital circuit such as Digital Signal Processor (DSP) and Field Programmable Gate Array (FPGA). However, in low power consumption applications such as wearable and implantable medical devices, the WT implementation by means of digital circuit is not suitable because such devices impose strict constraints on the power consumption. From the power perspective, an obvious alternative is to use analog circuits. Therefore, the analog circuit realization of the WT becomes a crucial research field.

Recently, there have been significant advances for implementing the WT in analog filter, the impulse response of which is the approximated wavelet base [4-16]. The solutions of the WT at various resolutions are obtained by the convolution of the signal with a dilated impulse response of the designed analog filter, mapping the signal onto a two dimensional function of time and frequency. The main ideal of the WT is to look at a signal at various windows and analyze it with various scales. However, these approaches mainly research the classical wavelet bases such as the first derivative of a Gaussian, Mexican hat and Morlet wavelet, etc. Moreover, these systems have to be high order filter in order to obtain good approximation, which leads to complex circuit architecture, since the shape of classical wavelet is not the natural impulse response of the analog filter. It is well known that different wavelet bases have various transform characteristics for a given signal, so constructing the new wavelet bases become a significant research topic in wavelet analysis theory and practical application. Subsequently, a special wavelet and its analog voltage-mode circuit realization are proposed in [17]. The wavelet base is the natural impulse response of the designed filter circuit, so the WT circuit is simple and easy to implementation. However, the considered second order system in the work is integer order rather than fractional order system. Furthermore, the analog WT implementation applies voltage-mode operational transconductance amplifier (OTA) circuit, this approach need huge capacitors, which is difficult to generate precisely. In addition, the designed voltage-mode circuit performance suffers as supply voltages are scaled down. To the authors' best knowledge, fractional wavelet based on fractional order system and analog fractional wavelet filter design have not yet been approached, which is the target of our work. In our work, a new fractional wavelet based on a fractional order system is constructed and the analog current-mode switched-current (SI) fractional wavelet filter is designed using SI single integrators.

The rest of this paper is organized as follows. In Section II, the theory of wavelet base is introduced, and then a fractional order system is analyzed and the impulse response of this system is proved as a wavelet base. In Section III, the fractional wavelet filter is implemented using single SI integrators with

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simply structure and few components. In section IV, the SI fractional wavelet circuit is simulated and analyzed. The simulation results verify the feasibility of the proposed method. Subsequently, an example is presented using numerical simulation method to demonstrate the fractional wavelet has a near performance with the traditional wavelet base in Section V. Finally, Section VI presents the conclusions.

II. WAVELET BASE AND FRACTIONAL WAVELET

A. Wavelet Base Theory

From the wavelet theory, we know that the wavelet analysis is performed using a wavelet base $\psi(t)$. The main characteristic of the wavelet base is given by

$$\int_{-\infty}^{\infty}\psi(t)dt=0 \tag{1}$$

This means that the wavelet base is oscillatory and has zero mean value. Also, this function needs to satisfy the admissibility condition as

$$C_\Psi=\int_{-\infty}^{\infty}\frac{|\Psi(\omega)|^2}{|\omega|}d\omega<\infty \tag{2}$$

where $\Psi(\omega)$ is the Fourier transform of the wavelet base. The admissible condition implies that the Fourier transform of the wavelet base must have a zero component at zero frequency. Therefore, a new wavelet base constructed must be satisfied with the admissibility condition. In order to construct a fractional wavelet, the fractional order system will be introduced and analyzed in the next content.

B. Fractional Order Systems and Fractional Wavelet Base

The research of fractional order calculus is an old topic in nonlinear system theory. But in the last two decades it has been gaining increasing attention in various fields. Traditionally, the Laplacian operator s is raised to an integer exponent such as s, s^2,…, s^n. However, a Laplacian of non-integer exponent s^α where $0<\alpha<1$ is mathematically valid and is representative of a fractional order system [18]. Unlike the integral order operator, the amplitude and phase characters of fractional order s^α are (20α) dB/dec and ($\alpha\pi/2$), respectively. They are not limited to some countable numbers such as 20dB/dec, 40dB/dec, … and $\pi/2$, π, …. Therefore, the fractional order system can describe a practical system better than the integer order system using their properties [19]. A fractional derivative may according to the Riemann-Liouville definition [20-22] be given by

$$\frac{d^\alpha}{dt^\alpha}f(t)\equiv D^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-\tau)^{-\alpha}f(\tau)d\tau \tag{3}$$

Where $0<\alpha<1$, $\Gamma(\cdot)$ is the gamma function. Zero initial conditions yield the Laplace transform of the above derivative as

$$L\left\{{}_0d_t^\alpha f(t)\right\}=s^\alpha F(s) \tag{4}$$

where s^α is the fractional Laplacian operator. Consider a fractional order system

$$H(s)=\frac{s^\beta}{s^\alpha+1} \tag{5}$$

From the stability point of view, this system is stable if and only if $\alpha<2$ while it will oscillate if and only if $\alpha=2$. The magnitude of this system is [21]

$$|H(j\omega)|=\frac{\omega^\beta}{\sqrt{\omega^{2\alpha}+2\omega^\alpha\cos(\alpha\pi/2)+1}} \tag{6}$$

The magnitude responses of the considered fractional order system for different values of α and β are shown in Fig.1. Note that for $\beta<\alpha$, $\lim_{\omega\to\infty}|H(j\omega)|=0$ and hence the filter is a band-pass filter. For $\alpha=\beta$, $\lim_{\omega\to\infty}|H(j\omega)|=1$, which makes the filter a high-pass filter. From Fig.1it is seen here that there is always a maximum point in the magnitude response if $\beta<\alpha$. When $\alpha=2\beta$, the maxima frequency ω_m is equal to the centre frequency ω_0. Furthermore, Note from Fig. 1 that the center frequencyω_0 is not necessarily equal to ω_m, which is significantly different from what is know in integer order system. It is well known that the wavelets are inherently bandpass filters in the Fourier domain. Next, we will prove that the impulse response of the fractional order system ($\beta<\alpha$) satisfies the admissibility condition of the WT, as is required to be considered a prototype wavelet. Now we split the expression in (2) as

$$C_\psi=2\int_0^{\omega_0}\frac{|H(\omega)|^2}{\omega}d\omega+2\int_{\omega_0}^{\infty}\frac{|H(\omega)|^2}{\omega}d\omega<\infty \tag{7}$$

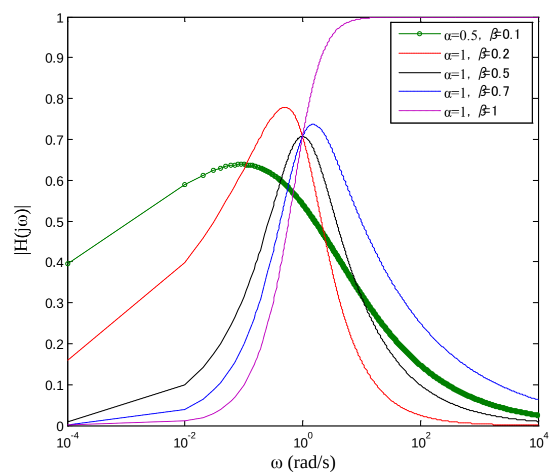

Fig.1 Magnitude response of the considered fractional order system for different values of α and β

In the Bode plot of Fig. 2, we can see that the integrand $|H(j\omega)|^2/\omega$ is bounded, so the first integral in (7) must be bounded. Hence the convergence of C_ψ is determined by the second integral in (7). Similarly, from Fig.2 it is easy to see that the curve of the integrand is convergent in the interval [ω_0, ∞] because $\lim_{\omega\to\infty}|H(j\omega)|=0$ when $\beta<\alpha$, so the second integral converges in (7). The above analysis proves the constructed wavelet satisfies the admissibility condition of the WT.

III. FRACTIONAL WAVELET FILTER IMPLEMENTATION USING SIMPLY CIRCUITS

Since design and analysis tools developed under the linear circuit theory are based on integer order differential equations, the fractional order system is usually implemented using an integer order transfer function approximation of the fractional order Laplacian operator s^{α}. There are many methods used to create an approximation of s^{α} that include continued fraction expansion (CFE) [21]. Using the CFE method we obtain the following approximation for the general Laplacian operator to first order

$$s^{\alpha} \approx \frac{(1-\alpha)+s(1+\alpha)}{(1+\alpha)+s(1-\alpha)} \tag{8}$$

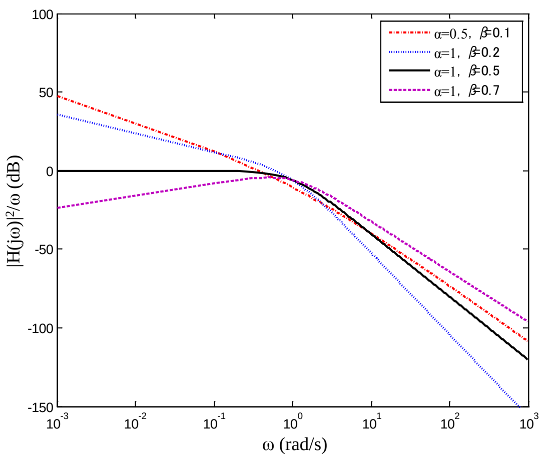

Fig. 2 Bode plot of the integrand

As an example, we set α=1, β=0.5 and a=b=1, the rational approximation of the fractional system in (5) is obtained as

$$H(s) = \frac{3s+1}{s^2+4s+3} \tag{9}$$

The impulse response of this system, before referred to as wavelet base, is shown in Fig.3. Next, this system is implemented using single SI integrator biquad stage. SI is an analogue sample-data signal processing technology. A key feature of using SI circuit for implementing the proposed system is that dilations of a given circuit may be easily and very precisely controlled. The SI integrator biquad stage [23] is shown in Fig.4. The z domain transfer function of this biquad stage is described as

$$H(z) = -\frac{(\alpha_5+\alpha_6)z^2+(\alpha_1\alpha_3-\alpha_5-2\alpha_6)z+\alpha_6}{(1+\alpha_4)z^2+(\alpha_2\alpha_3-\alpha_4-2)z+1} \tag{10}$$

The bilinear z transform $s \to 2(1-z^{-1})/[T(1+z^{-1})]$ is applied to (9), and then the coefficients of the z domain transfer function compare with those of (10) to obtain the coefficient values α_1-α_6. The current-mode wavelet circuit with single SI integrator biquad stage as main building block in ASIZ software [24] is shown in Fig.5. In Fig.5 the reference current source is omitted according to the requirement of the software. "①" and "②" are the network labels in the circuit and the same labels indicate the physical connection. The transistor M_{14} may use to adjust the output amplitude of the designed SI circuit.

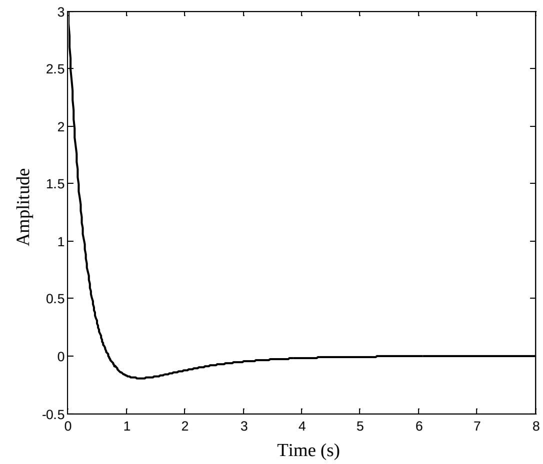

Fig. 3 Impulse response of the designed system

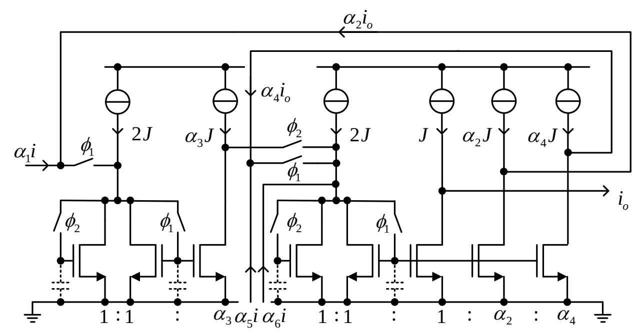

Fig. 4 SI integrator biquad stage

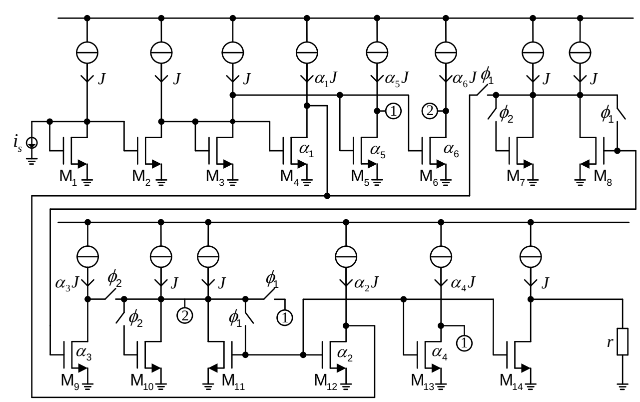

Fig. 5 SI fractional wavelet circuit

IV. Circuit Simulation and Analysis

In order to verify the feasibility of the designed SI fractional wavelet circuit in this work, the wavelet circuit has been simulated using ASIZ program. Firstly, the transistor transconductances (g_m) corresponding to the coefficients α_1-α_6 in the circuit are set according to the calculated values in Table 1. Then we select the input current source is 1A, the load resistance r=1Ω (ASIZ program prefers a normalized circuit, with all the capacitances, resistances and transconductances and so on with values close to 1). The transfer function in (9) can be denormalized to any desired centre frequency in the practical application. The sample frequency can be determined by the sample theorem. Herein, the center frequency is selected to be 0.24 Hz (scale a=1) as an example. By changing the clock frequency of the designed SI circuit, different scales wavelet functions can be gained with the same system architecture for implementing the WT, this characteristic is in general unachievable using conventional analogue designs. So the relevant sample frequencies are selected as 10 Hz, 5 Hz, 2.5 Hz and 1.25 Hz for scale a=1, 2, 4 and 8, respectively. The different scale impulse and frequency responses are shown in Fig.6 and Fig.7, respectively. In Fig. 6 the time domain waveforms achieve the positive peak value 0.37 A at t=0, which is closed to the ideal normalized value 0.3 A. In Fig. 7 the frequency domain waveform achieve the maximum value -2.52 dB at 1.52 *rad*/*s*, 0.76 *rad*/*s*, 0.38 *rad*/*s* Hz and 0.19 *rad*/*s*, respectively, which is small difference with ideal value -2.38 dB.

Table 1 The transistor transconductances of SI fractional wavelet circuit

Coefficient	Transistor	Transconductance (g_m)
α_1	M4	0.1238
α_2	M12	0.3715
α_3	M9	0.1000
α_4	M13	0.4954
α_5	M5	0.3715
α_6	M6	0.1827

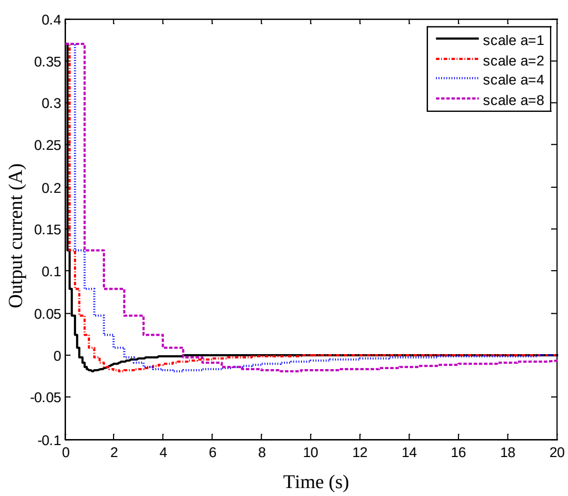

Fig. 6 Impulse responses of the fractional wavelet circuit at different scale a=1, 2, 4, 8

Secondly, to demonstrate the low sensitivity of the proposed SI fractional wavelet circuit, the sensitivity of the wavelet circuit using ASIZ program is analyzed. Now assuming uncorrelated ±3% random errors in all the transistor transconductances of the circuit with parasitic effects ignored, the error margins of frequency response at scale a=1 are plotted in Fig.8, which is computed by the statistical deviation of the gain. We are easy to see that the frequency points agree quite closely the nominal gain curve. The maximum gain error between nominal and practical gain is only 0.6382 dB, which shows that the designed fractional wavelet circuit has low sensitivity due to the simple structure and few components.

Thirdly, the imperfection simulation of the fractional wavelet circuit is made with the basic form, without enhancement circuit. The effect of finite ratio G_m/G_{ds} in the transistors, and parasitic C_{gd} capacitances are considered. Assuming G_m/G_{ds} and C_{gs}/C_{gd} ratio of 1000, with the biasing and signal current sources assumed as ideal, the frequency response of the circuit at scale a=1 is shown in Fig.9. It is seen that the gain of the imperfection circuit is small different from one of the perfection circuit. The maximum gain error between the imperfection and the perfection circuit is only 0.327 dB. Therefore, the designed circuit has litter effect in the sensitivity to the imperfection.

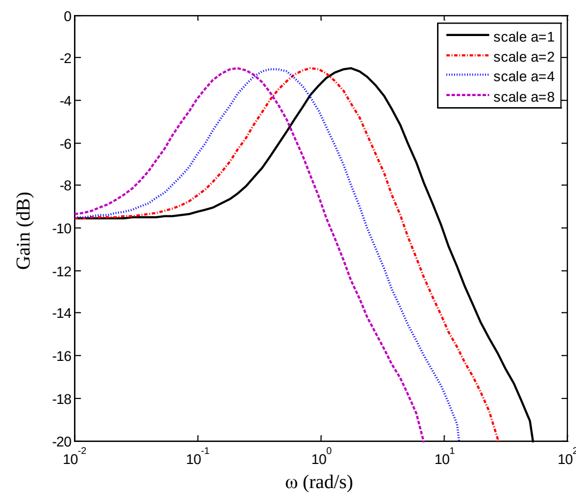

Fig. 7 Frequency responses of the fractional wavelet circuit at different scale a=1, 2, 4, 8

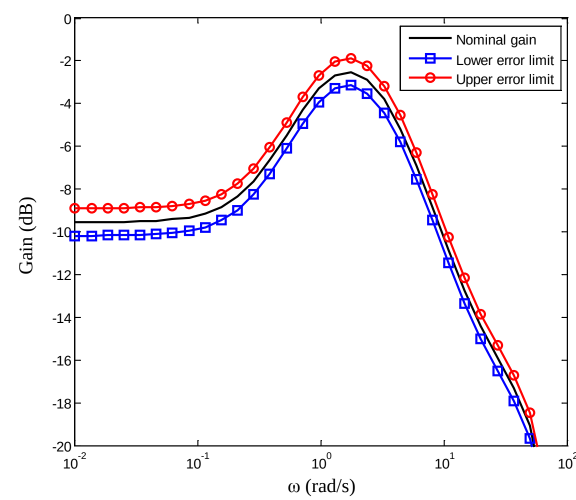

Fig. 8 Frequency response with error margins

V. Experiment Analysis and Verification

To verify the constructed fractional wavelet has a near performance with the traditional wavelet base, an example is presented using numerical simulation method in this section. The input non-stationary signal $s(t)$ is composed of three sinusoidal pulses of 40 *rad*/*s*, 20 *rad*/*s* and 10 *rad*/*s*, respectively. The envelope of this signal is a half sinusoidal wave of 1 *rad*/*s*.

The WT coefficients based on fraction wavelet is shown in Fig. 10. For comparison, Fig. 10 also showed the traditional Gaussian WT with the same input signal. We can see that the proposed fractional wavelet performs the WT and reflects the higher frequencies at lower scales and the lower frequencies at higher scales. The experiment results indicate the fractional WT has better time-frequency resolution for the non-stationary signal and may be employed to signal processing in the application.

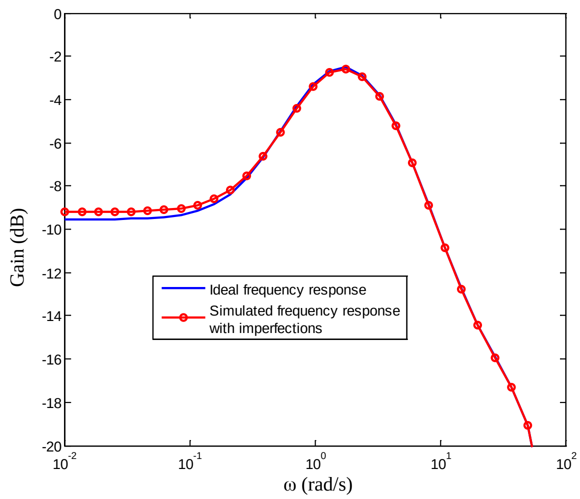

Fig. 9 Frequency response with imperfections

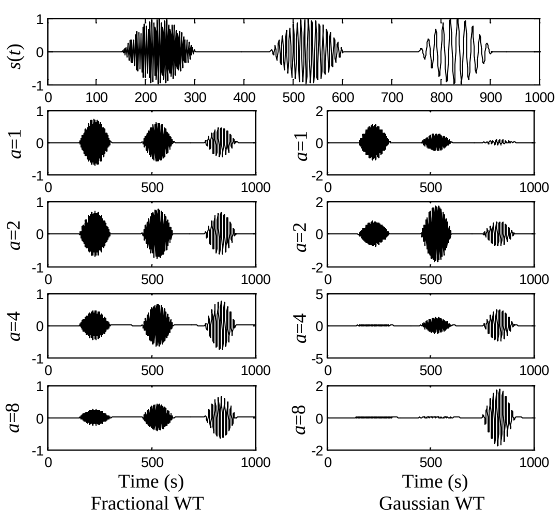

Fig. 10 Fractional WT example

VI. Conclusion

This paper has presented a new fractional wavelet based on a fractional order system and designed the fractional wavelet filter for implementing the fractional WT using current-mode SI circuit. First, a fractional order system is considered and analyzed. Meanwhile, the impulse response of this fractional order system is proved as a mother wavelet, namely a fractional wavelet. Then the transfer function of the fractional order system is approximated by applying CFE method and the fractional wavelet filter is implemented using single SI integrator with simple structure. Finally, the time and frequency domain responses of the designed SI fractional wavelet filter are simulated and analyzed. In additional, the low sensitivity and imperfection of the circuit are analyzed to verify the performance of the designed fractional wavelet circuit. Comparing reported methods in the literature for implementing the WT, the proposed method in this article has the following advantages: (i) a new fractional wavelet is presented based on a fractional order system rather than integer order system. (ii) the designed wavelet circuit has few components and simply structure. (iii) the shown approach can be applied to fractional order filter design using SI circuit.

References

[1] C. K. Chui, *Wavelet analysis and its application*. San Diego: Hartcourt Publishers, 1992.

[2] V. Martin, H, Cormac, "Wavelets and filter banks: theory and design," *IEEE Trans. Signal Processing*, vol. 40, pp. 2207-2231, 1992.

[3] S. Mallat, *A Wavelet Tour of Signal Processing*. New York: Academic, 1999.

[4] S. A. P. Haddad, B. Sumit, W. A. Serdijn, "Log-domain wavelet bases," *IEEE Trans. Circuits Systems*, vol. 52, pp. 2023-2032, 2005.

[5] P. R. Agostinho, S. A. P. Haddad, J. A. DE Lima, W. A. Serdijn, "An ultra low power CMOS PA/V transconductor and its application to wavelet filters," *Analog Integrated Circuits and Signal Processing*, vol. 57, pp.1-14, 2008.

[6] H. Li, Y. He, Y. C. Sun, "Detection of cardiac signal characteristic point using log-domain wavelet transform circuits," *Circuits, systems and signal processing*, vol. 27, pp. 683-698, 2008.

[7] A. N. Akansu, W. A. Serdijn, W. Selesnick, "Emerging applications of wavelets: A review," *Physical Communication*, vol. 3, pp. 1-18, 2010.

[8] W. Zhao, Y. Sun, Y. He, "Minimum component high frequency Gm-C wavelet filters based on Maclaurin series and multiple loop feedback," *Electronic Letters*, vol. 46, pp. 34-36, 2010.

[9] A. J. Casson, E. Rodriguez-Villegas, "A 60pW gm-C continuous wavelet transform circuit for portable EEG systems," *IEEE Journal of Solid-State Circuits*, vol. 46, pp. 1406-1415, 2011.

[10] J. M. H. Karel, S. A. P. Haddad, S. Hiseni, R. L. Westra, W. A. Serdijn, R. L. M. Peeters, "Implementing wavelets in continuous-time analog circuits with dynamic range optimization," *IEEE Trans. Circuits Systems*, vol. 59, pp. 2023-2032, 2012.

[11] M. Li, Y. G. He, "Analog VLSI implementation of wavelet transform using switched-current circuits," *Analog Integrated Circuits and Signal Processing*, vol. 71, pp. 283-291, 2012.

[12] W. S. Zhao, Y. G. He, "Realization of wavelet transform using switched-current filters," *Analog Integrated Circuits and Signal Processing*, vol. 71, pp. 571-581, 2012.

[13] M. Li, Y. G. He, "Analog wavelet transform using multiple-loop feedback switched-current filters and simulated annealing algorithms," *AEÜ-International Journal of Electronics and Communications*, vol. 68, pp. 388-394, 2014.

[14] W. S. Zhao, Y. G. He, Y. C. Sun, "Switched-current filter structure for synthesizing arbitrary characteristics based on follow-the-leader feedback configuration," *Analog Integrated Circuits and Signal Processing*, vol. 82, pp. 479-486, 2015.

[15] W. S. Zhao, Y. G. He, Y. C. Sun, "Structure and realization of pole-shared switched-current complex wavelet filter," *Analog Integrated Circuits and Signal Processing*, vol. 85, pp. 193-199, 2015.

[16] M. Li, Y. G. He, "A simple structure wavelet transform circuit employing function link neural networks and SI filters," *International Journal of Electronics*, vol. 103, pp. 2052-2063, 2016.

[17] M. A. Gurrola-Navarro, G. Espinosa-Flores-Verdad, "Analogue wavelet transform with single biquad stage per scale," *Electronics Letters*, vol. 46, pp. 616-618, 2010.

[18] T. J. Freeborn, B. Maundy, A. S. Elwakil, “Field programmable analogue array implementation of fractional step filters,” *IET Circuits, Devices and Systems*, vol. 4, pp. 514-524, 2010.
[19] L. A. Said, A. G. Radwan, A. H Madian, A. M. Soliman, “Fractional order oscillators based on operational transresistance amplifiers,” *AEÜ-International Journal of Electronics and Communications*, vol. 69, pp. 988-1003, 2015.
[20] B. T. Krishna, “Studies on fractional order differentiators and integrators: A survey,” *Signal Processing*, vol. 91, pp. 386-426, 2011.
[21] A. G. Radwan, A. M. Soliman, A. S. Elwakil, “First-order filters generalized to the fractional domain,” *Journal of Circuits, Systems, and Computers*, vol. 17, pp. 55-66, 2008.
[22] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: John Wiley and Sons, 1993.
[23] C. Toumazou, J. B. Hughes, N. C. Battersby, *Switched-currents: an analogue technique for digital technology*, London: Peter Peregrinus Ltd., 1993.
[24] A. C. M. De Queiroz, P. R. M. Pinheriro, L. P. Caloba, “Nodal analysis of switched-current filters,” *IEEE Trans. Circuits and Systems*, vol. 40, pp. 10-18, 1993.

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