Convergence of the Euler Method in Probability to SDEs under the Generalized Khasminskii-type Conditions

Hui Yu

***Abstract*—**The Euler method is introduced for stochastic differential equations (SDEs) with piecewise continuous arguments driven by Poisson process under the generalized Khasminskii-type conditions which cover more classes of such equations than classical conditions. To our known, few results are presented to such equations in current literature. Here, three results are obtained for such equations. Firstly, the existence and uniqueness of global solutions to such equations are proved by Itô formula and mathematical induction. Secondly, the Euler method with a given step-size is constructed. Lastly, the convergence of the Euler method in probability for such equations under the generalized Khasminskii-type conditions is investigated by means of the continuous-time Euler method. All the results show that on the basis of the existence of such equations, the Euler method is convergent in probability under the Khasminskii-type conditions. Moreover, some numerical examples are given to the results.

***Keywords*—**Stochastic differential equations, Poisson process, Piecewise continuous arguments, Convergence in probability, Euler method.

I. INTRODUCTION

THE SDEs, Poisson process and piecewise continuous arguments are important and are applied widely in mathematical modeling in many fields (see [1]). The explicit solutions of SDEs with piecewise continuous arguments driven by Poisson process can hardly be got. Therefore, it is significative in theory and in application to investigate appropriate numerical methods and their properties.

As far as I know, few results about convergence of the Euler method in probability to SDEs with piecewise continuous arguments driven by Poisson process, which are important and widely used, were presented under some conditions. In [2-4], the convergence of the Euler method and the implicit Euler method is proved respectively for SDEs with Poisson process. In [5], the exact convergence rate of the Euler method is considered for SDEs driven by a homogeneous Poisson process with intensity. Moreover, great progress has been made in the research work on the equations with piecewise continuous arguments. In [6], the Oscillatory and asymptotic behavior are presented for third order differential equations. Other authors also focused on the property of periodic solutions in [7, 8].

So far, there are many SDEs with piecewise continuous arguments driven by Poisson process, such as highly nonlinear equations, which do not satisfy the classical conditions and satisfy the generalized Khasminskii-type conditions (see [9]). The primary purpose here is to fill the gap in the convergence with the Euler method for such equations under the Khasminskii-type conditions. To show the major aim, the generalized Khasminskii-type conditions are presented in section II, the existence and the property of the global solution to such SDEs under the conditions are proved in section III, the Euler method with a given step-size is introduced and some properties are proved in section IV, and the convergence in probability of the Euler method under the generalized Khasminskii-type conditions is proved in section V.

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Hui Yu is with the College of Science, Heilongjiang Bayi Agricultural University, Daqing 163319, Heilongjiang, China (corresponding author; e-mail: yuhui163@163.com).

II. PRELIMINARY

Throughout this paper, $|\cdot|$ is the Euclidean norm in $\mathrm{R}^d, d\in\mathrm{N}$, If A is a matrix, its trace norm is denoted by $|A|=\sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$. $\mathrm{L}^2_{F_0}(\Omega;\mathrm{R}^d)$ and denotes the family of R^d -measurable random variables ξ with $\mathrm{E}|\xi|^2<\infty$. $u_1\vee u_2=\max\{u_1,u_2\}$, $u_1\wedge u_2=\min\{u_1,u_2\}$. The indicator function is $\mathrm{I}_A(x)=\begin{cases}1, & x\in A\\ 0, & x\notin A\end{cases}$. $\inf\Phi=\infty$, Φ denotes empty set. $[z]$ denotes the max integer which is less than or equal to z in R.

The following d-dimensional SDEs with piecewise continuous arguments driven by Poisson process is concerned in my paper

$$\begin{aligned}\mathrm{d}x(t)=&\ f(x(t^-),x([t^-]))\mathrm{d}t+g(x(t^-),x([t^-]))\mathrm{d}W(t)\\&+h(x(t^-),x([t^-]))\mathrm{d}\tilde{N}(t)\end{aligned}\tag{1}$$

where $t>0$, initial value $x_0\in\mathrm{L}^2_{F_0}(\Omega;\mathrm{R}^d)$ and $x(t^-)=\lim_{s\to t^-}x(s)$.

The drift coefficient $f:\mathrm{R}^d\times\mathrm{R}^d\to\mathrm{R}^d$, the diffusion coefficient $g:\mathrm{R}^d\times\mathrm{R}^d\to\mathrm{R}^d$, and the jump coefficient

$h:\mathrm{R}^d\times\mathrm{R}^d\to\mathrm{R}^d$ are assumed to be Borel measurable functions and the coefficients are sufficiently smooth.

A one-dimensional Brownian motion $W(t)$ is defined on a filtered probability space $(\Omega^W, F^W, (F^W{}_t)_{t\ge 0}, \mathrm{P}^W)$. A Poisson process $N(t)$ with intensity λ is defined on a filtered probability space $(\Omega^P, F^P, (F^P{}_t)_{t\ge 0}, \mathrm{P}^P)$ and $\tilde{N}(t)=N(t)-\lambda t$ is a martingale. The process $x(t)$ is thus defined on a product space $(\Omega, F, (F_t)_{t\ge 0}, \mathrm{P})$, where $\Omega=\Omega^W\times\Omega^P$, $F=F^W\times F^P$, $(F_t)_{t\ge 0}=(F_t)_{t\ge 0}{}^W\times(F_t)_{t\ge 0}{}^P$, $\mathrm{P}=\mathrm{P}^W\times\mathrm{P}^P$ and F_0 contain all P-null sets. The Brownian motion $W(t)$ and the Poisson process $N(t)$ are mutually independent.

In the following, the generalized Khasminskii-type conditions are given in the form of four assumptions.

Assumption 1 For each positive integer k, there is a positive constant C_k such that

$$\begin{aligned}&|f(x,y)-f(\bar{x},\bar{y})|^2\vee|g(x,y)-g(\bar{x},\bar{y})|^2\\&\le C_k(|x-y|^2+|\bar{x}-\bar{y}|^2)\end{aligned}\tag{2}$$

where $|x|\vee|y|\vee|\bar{x}|\vee|\bar{y}|\le k,\ x,y,\bar{x},\bar{y}\in\mathrm{R}^d$.

Assumption 2 There is a positive constant C such that

$$|h(x,y)-h(\bar{x},\bar{y})|^2\le C(|x-y|^2+|\bar{x}-\bar{y}|^2)\tag{3}$$

where $x,y,\bar{x},\bar{y}\in\mathrm{R}^d$.

Assumption 3 There exists a function $V\in C(\mathrm{R}^d;\mathrm{R}_+)$ and a constant μ such that

$$\lim_{|x|\to\infty}V(x)=\infty\tag{4}$$

and

$$LV(x,y)\le\mu(1+V(x)+V(y))\tag{5}$$

where the operator is defined by

$$\begin{aligned}LV(x,y)&=V_x(x)f(x,y)+\frac{1}{2}\mathrm{trace}(g^{\mathrm{T}}(x,y)V_{xx}(x)g(x,y))\\&\quad+V(x+h(x,y))-V(x)-V_x(x)h(x,y)\end{aligned}\tag{6}$$

and $x,y,\bar{x},\bar{y}\in\mathrm{R}^d$.

Assumption 4 There is a positive constant L such that

$$|h(0,0)|^2\le L\tag{7}$$

III. Existence of Global Solutions

In this section, the existence and the property of the global solution to (1) under the generalized Khasminskii-type conditions are given.

To demonstrate the existence of the global solution to (1), the following concepts are given.

Definition 1 Let $x(t)$ be an R^d-valued stochastic process. The process is said to be càdlàg if it is right and for almost all $\omega\in\Omega$ the left limit $\lim_{s\to t^-}x(s)$ exists and is finite for all $t\ge 0$.

Definition 2 Let τ_k be a stopping time such that $0\le\tau_k\le T$ a.s. An R^d -valued F_t -adapted and càdlàg process $\{x(t):0\le t<\tau_\infty\}$ is called a local solution of (1) on $t\ge 0$ if there is a nondecreasing sequence $\{\tau_k\}_{k\ge 1}$ of stopping times such that $0\le\tau_k\uparrow\tau_\infty$ a.s. and

$$\begin{aligned}x(t\wedge\tau_k)&=\int_0^{t\wedge\tau_k}f(x(s^-),x([s^-]))\mathrm{d}s\\&\quad+\int_0^{t\wedge\tau_k}g(x(s^-),x([s^-]))\mathrm{d}W(s)\\&\quad+\int_0^{t\wedge\tau_k}h(x(s^-),x([s^-]))\mathrm{d}\tilde{N}(s)\end{aligned}$$

holds any for $t\ge 0$ and $k\ge 1$ with probability 1. If, furthermore,

$$\limsup_{t\to\tau_\infty}|x(t)|=\infty,\quad \tau_\infty<T$$

then it is called a maximal local solution of (1) and τ_∞ is called the explosion time. A local solution $\{x(t):0\le t<\tau_\infty\}$ to (1) is called a global solution if $\tau_\infty=\infty$.

Lemma 1 Under Assumption 1 and 4, there exists a unique maximal local solution to (1).

Proof: For each integer $k\ge 1$, we define

$$Z^{(k)}=\frac{|z|\wedge k}{|z|}z,\quad 0^{(k)}=0$$

for $z\in\mathrm{R}^d$. And then we define the truncation functions

$$\begin{aligned}f_k(x,y)&=f(x^{(k)},y^{(k)})\\g_k(x,y)&=g(x^{(k)},y^{(k)})\\h_k(x,y)&=h(x^{(k)},y^{(k)})\end{aligned}$$

for $x,y\in\mathrm{R}^d$ and $k\ge 1$. Moreover, we define the following equation

$$\begin{aligned}\mathrm{d}x_k(t)&=f_k(x_k(t^-),x_k([t^-]))\mathrm{d}t+g_k(x_k(t^-),x_k([t^-]))\mathrm{d}W(t)\\&\quad+h_k(x_k(t^-),x_k([t^-]))\mathrm{d}\tilde{N}(t)\end{aligned}$$

on $t \in [0,T]$.

Obviously, the equation satisfies the global Lipschitz conditions and the linear growth conditions. Therefore according to [10], there is a unique global solution $x_k(t)$ and its solution is a càdlàg process. And the stopping time are defined by

$$\sigma_\infty := \lim_{k\to\infty} \sigma_k$$

and

$$\sigma_k := \inf\{t \ge 0 : |x(t)| \ge k, k \in \mathrm{N}_+\}$$

which means $\{\sigma_k\}_{k\ge1}$ is a nondecreasing sequence. We define

$$\{x(t) : 0 \le t < \sigma_\infty\}$$

and

$$x(t)=x_k(t), \quad t \in [\sigma_{k-1}, \sigma_k), \ k \ge 1, \ \sigma_0 = 0$$

We can have

$$\begin{aligned} x(t\wedge\sigma_k) = &\int_0^{t\wedge\sigma_k} f(x(s^-), x([s^-]))\mathrm{d}s \\ &+ \int_0^{t\wedge\sigma_k} g(x(s^-), x([s^-]))\mathrm{d}W(s) \\ &+ \int_0^{t\wedge\sigma_k} h(x(s^-), x([s^-]))\mathrm{d}\tilde{N}(s) \end{aligned}$$

for any $t \in [0,T]$ and $k \ge 1$ with probability 1. Moreover, if $\sigma_\infty < T$, then

$$\limsup_{t\to\sigma_\infty} |x(t)| \ge \limsup_{k\to\infty} |x(\sigma_k)| = \limsup_{k\to\infty} |x_k(\sigma_k)| = \infty$$

Therefore $\{x(t) : 0 \le t < \sigma_\infty\}$ is a maximal local solution to (1). To show the uniqueness of the solution to (1), let $\{\overline{x(t)} : 0 \le t < \overline{\sigma_\infty}\}$ be another maximal local solution. As the same proof as [10], we infer that

$$\mathrm{P}(x(t,\omega)=\bar{x}(t,\omega), \quad (t,\omega) \in [0, \sigma_k \wedge \overline{\sigma_k}) \times \Omega) = 1, \ k \ge 1$$

Taking $k \to \infty$, we get

$$\mathrm{P}(x(t,\omega)=\bar{x}(t,\omega), \quad (t,\omega) \in [0, \sigma_\infty \wedge \overline{\sigma_\infty}) \times \Omega) = 1, \ k \ge 1$$

Hence $x(t)$ is a unique local solution and then it is a unique maximal local solution to (1).

Theorem 1 Under Assumption 1-4, there exists a unique global solution to (1).

Proof: According to Lemma 1, there is a unique maximal local solution to (1) on $[0,\infty)$. Therefore, in order to prove that this local solution is a global one, it only needs to demonstrate $\sigma_\infty = \infty$ a.s.. Using Itô's formula to $V(x(t))$ (see [2]), we have

$$\begin{aligned} \mathrm{d}V(x(t)) = &LV(x(t^-), x([t^-]))\mathrm{d}t \\ &+ V_x(x(t^-))g(x(t^-), x([t^-]))\mathrm{d}W(t) \\ &+ (V(x(t^-) + h(x(t^-), x([t^-])) - V(x(t^-)))\mathrm{d}\tilde{N}(t) \end{aligned} \tag{8}$$

for $t \in [0, \sigma_\infty)$.

For any positive integer k and $0 \le t < 1$, by taking integration and expectations and using Assumption 2 and Assumption 3 to (8), we can have

$$\mathrm{E}V(x(t\wedge\sigma_k)) \le C_1 + \mu\int_0^{t\wedge\sigma_k} \mathrm{E}V(x((s\wedge\sigma_k)^-))\mathrm{d}s \tag{9}$$

where $C_1 = \mu + (\mu+1)\mathrm{E}V(x(0))$. Using the Gronwall inequality (see [9]) leads to

$$\mathrm{E}V(x(t\wedge\sigma_k)) \le C_1 e^{\mu t}, \quad 0 \le t < 1 \tag{10}$$

which means

$$\mathrm{E}V(x(1\wedge\sigma_k)) \le \lim_{t\to1} \mathrm{E}V(x(t\wedge\sigma_k)) \le C_1 e^{\mu} \tag{11}$$

by taking limit. Thus, from (10) and (11), we get

$$\mathrm{E}V(x(t\wedge\sigma_k)) \le C_1 e^{\mu t}, \quad 0 \le t \le 1 \tag{12}$$

Let $\varsigma_k := \inf_{|x|\ge k, 0\le t<\infty} V(x), \quad k \ge 1$, and we obtain

$$\varsigma_k \mathrm{P}(\sigma_k \le 1) \le \mathrm{E}(V(x(\sigma_k))\mathrm{I}_{\{\sigma_k \le 1\}}) \le \mathrm{E}V(x(1\wedge\sigma_k)) \le C_1 e^{\mu} \tag{13}$$

by taking $k \to \infty$ to (13), which gives $\mathrm{P}(\sigma_\infty \le 1)=0$. Hence we have

$$\mathrm{P}(\sigma_\infty > 1)=1 \tag{14}$$

It thus follows from (12) and (14) that

$$\mathrm{E}V(x(t)) \le C_1 e^{\mu t}, \quad 0 \le t \le 1.$$

So for any positive integer i, we repeat the similar analysis as above, then get

$$\mathrm{E}V(x(t\wedge\sigma_k)) \le C_i e^{\mu}, \quad i-1 \le t \le i$$

$$\mathrm{P}(\sigma_\infty > i)=1$$

and

$$\mathrm{E}V(x(t)) \le C_i e^{\mu}, \quad i-1 \le t \le i$$

where

$$C_i = \mu + (\mu+1)\mathrm{E}V(x(i-1)) < \infty .$$

So we can get $\mathrm{P}(\sigma_\infty = \infty)$=1. Therefore, (1) has a unique global solution $x(t)$ on $[0,\infty)$.

Lemma 2 Under Assumption 1-3, for any $\varepsilon \in (0,1)$ and $T > 0$, we can find a sufficiently large integer k^* such that

$$\mathrm{P}(\sigma_k \le T) \le \varepsilon, \quad \forall k \ge k^*.$$

Proof: For any $T > 0$, there exists a positive integer i such that $i-1 \le T \le i$. Therefore, according to Theorem 1, we have

$$\mathrm{E}V(x(T \wedge \sigma_k)) \le C_i e^{\mu} < \infty, \quad k > 1$$

which leads to

$$\varsigma_k \mathrm{P}(\sigma_k \le T) \le \mathrm{E}(V(x(\sigma_k))\mathrm{I}_{\{\sigma_k \le T\}}) \le \mathrm{E}V(x(T \wedge \sigma_k)) \le C_i e^{\mu}$$

Under Assumption 3, there exists a sufficiently large integer k^* such that

$$\mathrm{P}(\sigma_k \le T) \le \frac{C_i e^{\mu}}{\varsigma_k} \le \varepsilon, \quad \forall k \ge k^*.$$

IV. Euler Method

In this section, the Euler method is introduced to (1) under the generalized Khasminskii-type conditions.

Given a step size $\Delta t = \dfrac{1}{m} \in (0,1), m \in \mathrm{N}_+$, the Euler method applied (1) computes the approximation

$$\begin{aligned} X_{km+l+1} = X_{km+l} &+ f(X_{km+l}, X_{km})\Delta t + g(X_{km+l}, X_{km})\Delta W_{km+l} \\ &+ h(X_{km+l}, X_{km})\Delta \tilde{N}_{km+l} \end{aligned} \tag{15}$$

where $X_0 \approx x(0)$, $X_{km+l} \approx x(t_{km+l})$, $t_{km+l} = (km+l)\Delta t$, $\Delta W_{km+l} = W(t_{km+l+1}) - W(t_{km+l})$, $\Delta \tilde{N}_{km+l} = \tilde{N}(t_{km+l+1}) - \tilde{N}(t_{km+l})$, $k = 0,1, \cdots$, $l = 0,1,\cdots,m-1$.

The continuous-time Euler method is given by

$$\begin{aligned} \overline{X} := X_0 &+ \int_0^t f(Z_1(s), Z_2(s))\mathrm{d}s + \int_0^t g(Z_1(s), Z_2(s))\mathrm{d}W(s) \\ &+ \int_0^t h(Z_1(s), Z_2(s))\mathrm{d}\tilde{N}(s) \end{aligned} \tag{16}$$

where $Z_1 = X_{km+l}$, $Z_2 = X_{km}$, $t \in [t_{km+l}, t_{km+l+1})$.

In order to analyze the Euler method, two lemmas are given firstly.

Lemma 3 Under Assumption 1, 2 and 4, for any $T > 0$ and $\Delta t \in (0,1)$, we can find a positive constant K_1 such that the continuous-time Euler method (16) satisfies

$$\mathrm{E}|\overline{X}(t) \text{-} Z_1(t)|^2 \le K_1 \Delta t$$

where $0 \le t \le T \wedge \sigma_k \wedge \rho_k$, $\rho_k := \inf\{t \ge 0 : |\overline{X}(t)| \ge k\}$, $k \ge 1$.

Proof: For $0 \le t \le T \wedge \sigma_k \wedge \rho_k$, $k \ge 1$, there exists two positive integer k, l such that $t \in [(km+l)\Delta t, (km+l+1)\Delta t)$, (16) can leads to

$$\begin{aligned} \overline{X}(t) - Z_1(t) = X_{km+l} &+ \int_{t_{km+l}}^t f(Z_1(t), Z_2(t))\mathrm{d}s \\ &+ \int_{t_{km+l}}^t g(Z_1(t), Z_2(t))\mathrm{d}W(s) \\ &+ \int_{t_{km+l}}^t h(Z_1(t), Z_2(t))\mathrm{d}\tilde{N}(s) - X_{km+l} \end{aligned} \tag{17}$$

By taking expectations, the martingale properties and the Cauchy-Schwarz inequality to (17), we get

$$\begin{aligned} &\mathrm{E}|\overline{X}(t) \text{-} Z_1(t)|^2 \\ &\le 3\Delta t \mathrm{E}\int_{t_{km+l}}^t |f(Z_1(t), Z_2(t))|^2 \mathrm{d}s \\ &\quad + 3\mathrm{E}\int_{t_{km+l}}^t |g(Z_1(t), Z_2(t))|^2 \mathrm{d}s \\ &\quad + 3\mathrm{E}\int_{t_{km+l}}^t |g(Z_1(t), Z_2(t), v)|^2 \mathrm{d}s \end{aligned} \tag{18}$$

It follows from Assumption 1 and 2 that

$$\begin{aligned} &\mathrm{E}\int_{t_{km+l}}^t |f(Z_1(t), Z_2(t))|^2 \mathrm{d}s \\ &\le 2\mathrm{E}\int_{t_{km+l}}^t |f(Z_1(t), Z_2(t)) - f(0,0)|^2 \mathrm{d}s + 2\mathrm{E}\int_{t_{km+l}}^t |f(0,0)|^2 \mathrm{d}s \\ &\le 4k^2 C_k \Delta t + 2|f(0,0)|^2 \ \Delta t \end{aligned}$$

$$\mathrm{E}\int_{t_{km+l}}^t |g(Z_1(t), Z_2(t))|^2 \mathrm{d}s \le 4k^2 C_k \Delta t + 2|g(0,0)|^2 \ \Delta t$$

and

$$\mathrm{E}\int_{t_{km+l}}^t |g(Z_1(t), Z_2(t), v)|^2 \mathrm{d}s \le 4k^2 C_k \Delta t + 2\Delta t |g(0,0)|^2$$

From the above four inequalities, we have

$$\mathrm{E}|\overline{X}(t)\text{-}Z_1(t)|^2 \le K_1(k)\Delta t,\quad 0\le t\le T\wedge\sigma_k\wedge\rho_k,\ k\ge 1$$

Where

$$K_1(k)=24k^2C_k+12k^2C+6|f(0,0)|^2 \\ +6|g(0,0)|^2+6|h(0,0)|^2 .$$

Lemma 4 Under Assumption 1-4, for any $\varepsilon\in(0,1)$ and $T>0$, we can find a sufficiently large integer k^* and a sufficiently small Δt_1^* such that

$$\mathrm{P}(\rho_{k^*}\le T)\le\varepsilon,\quad \forall\Delta t\le\Delta t_1^*$$

Proof: Applying Itô's formula to $V(\overline{X}(t))$, we have

$$\begin{aligned} \mathrm{d}V(\overline{X}(t)) &= LV(\overline{X}(t),\overline{X}([t]))\mathrm{d}t \\ &\quad +a(\overline{X}(t),\overline{X}([t]),Z_1(t),Z_2(t))\mathrm{d}t \\ &\quad +V_x(\overline{X}(t))g(Z_1(t),Z_2(t))\mathrm{d}W(t) \\ &\quad +(V(\overline{X}(t)+h(Z_1(t),Z_2(t)))-V(\overline{X}(t)))\mathrm{d}\tilde{N}(t) \end{aligned} \tag{19}$$

where the function $a(x,y,Z_1,Z_2):\mathrm{R}^d\times\mathrm{R}^d\times\mathrm{R}^d\times\mathrm{R}^d\to\mathrm{R}$ is defined by

$$\begin{aligned} &a(x,y,Z_1,Z_2) \\ &=V_x(x)(f(Z_1,Z_2)-f(x,y)) \\ &\quad+\frac{1}{2}\mathrm{trace}(g^T(Z_1,Z_2)V_{xx}(x)g(Z_1,Z_2)-g^T(x,y)V_{xx}(x)g(x,y)) \\ &\quad+(V_x(x)h(x,y,v)-V_x(x)h(Z_1,Z_2,v) \\ &\quad+V(x+h(Z_1,Z_2,v))-V(x+h(x,y,v))) \end{aligned}$$

From Assumption 1-3, it is obvious that

$$a(x,y,Z_1,Z_2)\le L_k(|x-Z_1|+|y-Z_2|) \tag{20}$$

where $L_k>0$. when $0\le t\le T,\ k\ge 1$, it follows from Lemma 3 and (20) that

$$\begin{aligned} &\mathrm{E}\int_0^{t\wedge\rho_k} a(\overline{X}(s),\overline{X}([s]),Z_1(s),Z_2(s))\mathrm{d}s \\ &\le L_k\mathrm{E}\int_0^{t\wedge\rho_k}|\overline{X}(s)\text{-}Z_1(s)|\mathrm{d}s+L_k\mathrm{E}\int_0^{t\wedge\rho_k}|\overline{X}([s])\text{-}Z_2(s)|\mathrm{d}s \\ &\le L_k\int_0^t(\mathrm{E}|\overline{X}(s\wedge\rho_k)\text{-}Z_1(s\wedge\rho_k)|^2)^{1/2}\mathrm{d}s+0 \\ &\le L_kT\sqrt{K_1(k)\Delta t} \end{aligned} \tag{21}$$

Therefore, by means of taking expectations and integration to (19), we can have

$$\begin{aligned} &\mathrm{E}V(\overline{X}(t\wedge\rho_k)) \\ &=\mathrm{E}V(X_0)+\mathrm{E}\int_0^{t\wedge\rho_k}LV(\overline{X}(s),\overline{X}([s]))\mathrm{d}s \\ &\quad+\mathrm{E}\int_0^{t\wedge\rho_k}a(\overline{X}(s),\overline{X}([s]),Z_1(s),Z_2(s))\mathrm{d}s \\ &\le\mathrm{E}V(X_0)+L_kT\sqrt{K_1(k)\Delta t} \\ &\quad+\mathrm{E}\int_0^{t\wedge\rho_k}\mu(1+V(\overline{X}(s))+V(\overline{X}([s])))\mathrm{d}s \end{aligned} \tag{22}$$

When $0\le t<1,\ k\ge 1$, it follows from (22) that

$$\begin{aligned} \mathrm{E}V(\overline{X}(t\wedge\rho_k)) &\le\alpha_1+L_kT\sqrt{K_1(k)\Delta t} \\ &\quad+\mu\mathrm{E}\int_0^t V(\overline{X}(s\wedge\rho_k))\mathrm{d}s \end{aligned} \tag{23}$$

where $\alpha_1=\mu+(\mu+1)\mathrm{E}V(X_0)<\infty.$ Using the Gronwall inequality to (23) , we get

$$\mathrm{E}V(\overline{X}(t\wedge\rho_k))\le(\alpha_1+L_kT\sqrt{K_1(k)\Delta t})e^{\mu t},\ 0\le t<1,\ k\ge 1 \tag{24}$$

by taking limit, which leads to

$$\mathrm{E}V(\overline{X}(1\wedge\rho_k))=\lim_{t\to 1}\mathrm{E}V(\overline{X}(t\wedge\rho_k))\le(\alpha_1+L_kT\sqrt{K_1(k)\Delta t})e^{\mu} \tag{25}$$

From (24) and (25), we obtain

$$\mathrm{E}V(\overline{X}(t\wedge\rho_k))\le(\alpha_1+L_kT\sqrt{K_1(k)\Delta t})e^{\mu t} \tag{26}$$

Let $\upsilon_k:=\inf_{|X|\ge k}V(\overline{X}),\ k\ge 1,\ \rho_\infty:=\lim_{k\to\infty}\rho_k$. It follows from that

$$\begin{aligned} &\upsilon_k\mathrm{P}(\rho_k\le 1)\le\mathrm{E}(V(\overline{X}(\rho_k))\mathrm{I}_{\{\rho_k\le 1\}}) \\ &\le\mathrm{E}V(\overline{X}(1\wedge\rho_k))\le(\alpha_1+L_kT\sqrt{K_1(k)\Delta t})e^{\mu} \end{aligned}$$

which gives that $\mathrm{P}(\rho_\infty\le 1)=0$, that is,

$$\mathrm{P}(\rho_\infty>1)=1 . \tag{27}$$

In (26), by taking $k\to\infty$ and using (27), we have

$$\mathrm{E}V(\overline{X}(t))\le(\alpha_1+L_kT\sqrt{K_1(k)\Delta t})e^{\mu}<\infty,\quad 0\le t\le 1,\ k\ge 1.$$

For any positive integer i, we repeat the similar analysis as above, then we have

$$\mathrm{E}V(\overline{X}(t\wedge\rho_k))\le(\alpha_i+L_kT\sqrt{K_1(k)\Delta t})e^{\mu t} \tag{28}$$

where $i-1\le t\le i,\ k\ge 1,\ \alpha_i=\mu+(\mu+1)\mathrm{E}V(\overline{X}(i-1))<\infty.$

For any $T>0$, there exists a positive integer i such that $i-1\le T\le i$. It follows from (28) that

$$\begin{aligned}&\upsilon_k \mathrm{P}(\rho_k\le T)\le \mathrm{E}(V(\overline{X}(\rho_k))\mathrm{I}_{\{\rho_k\le T\}})\\&\le \mathrm{E}V(\overline{X}(T\wedge\rho_k))\le(\alpha_i+L_kT\sqrt{K_1(k)\Delta t})e^{\mu},\ k\ge 1\end{aligned}\tag{29}$$

Moreover, for any $\varepsilon\in(0,1)$, there are a sufficiently large integer k^* and a sufficiently small Δt_1^* such that

$$\frac{\alpha_i e^{\mu}}{\upsilon_{k^*}}\le\frac{\varepsilon}{2}\tag{30}$$

and

$$\frac{e^{\mu}L_{k^*}T\sqrt{K_1(k^*)\Delta t_1^*}}{\upsilon_{k^*}}\le\frac{\varepsilon}{2}\tag{31}$$

So, from (29)-(31), we can obtain that

$$\mathrm{P}(\rho_{k^*}\le T)\le\varepsilon,\quad \forall\Delta t\le\Delta t_1^*.$$

V. CONVERGENCE IN PROBABILITY

In this section, we firstly give the following lemma in order to prove the convergence in probability of the Euler method (15) under the generalized Khasminskii-type conditions (2)-(7).

Lemma 5 Under Assumption 1, 3 and 4, for any $T>0$ and $\Delta t\in(0,1)$, we can find a positive constant $K_2(k)$ such that (1) and (16) satisfy

$$\mathrm{E}(\sup_{0\le t\le T}|x(t\wedge\sigma_k\wedge\rho_k)-\overline{X}(t\wedge\sigma_k\wedge\rho_k)|^2)\le K_2(k)\Delta t,\quad k\ge 1$$

Proof: For $0\le t'\le T$ and $k\ge 1$, it follows from (1) and (16) that

$$\begin{aligned}&\mathrm{E}(\sup_{0\le t\le t'}|x(t\wedge\sigma_k\wedge\rho_k)-\overline{X}(t\wedge\sigma_k\wedge\rho_k)|^2)\\&\le 3\mathrm{E}(\sup_{0\le t\le t'}|\int_0^{t\wedge\sigma_k\wedge\rho_k}(f(x(s^-),x([s^-]))-f(Z_1(s),Z_2(s)))\mathrm{d}s|^2)\\&+3\mathrm{E}(\sup_{0\le t\le t'}|\int_0^{t\wedge\sigma_k\wedge\rho_k}(g(x(s^-),x([s^-]))-g(Z_1(s),Z_2(s)))\mathrm{d}W(s)|^2)\\&+3\mathrm{E}(\sup_{0\le t\le t'}|\int_0^{t\wedge\sigma_k\wedge\rho_k}(h(x(s^-),x([s^-]))-h(Z_1(s),Z_2(s)))\mathrm{d}\tilde{N}(s)|^2)\end{aligned}$$

By means of the Cauchy-Schwarz inequality, (2), (6) and Fubini's Theorem, we can have

$$\begin{aligned}&\mathrm{E}(\sup_{0\le t\le t'}|\int_0^{t\wedge\sigma_k\wedge\rho_k}(f(x(s^-),x([s^-]))-f(Z_1(s),Z_2(s)))\mathrm{d}s|^2)\\&\le\mathrm{E}(\sup_{0\le t\le t'}\int_0^{t\wedge\sigma_k\wedge\rho_k}1^2\mathrm{d}s\int_0^{t\wedge\sigma_k\wedge\rho_k}|f(x(s^-),x([s^-]))\\&\quad-f(Z_1(s),Z_2(s))|^2\,\mathrm{d}s)\\&\le 2TC_k\mathrm{E}(\int_0^{t'\wedge\sigma_k\wedge\rho_k}|x(s^-)-\overline{X}(s)|^2\,\mathrm{d}s)\\&+2TC_k\mathrm{E}(\int_0^{t'\wedge\sigma_k\wedge\rho_k}|\overline{X}(s)-Z_1(s)|^2\,\mathrm{d}s)\\&+2TC_k\mathrm{E}(\int_0^{t'\wedge\sigma_k\wedge\rho_k}|x([s^-])-\overline{X}([s])|^2\,\mathrm{d}s)\\&+2TC_k\mathrm{E}(\int_0^{t'\wedge\sigma_k\wedge\rho_k}|\overline{X}([s])-Z_2(s)|^2\,\mathrm{d}s)\\&\le 4TC_k\int_0^{t'}\mathrm{E}(\sup_{0\le u\le s}|x(u\wedge\sigma_k\wedge\rho_k^-)-\overline{X}(u\wedge\sigma_k\wedge\rho_k)|^2)\mathrm{d}s\\&+2T^2C_kK_1(k)\Delta t\end{aligned}\tag{32}$$

Moreover, using the property of martingale of $\mathrm{d}W(t)$ and $\mathrm{d}\tilde{N}(t)$, we get

$$\begin{aligned}&\mathrm{E}(\sup_{0\le t\le t'}|\int_0^{t\wedge\sigma_k\wedge\rho_k}(g(x(s^-),x([s^-]))-g(Z_1(s),Z_2(s)))\mathrm{d}W(s)|^2)\\&\le 16C_k\int_0^{t'}\mathrm{E}(\sup_{0\le u\le s}|x(u\wedge\sigma_k\wedge\rho_k^-)-\overline{X}(u\wedge\sigma_k\wedge\rho_k)|^2)\mathrm{d}s\\&+8TC_kK_1(k)\Delta t\end{aligned}\tag{33}$$

and

$$\begin{aligned}&\mathrm{E}(\sup_{0\le t\le t'}|\int_0^{t\wedge\sigma_k\wedge\rho_k}(h(x(s^-),x([s^-]))-h(Z_1(s),Z_2(s)))\mathrm{d}\tilde{N}(s)|^2)\\&\le 16C\int_0^{t'}\mathrm{E}(\sup_{0\le u\le s}|x(u\wedge\sigma_k\wedge\rho_k^-)-\overline{X}(u\wedge\sigma_k\wedge\rho_k)|^2)\mathrm{d}s\\&+8TCK_1(k)\Delta t\end{aligned}\tag{34}$$

Therefore, from (32)-(34), we have the result

$$\mathrm{E}(\sup_{0\le t\le T}|x(t\wedge\sigma_k\wedge\rho_k)-\overline{X}(t\wedge\sigma_k\wedge\rho_k)|^2)\le K_2(k)\Delta t,\quad k\ge 1$$

where

$$\begin{aligned}K_2(k)=&(6T^2C_kK_1(k)+24TC_kK_1(k)+24TCK_1(k))\\&\cdot\exp(12T^2C_k+48TC_k+48TC).\end{aligned}$$

In the following lemma and theorem, we demonstrate the convergence in probability of the Euler method (15) under the generalized Khasminskii-type conditions (2)-(7).

Lemma 6 Under Assumption 1-4, for any $T>0$ and sufficiently small $\varepsilon,\varsigma\in(0,1)$, we can find Δt^* such that

$$\mathrm{P}(\sup_{0\le t\le T}|x(t)-\overline{X}(t)|^2\ge\varsigma)\le\varepsilon,\quad \Delta t<\Delta t^*.$$

Proof: For sufficiently small $\varepsilon, \varsigma \in (0,1)$, we give the definition that

$$\overline{\Omega}=\{\omega: \sup_{0\le t\le T} |x(t)-\overline{X}(t)|^2 \ge \varsigma\}$$

According to Lemma 2 and Lemma 4, we can find k^* and Δt_1^* such that

$$P(\sigma_{k^*} \le T) \le \frac{\varepsilon}{3}$$

and

$$P(\rho_{k^*} \le T) \le \frac{\varepsilon}{3}, \ \forall \Delta t \le \Delta t_1^*,$$

which give

$$\begin{aligned} P(\overline{\Omega}) &\le P(\overline{\Omega}\cap\{\sigma_{k^*}\wedge\rho_{k^*}>T\})+P(\sigma_{k^*}\wedge\rho_{k^*}\le T) \\ &\le P(\overline{\Omega}\cap\{\sigma_{k^*}\wedge\rho_{k^*}>T\})+P(\sigma_{k^*}\le T)+P(\rho_{k^*}\le T) \\ &\le P(\overline{\Omega}\cap\{\sigma_{k^*}\wedge\rho_{k^*}>T\})+\frac{2\varepsilon}{3} \end{aligned}$$

where $\Delta t \le \Delta t_1^*$. Moreover, it follows from Lemma 5 that

$$\begin{aligned} &\varsigma P(\overline{\Omega}\cap\{\sigma_{k^*}\wedge\rho_{k^*}>T\}) \\ &\le E(I_{\{\sigma_{k^*}\wedge\rho_{k^*}>T\}} \sup_{0\le t\le T}|x(t)-\overline{X}(t)|^2) \\ &\le E(\sup_{0\le t\le T}|x(t\wedge\sigma_{k^*}\wedge\rho_{k^*})-\overline{X}(t\wedge\sigma_{k^*}\wedge\rho_{k^*})|^2) \\ &\le K_2(k^*)\Delta t \end{aligned}$$

where $\Delta t \le \Delta t_2^*$. Therefore, we get

$$P(\overline{\Omega}\cap\{\sigma_{k^*}\wedge\rho_{k^*}>T\}) \le \frac{\varepsilon}{3}, \quad \forall \Delta t \le \Delta t_2^*$$

From the inequalities above, we can obtain

$$P(\overline{\Omega}) \le \varepsilon, \quad \forall \Delta t \le \Delta t^*$$

where $\Delta t^* = \min\{\Delta t_1^*, \Delta t_2^*\}$.

Theorem 2 Under Assumption 1-4, for any $T>0$ and sufficiently small $\varepsilon, \varsigma \in (0,1)$, we can find Δt^* such that

$$P(|x(t)-Z(t)|^2 \ge \varsigma, 0\le t\le T) \le \varepsilon, \ \ \Delta t < \Delta t^*$$

Proof: For sufficiently small $\varepsilon, \varsigma \in (0,1)$, we define

$$\tilde{\tilde{\Omega}}=\{\omega: |x(t)-Z(t)|^2 \ge \varsigma, 0\le t\le T\}$$

we repeat the similar analysis as Lemma 6 and then have

$$P(\tilde{\tilde{\Omega}}) \le P(\tilde{\tilde{\Omega}}\cap\{\sigma_{k^*}\wedge\rho_{k^*}>T\})+\frac{2\varepsilon}{3}$$

Moreover, we get

$$\begin{aligned} &\varsigma P(\tilde{\tilde{\Omega}}\cap\{\sigma_{k^*}\wedge\rho_{k^*}>T\}) \\ &\le E(|x(T)-Z(T)|^2 I_{\{\sigma_{k^*}\wedge\rho_{k^*}>T\}}) \\ &\le E|x(T\wedge\sigma_{k^*}\wedge\rho_{k^*})-Z(T\wedge\sigma_{k^*}\wedge\rho_{k^*})|^2 \\ &\le 2E(\sup_{0\le t\le T}|x(t\wedge\sigma_{k^*}\wedge\rho_{k^*})-\overline{X}(t\wedge\sigma_{k^*}\wedge\rho_{k^*})|^2) \\ &\quad +2E|\overline{X}(t\wedge\sigma_{k^*}\wedge\rho_{k^*})-Z(T\wedge\sigma_{k^*}\wedge\rho_{k^*})|^2 \\ &\le 2K_1(k^*)\Delta t+2K_2(k^*)\Delta t \end{aligned}$$

According to Lemma 3 and Lemma 5, for sufficiently small Δt, we obtain

$$P(\tilde{\tilde{\Omega}}\cap\{\sigma_{k^*}\wedge\rho_{k^*}>T\}) \le \frac{\varepsilon}{3}$$

which leads to

$$P(\tilde{\tilde{\Omega}}) \le \varepsilon.$$

VI. Numerical Example

In this section, we analyze the following example I

$$\begin{aligned} dx(t) &= (x(t^-)+x([t^-]))dt+(2x(t^-)+x([t^-]))dW(t) \\ &\quad +vx([t^-])d\tilde{N}(t) \end{aligned} \tag{35}$$

where $d=m=r=1$, $\lambda=8$.

Let $V(x)=|x|^2$, we have

$$\begin{aligned} LV(x,y) &= 2x(x+y)+(2x+y)+v^2y^2\lambda \\ &\le 85(1+x^2+y^2) \end{aligned}$$

It is obvious that (35) does not satisfy the classical conditions but satisfies the generalized Khasminskii-type conditions (2)-(7).

On the basis of Theorem 1 and 2**,** (35) has a unique global solution and has its Euler method

$$X_{km+l+1} = X_{km+l} + (X_{km+l} + X_{km})\Delta t + (2X_{km+l} + X_{km})\Delta W_{km+l} + X_{km}\Delta N_{km+l} \quad (36)$$

which is convergent in probability.

Let $Z_n = X_{km+l}$, $\Delta t = \frac{1}{2^5}$, and the sample number is 10^4. In Matlab, the Brownian motion $W(t)$ and Poisson process $N(t)$ are simulated by the Matlab commands

```
Winc = sum(dW(R*(j-1)+1:R*j))
```

and

```
Ninc = sum(dN(R*(j-1)+1:R*j))
```

where

```
BROW=randn(1,L)
 dW = sqrt(dt)*BROW
POISSON=poissrnd(lambda,1,L)
 dN=dt*POISSON
L=2^5,  j=1,2, …,L, R=2.
```

In Fig.1, the trajectories of (35) and (36) are described in mean in Matlab. The observations show that the numerical example is consistent with the results of Theorem 2 in my paper.

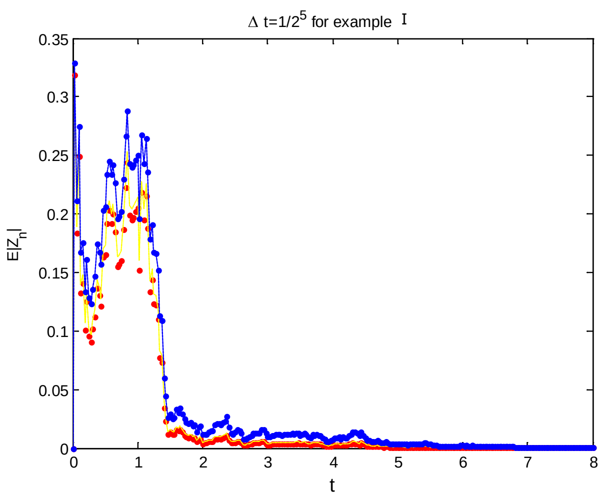

Fig. 1. Trajectories of (35) and (36) in mean

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Hui Yu was born on April 17, 1979. She received the PhD degree in fundamental mathematics from Harbin Institute of Technology. Currently, she is a lecturer in Heilongjiang Bayi Agricultural University, China. Her major research focuses on stochastic differential equations and their numerical methods. She has published many papers in related journals.