Spectral Identification of Nonlinear System

A. Brouri, and M. Benyassi

***Abstract*—** Presently determination approach of nonlinear system is suggested. The studied nonlinear system can be described by a linear element followed by a system nonlinearity. This type of non-linear systems is called Wiener models. In the proposed study, a spectral analysis is used to determine the parameters of the nonlinear systems. The modelling and determination of system parameters is based upon sample signals. Furthermore, the algorithm is easy to implement. The nonlinear element is not necessary polynomial function, but it is supposed to be continuous and smooth in a small interval. In this work, a spectral method is developed allowing the estimates of the complex frequency gain as well as the estimates of nonlinear block parameters.

***Keywords*—**Nonlinear systems, Fourier decomposition, nonparametric system, nonlinear system identification, nonlinear modelling.

I. Introduction

Nonlinear system modelling and identification has been an active research area, especially over the last two decade [1]-[5].

Grey box modelling is one of the most commonly used technics in nonlinear systems parameters determination. The nonlinear element may occurs in the input of system or in the output [6]-[8]. As it can manifest in the input and output, e.g. [5], [8]. Then, this leads to four popular models allowing to modelling nonlinear systems.

Presently, a nonlinear approach is proposed based on the use of spectral technic to determine the nonlinear systems parameters. The considered nonlinear system in this work can be described by Wiener model. This latter is one of most popular models. It consists of linear element followed by nonlinearity block (Fig. 1). This model is more difficult than Hammerstein model (nonlinearity element followed by linear dynamic element).

The usefulness of this nonlinear model has practically been confirmed in various domains fields [9]-[12].

Roughly, these solutions can be classified into four categories. The recursive method [18], the frequency solution [15]-[17], the stochastic methods [19]-[20], and the blind-identification methods [21]-[22].

Then, the estimation of Wiener models parameters is proposed without any a priori structural knowledge of the nonlinearity.

In this study, an approach is suggested using spectral and Fourier series representation. Then, the relationship between the spectral decomposition of the output signal and input signal is exploited.

Moreover, because of the Fourier decomposition, the amplitude and phase information of the unknown linear element as well as the coefficient of the nonlinear block can be easily extracted based on the system output measurements and spectral representation of input and output signal.

The considered problem of parameters determination of nonlinear system (Fig. 1) is addressed in the case where the linear dynamic block is nonparametric. The present solution can also be applied in the case where the linear element is parametric.

On the other hand, the system nonlinearity $f(.)$ (Fig. 1) can be nonparametric. This latter is supposed to be memorless continuous function. Then, it can be approximated by a polynomial function. The polynomial degree n can vary from one interval to another. To the author's knowledge, the identification approach is performed using only one stage unlike most of other methods [19].

The proposed parameters determination solution involves periodic input signals. Then, in the steady state, the undisturbed system output $w(t)$ is also periodic signal of the same period of $u(t)$ (Fig. 1). Accordingly, the internal signal $w(t)$ is decomposable of Fourier series. The nonlinear system parameters (i.e. the nonlinearity block parameters as well as the linear subsystem parameters) can be determined using $u(t)$ and the system output.

The remaining parts of paper are organized as follows: the identification problem is formulated in Section 2; the identification method of linear and nonlinear elements is presented in Section 3; some conclusions of these method performances are presented in Section 4.

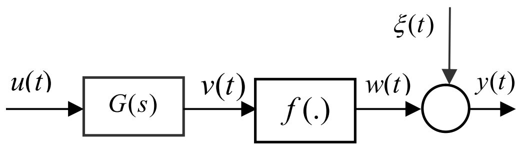

Fig. 1 Nonlinear system structured by Wiener model

II. Problem Statement

Nonlinear system described by Wiener models consist of series connection of linear block $G(s)$ and nonlinear static

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element *f*(.) (Fig. 1). In this work, we assume that, the nonlinear block is continuous function and smooth. Then, this statement means that, the nonlinear function *f*(.) can be approximated within any interval by a polynomial function of degree n, where n is any limited integer. Then, the system nonlinearity *f*(.) is allowed to be noninvertible. Furthermore, the linear dynamic element is not necessarily parametric.

One other hand, the considered nonlinear system in this work (Fig. 1) can be analytically described by the following equations:

$$v(t) = g(t) * u(t) \quad (1a)$$

$$g(t) = \mathcal{L}^{-1}(G(s)) \quad (1b)$$

$$w(t) = f\left(v(t)\right) \quad (1c)$$

$$y(t) = w(t) + \xi(t) \quad (1d)$$

where $\mathcal{L}^{-1}$ and * denote respectively Laplace transform-inverse and the convolution operator. Note that, the signals $u(t)$ and $y(t)$ represent respectively the input and the output system. The internal signals $v(t)$, $\xi(t)$ and, $w(t)$ are not accessible to measurement. Finally, note that the signal $\xi(t)$ refers to the system noise. It is supposed to be a random ergodic sequence of zero-mean and satisfying the stationarity property.

Recall that, that the nonlinear element $f(.)$ is continuous and smooth. Then, it can be modelled by a polynomial function of degree *n*. Let $\theta = \begin{bmatrix} a_0 & \cdots & a_n \end{bmatrix}^T$ denotes the parameters vector of the system nonlinearity $f(.)$. Then, one immediately gets:

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n = \begin{bmatrix} a_0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix} = \theta^T X \quad (2a)$$

where the vector *X* is as follows:

$$X = \begin{bmatrix} 1 & x & \cdots & x^n \end{bmatrix}^T \quad (2b)$$

On the other hand, it is important to emphasize that, this problem identification does not have a unique solution (solution plurality). Indeed, if the couple $\left(G(s), f(v)\right)$ is solution of this problem, then any couple of the form $\left(G(s)/k, f(k\,v)\right)$ is also a model, for any $k \neq 0$. In this respect, the question that arises is how to choose the factor *k* ? This question will be dealt in the next section.

The aim of this study is to determine an accurate estimate of the parameters vector θ of the nonlinearity $f(.)$ as well as the complex frequency gain $G(j\omega)$ for any frequency ω (i.e. the phase $\varphi(\omega) = \arg\left(G(j\omega)\right) = \angle G(j\omega)$ and the modulus gain $\left|G(j\omega)\right|$). Roughly, to determine the parameters of linear dynamic block, two statements can arise. Firstly, if the linear element is parametric, then this latter has a limited number *m* of unknown parameters, where *m* is the sum of the numbers of the numerator and denominator parameters. The identification of this latter consists thus to estimate the complex frequency gain $G(j\omega)$ for *m* frequencies $\omega \in \{\omega_1, \ldots, \omega_m\}$, arbitrarily chosen by the user. The objective in the second case (i.e. the linear block is nonparametric) is to determine the complex gain $G(j\omega)$ for any frequency ω. In this paper, the considered linear element is nonparametric and can be of unknown structure.

III. System identification scheme

In this section, a solution of system parameters determination is proposed. In this method, the nonlinear system described by (1a-d) is excited by a sine signal.

If the linear block is parametric of m unknown parameters, we will need to estimate the complex frequency gain $G(j\omega_l)$ for m frequencies $\omega_l \in \{\omega_1, \ldots, \omega_m\}$ (the frequencies are arbitrarily chosen). In the other case, i.e. the linear element is nonparametric, the aim is to estimate the complex gain $G(j\omega)$ for any frequency ω or for a set of frequencies. Presently, the linear element is nonparametric, we seek thus the estimation of $G(j\omega)$ for a set of frequencies.

Presently, the determination of the nonlinear system parameters, characterized by (1a-d), will be done using spectral analysis method, which amounts to accurately estimating the nonlinear element parameters as well as the frequency complex gain $G(j\omega_l)$ for an arbitrarily set of frequencies $\{\omega_1, \ldots, \omega_N\}$.

The suggested solution, allowing to estimate the nonlinear system parameters, is based on input sine signals:

$$u(t) = U \sin(\omega_l\, t) \quad (3)$$

where $\omega_l \in \{\omega_1, \ldots, \omega_N\}$. Using the fact that, the linear block is stable, the output of this element is also sine signal (when it is excited with a sine signal). Specifically, using (1a-b) and (3), one immediately gets in the steady state the following expression for the output signal of linear element $v(t)$:

$$v(t) = U\left|G(j\omega_l)\right| \sin\left(\omega_l t + \varphi(\omega_l)\right) \quad (4)$$

Accordingly, it readily follows from (2a-b) and (4) that, the output signal of nonlinear element $w(t)$ can be written in the following form:

$$w(t) = \theta^T \begin{bmatrix} 1 \\ v(t) \\ \vdots \\ v(t)^n \end{bmatrix} = \sum_{k=0}^{n} a_k \left(U \left| G(j\omega_l) \right| \right)^k \sin^k \left(\omega t + \varphi(\omega_l) \right) \quad (5)$$

On the other hand, it turns out that equation (5) involves ($n+3$) unknown quantities. Which are the parameters vector $\theta = \begin{bmatrix} a_0 & \cdots & a_n \end{bmatrix}^T$ of the nonlinear element and the linear block parameters $\left(\left| G(j\omega_l) \right|, \varphi(\omega_l) \right)$, i.e. respectively the modulus gain and the phase.

It is interesting to recall that, the following results can be obtained immediately using the power linearization formulas:

$$\sin^{2l} \left(\omega_l t + \varphi(\omega_l) \right) = \frac{1}{2^{2l}} C_l^{2l} + \frac{2}{2^{2l}} \sum_{i=0}^{l-1} (-1)^{l-i} C_i^{2l} \sin \left(2(l-i) \left(\omega_l t + \varphi(\omega_l) \right) + \frac{\pi}{2} \right) \quad (6a)$$

$$\sin^{2l+1} \left(\omega_l t + \varphi(\omega_l) \right) = \frac{1}{2^{2l}} \sum_{i=0}^{l-1} (-1)^{l-i} C_i^{2l+1} \sin \left((2l+1-2i) \left(\omega_l t + \varphi(\omega_l) \right) \right) \quad (6b)$$

Then, it is readily seen using (5) and (6a-b) that, the undisturbed output signal $w(t)$ can be rewritten as follows:

$$w(t) = \sum_{k=0}^{n} S_k \left(\theta, \left| G(j\omega_l) \right| \right) \sin \left(k\,\omega_l t + \lambda_k \left(\varphi(\omega_l) \right) \right) \quad (7)$$

where the amplitudes S_k ($k = 1, \cdots, n$) and the DC component S_0 depend on the system nonlinearity parameters $\left(a_0, \cdots, a_n \right)$ and the modulus gain $\left| G(j\omega_l) \right|$; as for the phase λ_k ($k = 1, \cdots, n$), it depends only on the linear element argument $\varphi(\omega_l)$.

This last result (equation (7)) implies that the inner signal $w(t)$ (not accessible to measurement) is equivalent to sum of n sine signals and a DC component. Accordingly, it readily follows by combining (1d) and (7) that, the output signal $y(t)$ of nonlinear system described by (1a-d) can be expressed as:

$$y(t) = \sum_{k=0}^{n} S_k \left(\theta, \left| G(j\omega_l) \right| \right) \sin \left(k\omega_l t + \lambda_k \left(\varphi(\omega_l) \right) \right) + \xi(t) \quad (8)$$

On the other hand, it is interesting to note that for any frequency $\omega_l \left(k = 1, \ldots, N \right)$, the undisturbed output signal $w(t)$ (not accessible to measurement) is periodic of the same period $T_l = 2\pi / \omega_l$ of the input signal $u(t)$. This result can be easily checked using (7).

Likewise, the spectrum of the inner signal $w(t)$ is characterized by n components of frequency $k\omega_l$ ($k = 1, \cdots, n$) and a DC component (see (7)).

If the undisturbed system output $w(t)$ is accessible to measurement, an accurate estimate of the frequency component amplitudes S_k ($k = 0, \cdots, n$) as well as of the component phases $\lambda_k(.)$ can be obtained using the spectrum of the signal $w(t)$. In this respect, note that the unique accessible signal to measurement is the system output signal $y(t)$. This latter is $w(t)$ mixed up to the noise signal $\xi(t)$. Obviously, the problem that arises is how to measure the inner signal $w(t)$?

Fortunately, such an accurate estimation can be available thanks to the ergodicity property of the noise signal $\xi(t)$ and the steady-state periodic nature of the inner signal $w(t)$. The ergodicity allows the substitution of arithmetic averages to probabilistic means, making simpler forthcoming developments. These remarks suggest that an accurate estimate $\hat{w}(t)$ of $w(t)$ can be obtained using the following periodical averaging:

$$\hat{w}(t) = \frac{1}{M} \sum_{k=1}^{M} y(t + kT_l) \text{ for } t \in [0\ T_l = 2\pi / \omega_l) \quad (9a)$$

$$\hat{w}(t + kT_l) = \hat{w}(t) \qquad \text{for } k = 1, 2, \ldots \quad (9b)$$

where M is any sufficiently large integer. It is readily shown that, the suggested estimator (11a-b) is consistent, i.e. the inner signal estimate $\hat{w}(t)$ converges with probability 1 to the true signal $w(t)$. Indeed, one immediately gets using (1d) and (9a-b) for any time t:

$$\hat{w}(t) = \frac{1}{M} \sum_{k=1}^{M} w(t + kT_l) + \frac{1}{M} \sum_{k=1}^{M} \xi(t + kT_l) \quad (10)$$

Then, for any time t, this latter becomes using the fact that $w(t)$ is periodic of the same period $T_l = 2\pi / \omega_l$ of the input signal $u(t)$:

$$\hat{w}(t) = w(t) + \frac{1}{M} \sum_{k=1}^{M} \xi(t + kT_l) \quad (11)$$

Accordingly, the noise $\xi(t)$ is presently supposed to be a zero-mean ergodic stochastic process featuring the T_l-periodic stationarity (on the set of T_l 's of interest). The periodic stationarity of noise $\xi(t)$ implies that, for any time t and any integer k, one has:

$$E\left(\xi(t + kT_l) \right) = E(\xi(t)) \quad (12)$$

Finally, it immediately follows from (12) that the last term in (11) boils down to zero (w.p.1). Then, it readily follows by combining (11)-(12) for any time t:

$$\lim_{M \to \infty} \hat{w}(t) = w(t) \quad (13)$$

This result means that, the inner signal $w(t)$ is equal to its T_l-periodic average versions obtained by (9a-b).

On the other hand, it is important to emphasize that, the determination of spectrum of the inner signal $w(t)$ using (9a-b) allows to estimate the DC component S_0, the frequency harmonic amplitudes S_k ($k = 1, \cdots, n$), and the component phases λ_k ($k = 1, \cdots, n$).

Furthermore, it follows from the estimate of the DC component S_0 and the harmonic amplitudes S_k ($k = 1, \cdots, n$) that, $(n+1)$ equations are given which involve $(n+2)$ unknown parameters, i.e. the nonlinearity coefficients $(a_0, \cdots, a_n)$ and the modulus gain $|G(j\omega_l)|$ (where $\omega_l \in \{\omega_1, \ldots, \omega_N\}$).

We have thus obtained a system of equations with fewer equations than unknown parameters.

At the same time, n equations are given using the component phases λ_k ($k = 1, \cdots, n$) with only one involved unknown. Specifically, the linear element argument $\varphi(\omega_l)$ where $\omega_l \in \{\omega_1, \ldots, \omega_N\}$. Then, for any input sine signal (3) of frequency $\omega_l \in \{\omega_1, \ldots, \omega_N\}$, the phase of linear block $\varphi(\omega_l)$ can be easily determined using the phase estimate of any chosen harmonic component λ_k ($k = 1, \cdots, n$).

Accordingly, the determination of the modulus gain $|G(j\omega_l)|$ (for any $\omega_l \in \{\omega_1, \ldots, \omega_N\}$) and the nonlinearity parameters $(a_0, \cdots, a_n)$ needs to generate more equations involving the nonlinearity parameters $(a_0, \cdots, a_n)$ or/and the modulus gain $|G(j\omega_l)|$ of linear element. For convenience, to resolve this identification problem the system is excited by the sine input signal (3) with another frequency $\omega_p \in \{\omega_1, \ldots, \omega_N\}$ and $\omega_p \neq \omega_l$.

Therefore, it follows using these two experiments and using uniquely the DC component S_0 and the harmonic amplitudes S_k ($k = 1, \cdots, n$) that, $2(n+1)$ equations are generated involving $(n+3)$ unknowns, i.e. the nonlinearity block parameters $(a_0, \cdots, a_n)$ and the frequency modulus gains $(|G(j\omega_l)|, |G(j\omega_p)|)$, with $(l, p) \in \{1, \ldots, N\}$ and $l \neq p$.

Finally, knowing that $2(n+1) > (n+3)$ for any integer $n \geq 2$, then an accurate estimate of nonlinear system parameters, described by (1a-d), can be immediately determined using the data acquisition collected from these two experiments. If necessary excite the nonlinear system with the input signal (3) for other frequencies.

IV. Conclusion

We have developed a new one-stage identification method to deal with continuous-time Wiener systems involving nonparametric system nonlinearity. The originality of the present study lies in the fact that, the parameters determination method is done is one simple stage using simple sine signal.

Furthermore, the linear subsystem is nonparametric and of unknown structure. Accordingly, the linear subsystem is not necessarily finite order and the nonlinear element can be discontinuous noninvertible.

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