Dynamics and Circuit Emulation of an Extreme Multistable System of Two Linear and One Nonlinear Coupled Oscillators with Hidden Attractors

Jamal-Odysseas Maaita, Efthymia Meletlidou, and Christos Volos

***Abstract*—**A system of two linear oscillators coupled to a damped nonlinear oscillator that has multiple stability and hidden chaotic attractors, is studied in this work. The unperturbed Hamiltonian part contains, apart from the quadratic harmonic oscillations, a nonlinear fourth order term with extra linear part with respect to the first two oscillators. As a consequence the proposed system has only one equilibrium point that is non-hyperbolic. Also, the chaotic attractors of the full system are hidden i.e. their basin of attraction does not have any unstable equilibrium point. Furthermore, the electronic realization of the system is presented and its dynamical behavior is studied in order to confirm the feasibility of the theoretical model.

***Keywords*—**Bifurcation diagram, chaos, Extreme multistability, hidden attractors, Lyapunov exponent, Phase Portrait, Poincaré map.

I. Introduction

IN recent years chaotic behavior were discovered on systems with no equilibrium points. The chaotic attractors of such systems are named "Hidden Chaotic Attractors".

There is a great interest in studying hidden attractors in applied systems. In the 1950-1960s hidden attractors were discovered and observed in various nonlinear control systems [1]. Since then a lot of work has been done by various scientists in many fields of science and engineering [2].

In 1997 Lauvdal *et al*. [3] presented numerical difficulties in simulation of aircraft control systems (anti-windup scheme) that were connected to hidden oscillations.

Studies on hidden oscillations have been done for electrical machines as drilling systems [4]-[6]. These studies try to solve problems and failures of drill string systems that cause huge cost loses for the drilling companies.

A great work has been done by various scientists in the field of nonlinear electronic circuits. In 2010 Leonov and Kuznetsov discovered chaotic hidden attractor in Chua's circuit [7]-[9]. Since then a lot of work has been done and many systems with hidden attractors were emulated by nonlinear electronic circuits [10]-[12]. Furthermore, techniques and methods that are used in hidden attractors may be useful to construct adequate nonlinear models for phase-locked loop (PLL) systems that are used in radio, telecommunications, computers and other electronic applications [13].

In our paper we present a system of two linear oscillators coupled to a damped third order nonlinear oscillator with a mass much smaller than the linear oscillators.

A similar system to the one presented here, has been exhaustively studied in [14]-[16]. Because the mass of the nonlinear attachment is much smaller, in comparison to the linear oscillators, the system is singular, and its dynamics are governed by different time scales.

In our previous works we studied the system with the use of methods, such as singularity analysis or multiple scale analysis [17]-[18] and showed that the slow flow of the system and the Slow Invariant Manifolds obtained in the singular limit play a very important role in the dynamics. Furthermore, we confirmed that the Slow Invariant Manifold, its bifurcations, and the dynamics of the slow flow play an essential role in the energy transfer, from the linear to the nonlinear oscillator, and dissipation of the system.

In this work we study a slightly altered system. The unperturbed Hamiltonian part contains, apart from the quadratic harmonic oscillations, the nonlinear fourth order term an extra linear part with respect to the first two oscillators. This results to having only one equilibrium point that is non-hyperbolic. Thus, the chaotic attractors of the full system are hidden i.e. their basin of attraction does not have any unstable equilibrium point. The detailed study of the system showed that it possesses a very interesting multistable dynamical behavior.

The paper is organized as follows. In the next section the proposed system and its basic properties are presented. The simulation results of system's dynamics are discussed in the third section. In the fourth section a circuit emulation of the system is presented and its dynamical behavior is discussed. Finally, we conclude in the fifth section.

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II. Configuration of the System

The system consists of a damped nonlinear oscillator coupled to two linear oscillators and is described by

$$\begin{aligned}&\varepsilon\ddot{y}+\varepsilon\lambda(\dot{y}-\dot{x}_0)+C(y-x_0)^3+\varepsilon^2=0,\\ &\ddot{x}_0+d(x_o-x_1)=\varepsilon\lambda(\dot{y}-\dot{x}_0)+c(y-x_o)^3+\varepsilon^2,\\ &\ddot{x}_1+\alpha\,x_1+d(x_1-\mathrm{x}_0)=0,\end{aligned}\quad(1)$$

where a, d, C and $\varepsilon << 1$ are the parameters of the system and λ is the damping parameter.

After applying the linear singular transformation $v=\epsilon^{-1/2}x_0+\epsilon^{1/2}\,y$, $u=\epsilon^{-1/2}x_0-\epsilon^{-1/2}y$ and $x\rightarrow\epsilon^{1/2}x_1$, the system assumes the form:

$$\begin{aligned}&\ddot{u}=-\lambda(1+\epsilon)\dot{u}-C(1+\epsilon)u^3-d\frac{v+\epsilon u}{1+\epsilon}+dx+\epsilon^{\frac{1}{2}}+\epsilon^{\frac{3}{2}}\\ &\ddot{v}=-d\frac{v+\epsilon u}{1+\epsilon}+d\mathrm{x},\\ &\ddot{x}=-(a+d)\mathrm{x}+d\frac{v+\epsilon u}{1+\epsilon}.\end{aligned}\quad(2)$$

The system has only one equilibrium point

$$\mathrm{v}=-\frac{\epsilon^{\frac{7}{6}}}{C^{\frac{1}{3}}},\ \mathrm{u}=\frac{\epsilon^{\frac{1}{6}}}{C^{\frac{1}{3}}},\ \mathrm{x}=\mathrm{p_x}=\mathrm{p_v}=\mathrm{p_u}=0,$$

where $p_x=\dot{x}, p_u=\dot{u}, p_v=\dot{v}$.

Furthermore, $div\bar{f}=-\lambda(1+\epsilon)$, where $\bar{f}$ is the vector field, and the system is dissipative.

III. Numerical Investigation

The dynamics of the system were exhaustively investigated with the help of numerical tools such as bifurcations diagrams, maximal Lyapunov exponent, phase portraits and Poincaré sections.

The numerical investigation of this system is not a simple task due to the big number of its equations (six equations for the numerical solution of the system and twelve for the maximal Lyapunov exponent).

The dynamical behavior of the oscillators, as it is expected, depends on the parameters and the initial conditions. The system may oscillate regularly (with periodic and quasiperiodic orbits) or has chaotic motion.

For the parameters $a=0.4$, $d=0.1$, $C=5.5$, $\epsilon=0.01$ and initial conditions $u_0=0.1, v_0=pv_0=pu_0=px_{10}=0$ and $x_0=30$, the bifurcation diagram (Fig. 1) shows that the system has chaotic behavior for $\lambda\leq 0.2$ and quasiperiodic for $\lambda>0.2$. The chaotic behavior is proven with the help of maximal Lyapunov exponent and the chaotic motion can be seen in the phase portrait of the system. Indeed, for $\lambda=0.1$ (Fig. 2) shows that the system has a positive maximal Lyapunov exponent and Fig. 3 shows the phase portrait of the system.

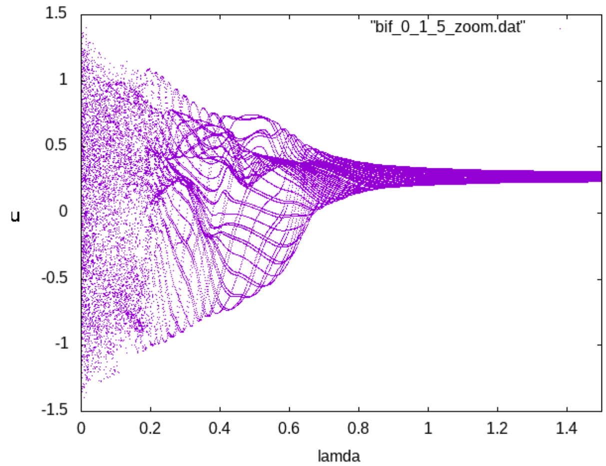


Fig. 1 Bifurcation diagram of u versus λ

The chaotic attractor of the system for $\lambda=0.1$ is given in the Poincaré diagrams (Fig. 4) and since the system has only one non-hyperbolic equilibrium point then the chaotic attractor is hidden.

Another example of chaotic behavior is given for the parameters: $\mathrm{a}=0.35$, $\mathrm{d}=0.1$, $\mathrm{C}=3.5$, $\epsilon=0.01, \lambda=0.1$ and initial conditions $u_0=0.5, v_0=pv_0=pu_0=px_{10}=10$ and $x_0=40$. Fig. 5 shows the maximal Lyapunov Exponent of the system that is positive, and Fig. 6 shows the phase portrait of the system.

The bifurcation diagram for different initial conditions x_0 (Fig.7) show that the system changes its dynamical behavior in relation to the initial condition. So, the system is extremely multistable, that is, the system has an infinite number of attractors.

Indeed, for the parameters $a=0.4$, $d=0.1$, $C=5.5$, $\epsilon=0.01$ and initial conditions $u_0=v_0=pv_0=pu_0=px_{10}=0$ we have: i) for $x_0=12$, $\mathrm{u}_0=0.5$ the system oscillates quasi-periodically (Fig. 8), ii) for $x_0=2$, $\mathrm{u}_0=0.5$ the system tends to its equilibrium point (Fig. 9) and iii) for $x_0=40$, $\mathrm{u}_0=0.5$ the system oscillates chaotically (Fig. 10).

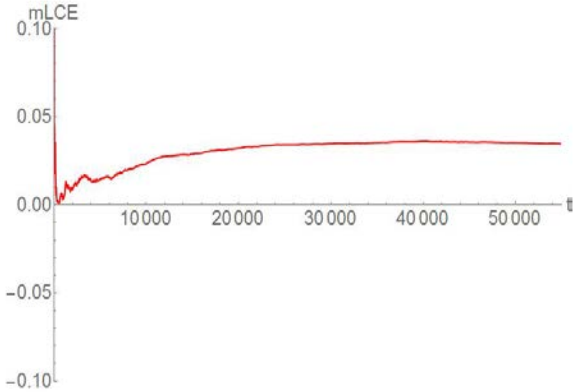


Fig. 2 Maximal Lyapunov Exponent

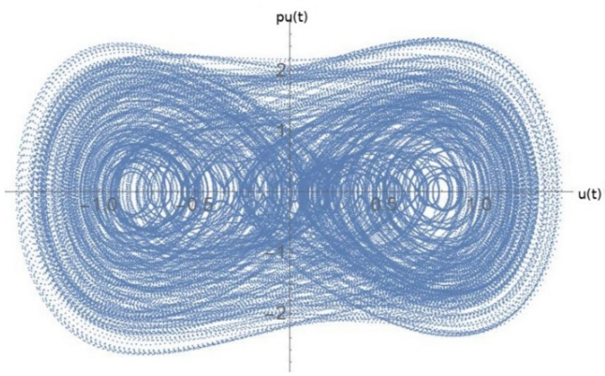


Fig. 3 Phase portrait of p_u versus u

IV. Circuit Emulation

The classical approach for the verification of the feasibility of theoretical chaotic models is the physical realization through electronic circuits [14]-[16]. Furthermore, the circuital realization of chaotic systems has been applied in numerous engineering applications, for example in secure communications [17], liquid mixing [18], robotics [19], image encryption process [20], audio encryption scheme [21], target detection [22] or random signal generation [23], [24]. For this reason, analog and digital approaches have been applied to realize chaotic oscillators by using different kinds of electronic devices such as common off-the-shelf electronic components [25], [26], integrated circuit technology [27], microcontroller [28] or field-programmable gate array (FPGA) [29]-[31]. Therefore, we will confirm the feasibility of the proposed system by discussing its circuital realization by using the general operational amplifier-based approach.

Figure 11 shows the schematic of the circuit for realizing the proposed system. As shown in this figure, the circuit includes thirty-one (31) resistors, six (6) capacitors, twelve (12) operational amplifiers (TL081) and two (2) analog multipliers (AD633). By applying Kirchhoff's circuit laws into the designed circuit, we get the following circuital equation:

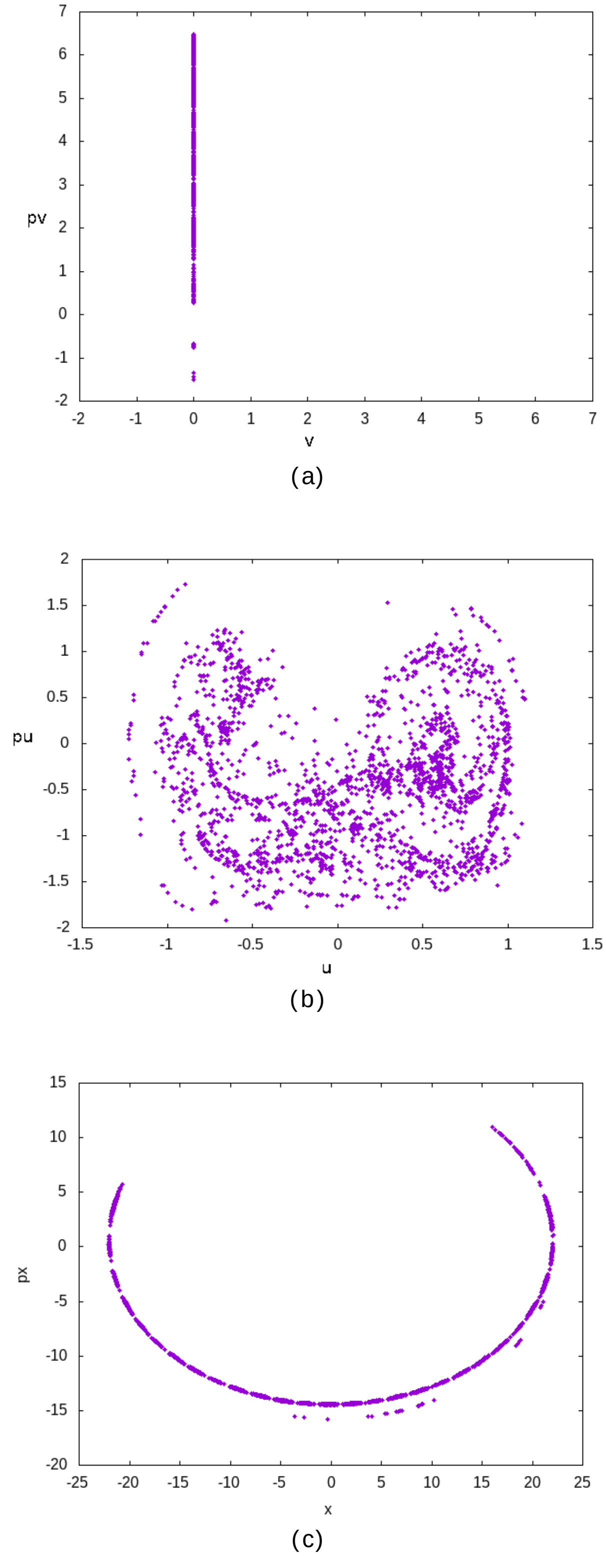


Fig. 4 Poincaré maps in three different planes

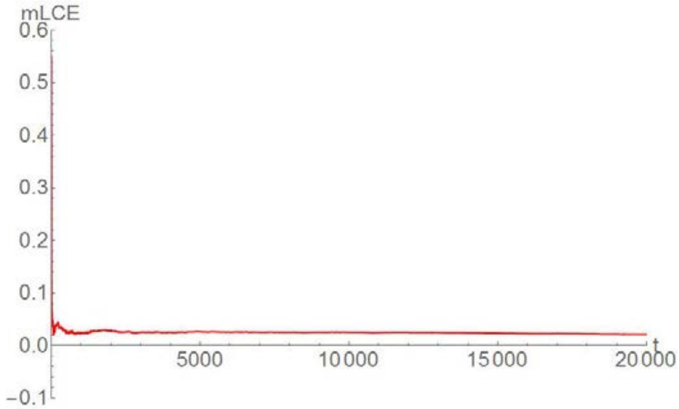


Fig. 5 Maximal Lyapunov Exponent

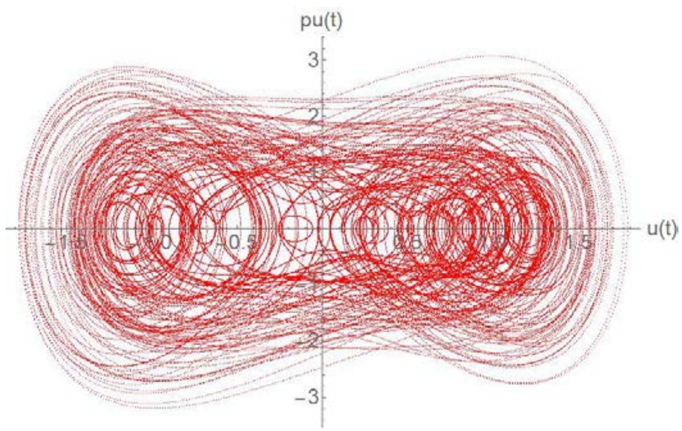


Fig. 6 Phase portrait of p_u versus u

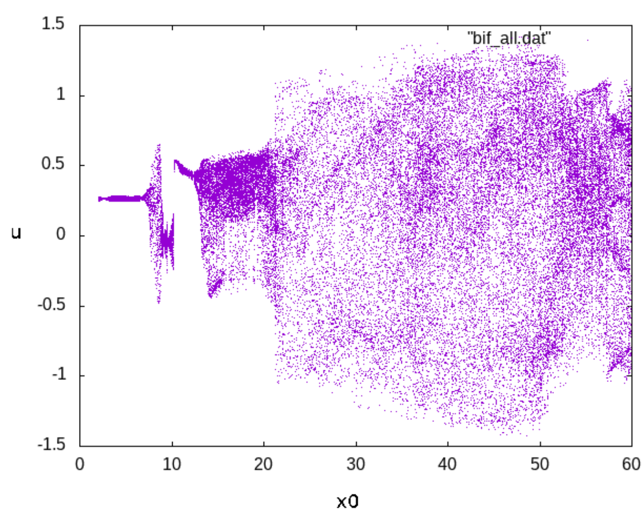


Fig. 7 Bifurcation-like diagram of u versus the initial condition of x_0

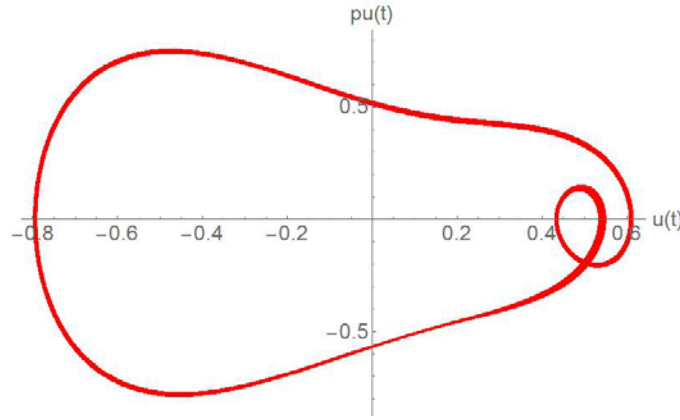


Fig. 8 Phase portrait of p_u versus u (Quasi-periodic oscillations)

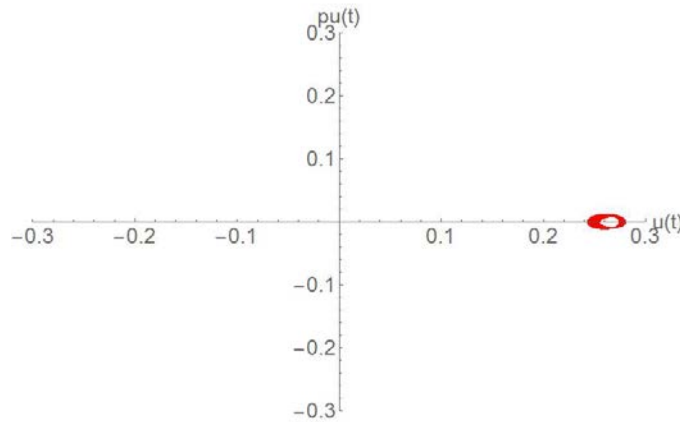


Fig. 9 Phase portrait of p_u versus u (The system tends to its equilibrium point)

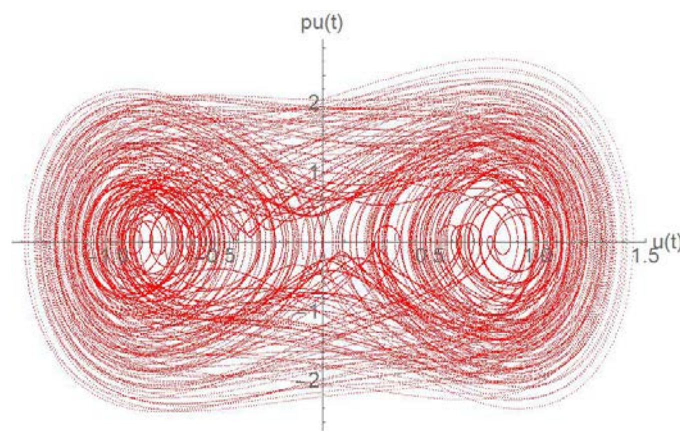


Fig. 10 Phase portrait of p_u versus u (Chaotic oscillations)

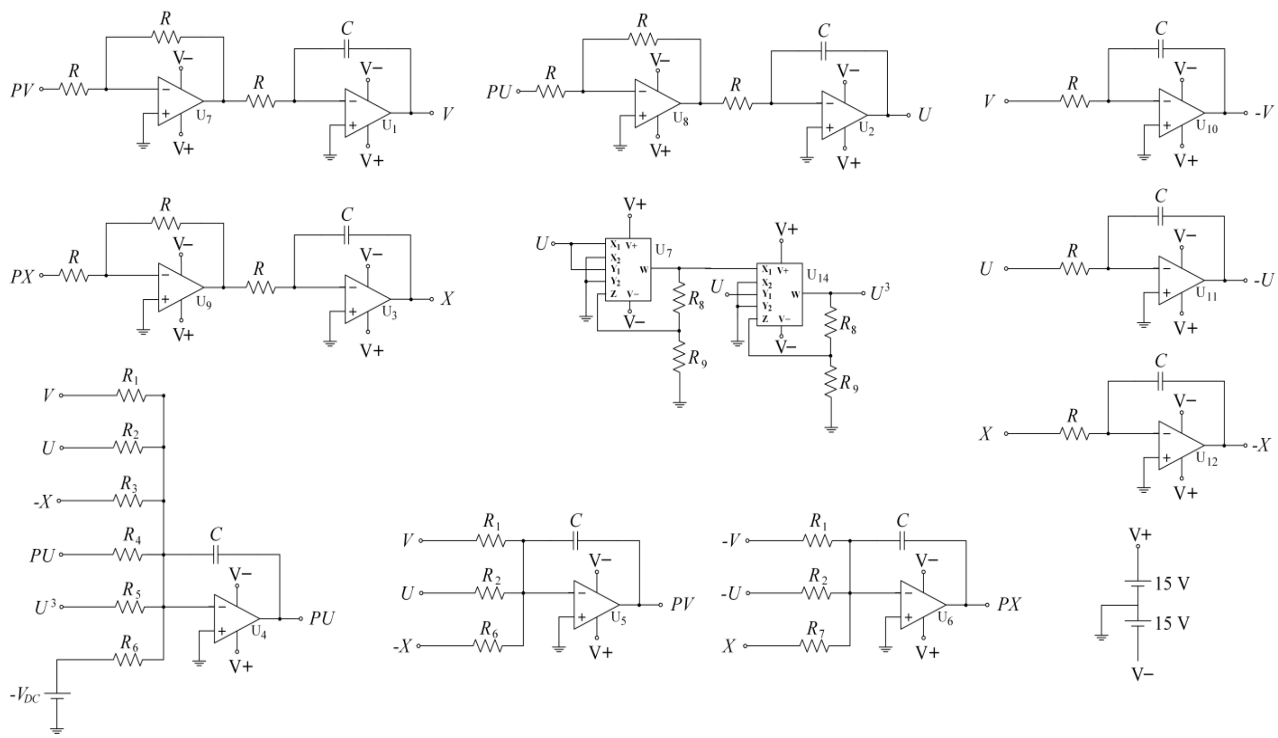
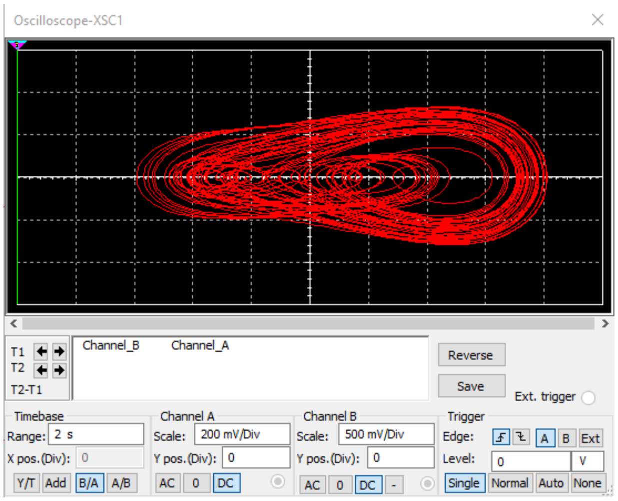


Fig. 11 Schematic of the circuit

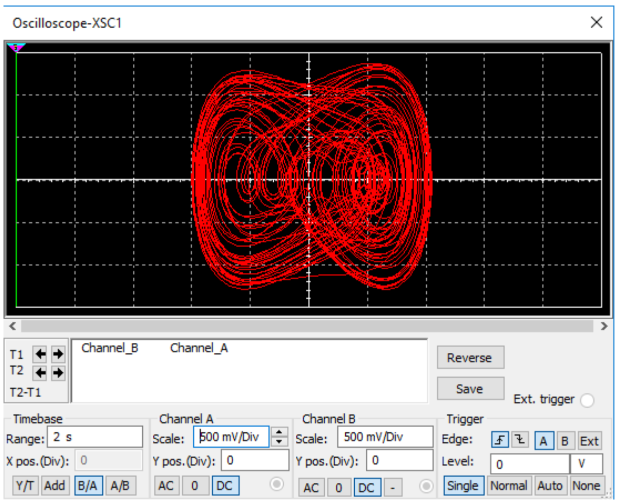
$$\frac{dU}{dt} = \frac{1}{\mathrm{RC}} PU,$$
$$\frac{dV}{dt} = \frac{1}{RC} PV,$$
$$\frac{dX}{dt} = \frac{1}{RC} PX, (3)$$
$$\frac{dPU}{dt} = \frac{1}{RC}\Big(-\frac{R}{R_1}V - \frac{R}{R_2}U + \frac{R}{R_3}X - \frac{R}{R_4}PU - \frac{R}{R_5(1Volt)^2}U^3 + \frac{R}{R_6}V_{DC}\Big),$$
$$\frac{dPV}{dt} = \frac{1}{RC}\Big(-\frac{R}{R_1}V - \frac{R}{R_2}U + \frac{R}{R_3}X\Big),$$
$$\frac{dPX}{dt} = \frac{1}{RC}\Big(\frac{R}{R_1}V + \frac{R}{R_2}U - \frac{R}{R_7}X\Big),$$

In system (3), (V, U, X, PV, PU, PX) correspond to the voltages on the integrators $(U_1 - U_6)$, respectively, while the power supply is ± 15 Volt. The system has also six (6) inverting amplifiers $(U_7 - U_{12})$ and two multipliers (U_{13}, U_{14}). System (3) is normalized by using $\tau = \frac{t}{RC}$. It can thus be suggested that system (3) is equivalent to the proposed system (3), with $\frac{R}{R_1} = \frac{d}{1+\epsilon}$, $\frac{R}{R_2} = \frac{\epsilon d}{1+\epsilon}$, $\frac{R}{R_3} = d$, $\frac{R}{R_4} = \lambda(1+\epsilon)$, $\frac{R}{R_5} = C(1+\epsilon)$, $\frac{R}{R_6} = \sqrt{\epsilon} + \sqrt{\epsilon^3}$ and $\frac{R}{R_7} = a + d$.

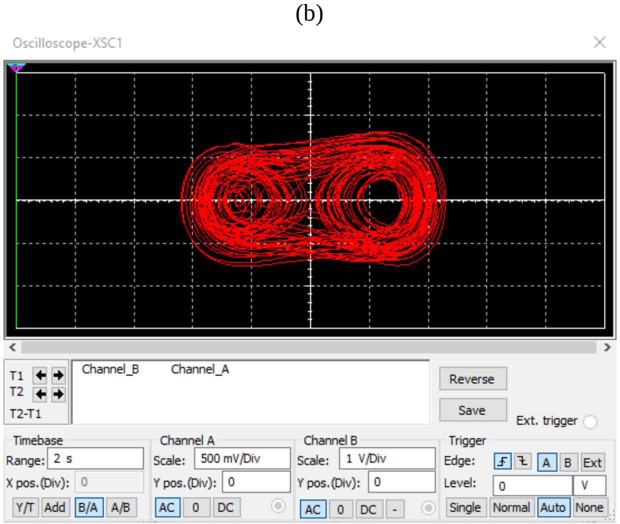
So, for the same set of parameter values, the values of circuit components are: $R = 10\,\mathrm{k\Omega}$, $\mathrm{R}_1 = 101\mathrm{k\Omega}$, $R_2 = 10.101\,\mathrm{M\Omega}$, $R_3 = 100\,\mathrm{k\Omega}$, $R_4 = R_6 = 99.01\,\mathrm{k\Omega}$, $R_5 = 1.8\,\mathrm{k}\Omega$, $R_7 = 24.39\,\mathrm{k\Omega}$, $R_8 = 10\,\mathrm{k\Omega}$, $R_9 = 90\,\mathrm{k\Omega}$ and p_u The designed circuit has been implemented in Multisim and PSpice results for selected values of the initial condition of variable X, while the rest initial conditions are equal to zero, are reported in Fig.12. It is easy to see the agreement between the circuit's simulation results (Fig.12) and numerical results (Fig.13).



(a)

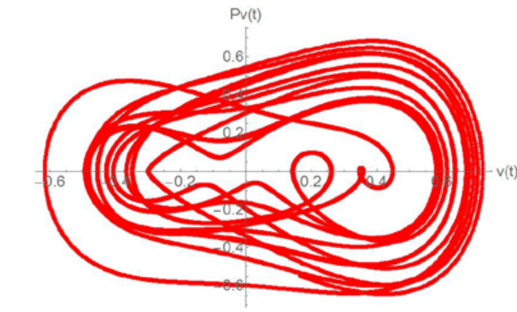


(b)

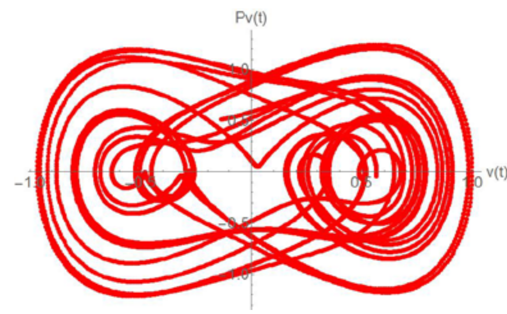


(c)

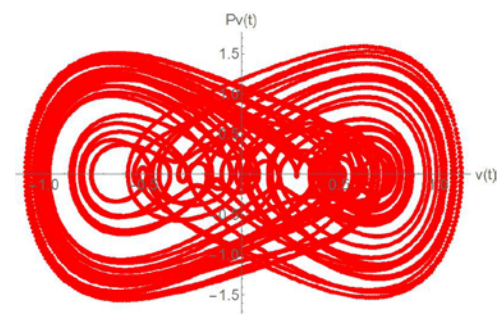
Fig. 12 PSpice chaotic attractors of the designed circuit in *U - PU* plane, for (a) $X(0)$ = 3 Volt, (b) $X(0)$ = 10 Volt, (c) $X(0)$ = 14 Volt, while the rest initial conditions are equal to zero



(a)



(b)



(c)

Fig. 13 Numerical results for (a) $X(0)$ = 3 Volt,(b) $X(0)$ = 10 Volt, (c) $X(0)$ = 14 Volt, while the rest initial conditions are equal to zero

V. Conclusions

We presented the dynamics of a system of two linear oscillators to a damped third order nonlinear oscillator with a mass much smaller than the linear oscillators that contain an extra linear part. The system has only one equilibrium point that is non-hyperbolic. Thus, the chaotic attractors of the full system are hidden. Furthermore, the system is extremely multistable and so has an infinite number of attractors. Finally, the system is experimentally emulated by an electronic circuit and its dynamical behavior confirms the feasibility of the theoretical model.

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