Construction of exponentially fitted explicit peer methods

Dajana Conte, Beatrice Paternoster, Leila Moradi and Fakhrodin Mohammadi

Abstract—It is the purpose of this work to present exponentially fitted explicit two-step peer methods for the numerical integration of ordinary differential equations exhibiting oscillatory solution.

We will use a problem oriented approach based on exponential fitting, in order to exploit a-priori known information about the qualitative behavior of the solution. Moreover the constructed methods have inherent method parallelism, therefore they are suitable for the numerical solution of high dimension ordinary differential systems arising for example in the semi-discretization in space of partial differential equations.

The construction of methods with 2 and 3 stages is provided. Numerical tests show that the error of EF peer methods is smaller with respect to that of classical peer methods, as the frequency of oscillation increases, thus confirming the effectiveness of this problem-oriented approach.

Keywords—peer methods, exponential fitting, oscillatory problems, ordinary differential equations

I. Introduction

We consider initial value problems for ordinary differential equations

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0 \in \mathbb{R}^d, \qquad t \in [t_0, T], \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is smooth enough to guarantee the existence and the uniqueness of the solution. We are interested in solving the problem (1) in the case when the solution exhibits a pronounced oscillatory character. Systems of this kind arise in the semi-discretization in space of partial differential equations in many applications (compare [1], [63], [64], [85], [74] and references therein). Classical numerical methods may require a very small stepsize in order to accurately reproduce the qualitative behavior of the solution, therefore it is convenient to use *special purpose* formulae, i.e. numerical methods adapted to the problem, constructed in order to be exact on functions other than polynomials.

Following the general idea shown in the classical monograph [71] (compare also the review paper on the topic [79]), the adaptation of already existing schemes has led to exponentially fitted methods for a wide range of problems such as interpolation, numerical differentiation and quadrature [34], [32], [36], [37], [67], [68], [73], [72], [89], numerical solution of integral equations [18], [19], [20], [21], partial differential equations [46], [58], [57], [59] and ordinary differential equations [4], [40], [41], [42], [45], [71], [68], [90]. In particular, 2-step hybrid exponentially fitted methods for the integration of second order differential equations have been presented in [38], [42], [43], whereas different estimates for the parameter characterizing the fitting space have been proposed in [40], [45]. Adapted Runge Kutta methods have been constructed in [39], [44], [58], [60], [65], [71], [68], [77], [78], [86], [87], while adapted peer methods in [22], [6], [76].

Peer methods are characterized by several stages like Runge Kutta but all of these stages exhibit the same properties, such as accuracy and stability (see [6], [75], [83], [91] and references therein). Such methods, combining the advantages of Runge-Kutta and multistep methods, achieve good stability features and have no order reduction for very stiff systems [82]. Moreover, two-step peer methods have inherent method parallelism when the actual stages rely only on the previous ones [5], [80], [82], [84].

In [6], [76], it has been shown that it is possible to construct explicit two-step s stage peer methods adapted to fitting space of high dimension $2s$. In particular, authors have derived explicit peer methods having 2 and 3 stages and fitted to trigonometric spaces. In [22] the authors derived a general class of exponentially fitted two-step explicit peer method having order s, by employing the six-step procedure described in [71].

It is the purpose of this paper to describe the practical construction of parallel explicit exponentially fitted peer methods, as a sublcass of methods introduced in [22]. In summary, in Section II we recall classical parallel explicit peer methods, in Section III we derive exponentially fitted parallel explicit peer methods. Section IV contains the construction of methods with 2 and 3 stages while in Section V we show some numerical experiments. Section VI is devoted to conclusions. Finally the Appendix contains informations about the $\eta_m(Z)$ functions, which are used to construct the coefficients of the methods.

II. Classical explicit peer methods

With the aim of constructing peer methods having inherent method parallelism (see [80], [81]), we consider s-stage two-step peer method with fixed stepsize h having the form

$$Y_{ni} = \sum_{j=1}^{s} b_{ij}\, Y_{n-1,j} + h \sum_{j=1}^{s} a_{ij}\, f(t_{n-1,j}, Y_{n-1,j}), \tag{2}$$

$i = 1, \ldots, s$, where

$$Y_{ni} \approx y(t_{ni}), \quad t_{ni} = t_n + c_i\, h, \qquad i = 1, \ldots, s.$$

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The fixed nodes $c_i < 1$ for $i = 1, \ldots, s$ are assumed to be distinct and we set $c_s = 1$, so Y_{ns} is the approximation of the solution at grid point t_{n+1}. By using the notation:

$$Y_n = [Y_{ni}]_{i=1}^s\,,\ F(Y_n) = [f(t_{ni}, Y_{ni})]_{i=1}^s\,,$$
$$A = [a_{ij}]_{i,j=1}^s\,,\ B = [b_{ij}]_{i,j=1}^s\,,$$

the method (2) can be rewritten in a more compact form

$$Y_n = (B \otimes \mathbb{I})\, Y_{n-1} + h\, (A \otimes \mathbb{I})\, F(Y_{n-1}), \tag{3}$$

where $\mathbb{I}$ is the identity matrix of dimension d. The matrices of coefficients A and B are constructed in order by means of the order conditions:

Theorem 2.1: The peer method (2) has order $p = s$ if

$$B\,\mathbf{1} = \mathbf{1}, \tag{4a}$$
$$AV_1D \ = CV_0 - B\,(C - \mathbb{I})V_1, \tag{4b}$$

where $\mathbf{1} = [1, 1, \ldots, 1]^T$, $C = \text{diag}(c_1, \ldots, c_s)$, $D = \text{diag}(1, \ldots, s)$ and

$$V_0 = \begin{bmatrix} 1 & c_1 & \ldots & c_1^{s-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & c_s & \ldots & c_s^{s-1} \end{bmatrix},$$

$$V_1 = \begin{bmatrix} 1 & (c_1 - 1) & \ldots & (c_1 - 1)^{s-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (c_s - 1) & \ldots & (c_s - 1)^{s-1} \end{bmatrix}.$$

We associate to the numerical scheme (2) the linear operator defined as

$$\begin{aligned} \mathcal{L}_i[h, \mathbf{w}]\, y(t) &= y(t + c_i\, h) - \sum_{j=1}^s b_{ij}\, y(t + (c_j - 1)\, h) \\ &\quad - h \sum_{j=1}^s a_{ij}\, y'(t + (c_j - 1)\, h), \end{aligned} \tag{5}$$

$i = 1, \ldots, s$, where $\mathbf{w}$ contains the coefficients of the method. We observe as the order conditions of Theorem 2.1 are obtained by annihilating the difference operator (5) on the set polynomials

$$\left\{1, t, t^2, \ldots, t^s\right\}. \tag{6}$$

III. Exponentially fitted explicit peer methods

We now aim to construct exponentially fitted explicit two-step peer methods with a fixed stepsize h, which are particularly suitable for problems with hyperbolic or trigonometric solutions. We then consider the fitting space

$$\mathcal{F} = \left\{1, t, t^2, \ldots, t^K, e^{\pm\mu t}, t\, e^{\pm\mu t}, t^2 e^{\pm\mu t}, \ldots, t^P e^{\pm\mu t}\right\}, \tag{7}$$

where μ is a parameter characterizing the exact solution and it is real or complex if the exact solution belongs to the space spanned by hyperbolic functions or trigonometric functions, respectively. The coefficients matrices A and B are now derived by annihilating the difference operator (5) on the basis functions (7), with suitable choices for K and P. In particular, by following the six-step procedure of [71], we get the following results (see [22]).

Theorem 3.1: Assume s is even. The peer method (2) has order $p = s$ and is adapted to the fitting space

$$\mathcal{F} = \left\{1, e^{\pm\mu t}, t\, e^{\pm\mu t}, t^2 e^{\pm\mu t}, \ldots, t^{\frac{s}{2}-1} e^{\pm\mu t}\right\},$$

if the coefficient matrices A and B satisfy

$$B\,\mathbf{1} = \mathbf{1}, \tag{8a}$$
$$AD_3 \ = D_1 - B\,D_2, \tag{8b}$$

where $\mathbf{1} = [1, 1, \ldots, 1]^T$ and

$$D_1 = \begin{bmatrix} \ldots & \frac{1}{2^i}\, c_1^{2i}\, \eta_{i-1}\left(c_1^2\, Z\right) & \frac{1}{2^i} c_1^{2i+1}\, \eta_i\left(c_1^2\, Z\right) & \ldots \\ & \vdots & \vdots & \\ \ldots & \frac{1}{2^i}\, c_s^{2i}\, \eta_{i-1}\left(c_s^2\, Z\right) & \frac{1}{2^i} c_s^{2i+1}\, \eta_i\left(c_s^2\, Z\right) & \ldots \end{bmatrix},$$

$$D_2 = \begin{bmatrix} \ldots & \frac{1}{2^i}\, \hat{c}_1^{2i}\, \eta_{i-1}\left(\hat{c}_1^2\, Z\right) & \frac{1}{2^i} \hat{c}_1^{2i+1}\, \eta_i\left(\hat{c}_1^2\, Z\right) & \ldots \\ & \vdots & \vdots & \\ \ldots & \frac{1}{2^i}\, \hat{c}_s^{2i}\, \eta_{i-1}\left(\hat{c}_s^2\, Z\right) & \frac{1}{2^i} \hat{c}_s^{2i+1}\, \eta_i\left(\hat{c}_s^2\, Z\right) & \ldots \end{bmatrix},$$

$$D_3 = \begin{bmatrix} \ldots & \frac{i}{2^{i-1}}\, \hat{c}_1^{2i-1}\, \eta_{i-1}\left(\hat{c}_1^2\, Z\right) + \frac{1}{2^i}\, \hat{c}_1^{2i+1}\, Z\eta_i\left(\hat{c}_1^2\, Z\right) & \frac{1}{2^i} \hat{c}_1^{2i}\, \eta_{i-1}\left(\hat{c}_1^2\, Z\right) & \ldots \\ & \vdots & \vdots & \\ \ldots & \frac{i}{2^{i-1}}\, \hat{c}_s^{2i-1}\, \eta_{i-1}\left(\hat{c}_s^2\, Z\right) + \frac{1}{2^i}\, \hat{c}_s^{2i+1}\, Z\eta_i\left(\hat{c}_s^2\, Z\right) & \frac{1}{2^i} \hat{c}_s^{2i}\, \eta_{i-1}\left(\hat{c}_s^2\, Z\right) & \ldots \end{bmatrix},$$

where $i = 0, 1, \ldots, \frac{s}{2} - 1$, and $\hat{c}_j = 1 - c_j$, $j = 0, 1, \ldots, s$.

In case of odd number of stages we have the following.

Theorem 3.2: Assume s is odd. The peer method (2) has order $p = s$ and is adapted to the fitting space

$$\mathcal{F} = \left\{e^{\pm\mu t}, t\, e^{\pm\mu t}, t^2 e^{\pm\mu t}, \ldots, t^{\frac{s-1}{2}} e^{\pm\mu t}\right\},$$

if the coefficient matrices A and B satisfy

$$B\, v_0 = v_1 - ZAv_2, \tag{9a}$$
$$AF_3 \ = F_1 - B\,F_2, \tag{9b}$$

where F_i for $i = 1, 2, 3$ are obtained by deleting the first column to D_i for $i = 1, 2, 3$, and

$$v_0 = \left[\eta_{-1}\left(\hat{c}_1^2\, Z\right), \ldots, \eta_{-1}\left(\hat{c}_s^2\, Z\right)\right]^T, \tag{10a}$$
$$v_1 = \left[\eta_{-1}\left(c_1^2\, Z\right), \ldots, \eta_{-1}\left(c_s^2\, Z\right)\right]^T, \tag{10b}$$
$$v_2 = \left[\hat{c}_1\eta_0\left(\hat{c}_1^2\, Z\right), \ldots, \hat{c}_s\eta_0\left(\hat{c}_s^2\, Z\right)\right]^T. \tag{10c}$$

IV. Construction of EF peer methods

In this Section we describe the practical derivation of EF explicit peer methods by using Theorems 3.1-3.2.

A. Methods with two stages

Theorem 4.1: Let

$$c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \tag{11a}$$

$$A = \begin{bmatrix} 0 & 0 \\ \frac{1-\eta_{-1}(Z)}{Z\eta_0(Z)} & -\eta_{-1}(Z)\frac{1-\eta_{-1}(Z)}{Z\eta_0(Z)} + \eta_0(Z) \end{bmatrix}. \tag{11b}$$

Then the peer method (2) has order $p = 2$ and is adapted to the fitting space

$$\{1, e^{\pm\mu t}\}.$$

Proof: Matrix B clearly satisfies (8a). From $c_1 = 0$, $c_2 = 1$, we have $\hat{c}_1 = -1$, $\hat{c}_2 = 0$ and

$$D_1 = \begin{bmatrix} 1 & 0 \\ \eta_{-1}(Z) & \eta_0(Z) \end{bmatrix}, D_2 = \begin{bmatrix} \eta_{-1}(Z) & -\eta_0(Z) \\ 1 & 0 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} -\frac{1}{Z\eta_0(Z)} & \frac{\eta_{-1}(Z)}{Z\eta_0(Z)} \\ 0 & 1 \end{bmatrix}.$$

Then form (8b) we get $A = (D_1 - BD_2)D_3^{-1}$ and the thesis follows. ■

The corresponding classic peer method is obtained in the limit as $Z \to 0$ and has coefficients:

$$c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \tag{12a}$$

$$A = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}. \tag{12b}$$

B. Methods with three stages

Theorem 4.2: Let

$$c = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}, \quad A = F_1 F_3^{-1}, \quad B = H_1 - AH_2, \tag{13}$$

with

$$H_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \eta_{-1}(\frac{Z}{4}) \\ 0 & 0 & \eta_{-1}(Z) \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 0 & -Z\eta_0(Z) \\ 0 & 0 & -\frac{Z}{2}\eta_0(\frac{Z}{4}) \\ 0 & 0 & 0 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2}\eta_0(\frac{Z}{4}) & \frac{1}{8}\eta_0(\frac{Z}{4}) & \frac{1}{16}\eta_1(\frac{Z}{4}) \\ \eta_0(Z) & \frac{1}{2}\eta_0(Z) & \frac{1}{2}\eta_1(Z) \end{bmatrix},$$

$$F_3 = \begin{bmatrix} \eta_{-1}(Z) & -\eta_0(Z) - \frac{Z}{2}\eta_1(Z) & \frac{1}{2}\eta_0(Z) \\ \eta_{-1}(\frac{Z}{4}) & -\frac{1}{2}\eta_0(\frac{Z}{4}) - \frac{Z}{16}\eta_1(\frac{Z}{4}) & \frac{1}{8}\eta_0(\frac{Z}{4}) \\ 1 & 0 & 0 \end{bmatrix}.$$

Then the peer method (2) has order $p = 3$ and is adapted to the fitting space

$$\{e^{\pm\mu t}, te^{\pm\mu t}\}.$$

Proof: From $c_1 = 0$, $c_2 = 1/2$, $c_3 = 1$ we have $\hat{c}_1 = -1$, $\hat{c}_2 = -1/2$, $\hat{c}_3 = 0$ and the vectors in (10) assume the form

$$v_0 = \begin{bmatrix} \eta_{-1}(Z) \\ \eta_{-1}(\frac{Z}{4}) \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ \eta_{-1}(\frac{Z}{4}) \\ \eta_{-1}(Z) \end{bmatrix}, v_2 = \begin{bmatrix} -\eta_0(Z) \\ -\frac{1}{2}\eta_0(\frac{Z}{4}) \\ 0 \end{bmatrix}.$$

Thus the matrices H_1 and H_2 can be written as

$$H_1 = [\mathbf{0}\,|\,\mathbf{0}\,|\,v_1], \quad H_2 = [\mathbf{0}\,|\,\mathbf{0}\,|\,v_2],$$

where $\mathbf{0} = [0, 0, 0]^T$. Then we have

$$B = [\mathbf{0}\,|\,\mathbf{0}\,|\,v_1 - Av_2],$$

and, as the first two columns of B are zeros and the last row of v_0 is equal to 1, we get (9a). Then by substituting the expression $B = H_1 - AH_2$ in (9b) we obtain

$$A(F_3 - H_1F_2) = (F_1 - H_2F_2),$$

where

$$F_2 = \begin{bmatrix} -\eta_0(Z) & \frac{1}{2}\eta_0(Z) & -\frac{1}{2}\eta_1(Z) \\ -\frac{1}{2}\eta_0(\frac{Z}{4}) & \frac{1}{8}\eta_0(\frac{Z}{4}) & -\frac{1}{16}\eta_1(\frac{Z}{4}) \\ 0 & 0 & 0 \end{bmatrix}.$$

As $H_1F_2 = 0$ and $H_2F_2 = 0$ the matrix A satisfies (9b) and the thesis follows.

■

The corresponding classic peer method is obtained in the limit as $Z \to 0$ and has coefficients:

$$c = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{14a}$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ \frac{5}{24} & -\frac{2}{3} & \frac{23}{24} \\ \frac{7}{6} & -\frac{10}{3} & \frac{19}{6} \end{bmatrix}. \tag{14b}$$

V. Numerical experiments

We show the results obtained by integrating the following Prothero-Robinson problem [66]

$$\begin{aligned} &y'(t) = \lambda\,(\,y(t) - \sin(\omega\,t + t)\,) + (\omega + 1)\,\cos(\omega\,t + t), \\ &y(0) = 0, \end{aligned} \tag{15}$$

in the interval $t \in \left[0, \frac{\pi}{2}\right]$ with $\lambda = -1$. The exact solution is

$$y(t) = \sin(\omega\,t)\,\cos(t) + \cos(\omega\,t)\,\sin(t)$$

therefore we employ the exponentially fitted methods (11) and (13) with $\mu = \mathrm{i}\,\omega$. Fig. 1 shows that the exponentially fitted methods (11) and (13) are more accurate than their classic counterparts (12) and (14), which become totally inaccurate for highly oscillating problems.

Moreover, we estimate the order p of the exponentially fitted peer methods (11) and (13) employing the following relations

$$p = \lim_{h \to 0} p(h), \quad p(h) \approx \log_2\left(\frac{E(h)}{E(h/2)}\right), \tag{16}$$

where $E(h)$ and $E(h/2)$ are the errors with a stepsize h and $h/2$, respectively. Table I shows that the estimated order $p(h)$ of the exponentially fitted peer methods (11) and (13) are equal to 2 and 3, respectively.

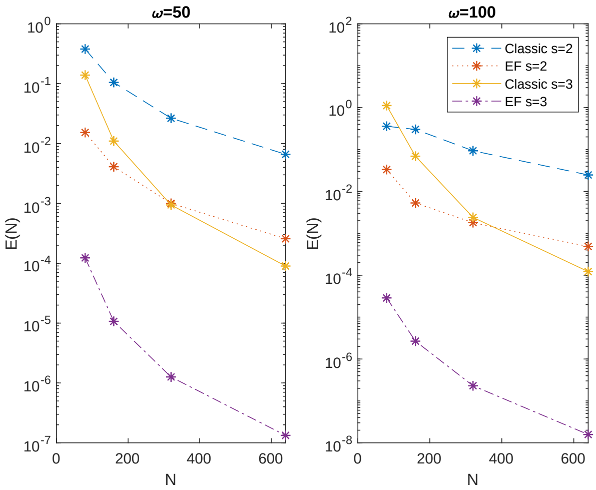

Fig. 1. Error $E(N)$ in the last end point of the interval obtained by integrating problem (15) with the classic peer methods (12) and (14) and the exponentially fitted ones (11) and (13) by using N grid points and different values for the frequency ω.

N	Exp. Fitted $s = 2$	Exp. Fitted $s = 3$
80	2.57	–
160	1.91	3.52
320	1.98	3.09
640	2.00	3.24

TABLE I
ESTIMATED ORDER OF THE EXPONENTIALLY FITTED PEER METHODS (11) AND (13) COMPUTED BY EQUATION (16) WITHIN THE INTEGRATION OF PROBLEM (15) WITH $\omega = 50$.

VI. Conclusions

We have constructed a general class of parallel EF peer methods. These methods are suitable for the numerical integration of ordinary differential equations having a solution with oscillatory behavior or an exponential decay. The adopted strategy is based on adapting existing methods in order to be exact (within round-off error) on trigonometric or hyperbolic functions. Numerical experiments have confirmed the effectiveness of the approach.

Future work will address the construction of implicit EF peer methods, suitable for the numerical integration of oscillatory stiff problems. Moreover we will employ the constructed methods for the numerical solution of ordinary differential systems arising in the semi-discretization in space of partial differential equations.

Appendix
$\eta_m(Z)$ functions

The set of functions $\eta_m(Z)$, $m = -1, 0, 1, 2, \dots$ has been originally introduced in [71] in the context of CP methods for the Schrödinger equation. The functions $\eta_m(Z)$ with $m = -1, 0$ are defined by

$$\eta_{-1}(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if } Z \le 0 \\ \cosh(Z^{1/2}) & \text{if } Z > 0 \end{cases} \tag{17}$$

$$\eta_0(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0 \\ 1 & \text{if } Z = 0 \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0 \end{cases} \tag{18}$$

and those with $m > 0$ are further generated by recurrence

$$\eta_m(Z) = \frac{1}{Z}[\eta_{m-2}(Z) - (2m-1)\eta_{m-1}(Z)], \quad m = 1, 2, 3, \dots \tag{19}$$

if $Z \neq 0$, and by following values at $Z = 0$:

$$\eta_m(0) = \frac{1}{(2m+1)!!}, \quad m = 1, 2, 3, \dots \tag{20}$$

The differentiation rule is

$$\eta'_m(Z) = \frac{1}{2}\eta_{m+1}(Z), \; m = -1, 0, 1, 2, 3, \dots \tag{21}$$

For more details on these functions see [34], [71] or the Appendix of [67].

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