Prime Geodesic Theorem for Compact Riemann Surfaces

Dženan Gušić

Abstract—As it is known, there have been a number of attempts to obtain precise estimates for the number of primes not exceeding x. A lot of them are related to the ones done by Chebyshev. Thus, a good deal is known about them and their limitations. The truth, or otherwise, of the Riemann hypothesis, however, has still not been established. In this paper we derive a prime geodesic theorem for a compact Riemann surface regarded as a quotient of the upper half-plane by a discontinuous group. We assume that the surface at case, considered as a compact Riemannian manifold, is equipped with classical Poincare metric. Our result follows from the standard theory of the zeta functions of Selberg and Ruelle. The closed geodesics in this setting are in one-to-one correspondence with the conjugacy classes of the corresponding group, so analysis conducted here is reminiscent of the relationship between the distribution of rational primes and Riemann zeta function. By analogy with the classical arithmetic case and the fact that the Riemann hypothesis is true in our setting, one would certainly expect to obtain an analogous error term in the prime geodesic theorem. Bearing in mind that the corresponding Selberg zeta funcion has much more zeros than the Riemann zeta, the latter is not satisfied however.

Keywords—Compact Riemann surfaces, prime geodesic theorem, upper half-plane, zeta functions, Laplace operator.

I. Introduction

THE positive integers other than 1 may be divided into two classes: prime numbers (which do not admit of resolution into smaller factors), and composite ones which do.

We shall denote by $\pi(x)$ the number of primes not exceeding x.

The statement

$$\frac{\pi(x)}{\frac{x}{\log x}} \to 1$$

as $x \to \infty$, is known as the prime number theorem and represents the central theorem in the theory of the distribution of primes. It is known that the problem of deciding its truth or falsehood captured particular attention of mathematicians for almost a hundred of years.

Any generalization of the prime number theorem to the more general situations is known as a prime geodesic theorem.

Almost all generalized papers treat the case of hyperbolic Riemann surfaces (see e.g., [14], [19]).

Gangolli [8] (see also, [7], [22]) and DeGeorge [4] independently proved that

$$\frac{\pi_\Gamma(x)}{\frac{x^{d-1}}{(d-1)\log x}} \to 1$$

as $x \to \infty$, for compact, d-dimensional locally symmetric spaces of real rank one, where $\pi_\Gamma(x)$ is the corresponding counting function, i.e., a yes function counting prime geodesics of the length not larger than $\log x$.

The same prime geodesic theorem was derived by Gangolli-Warner [9] when the underlying locally symmetric space is not necessarily compact but has a finite volume.

The first refinement of such prime geodesic theorem (in the case of non-compact, real hyperbolic manifolds with cusps) was given by [18] (see, [17] for a related work). Later, it was further improved by Avdispahić-Gušić [1].

In the case of compact hyperbolic Riemann surfaces (Γ is a co-finite discrete subgroup of $PSL(2,\mathbb{R})$), prime geodesic theorem with error terms is given by

$$\pi_\Gamma(x) = \sum_{\frac{3}{4}<s_n\leq 1} \mathrm{li}\left(x^{s_n}\right) + O\left(x^{\frac{3}{4}}(\log x)^{\alpha}\right)$$

as $x \to \infty$, where s_n is a zero of the corresponding Selberg zeta function, and $\alpha = -\frac{1}{2}$ [14] ($\alpha = -1$ [19]).

Note that there have been many works: Iwaniec [15], Luo-Sarnak [16], Cai [3], Soundararajan-Young [21], to achieve better error terms for a specific arithmetic discrete subgroup $\Gamma \subseteq PSL(2,\mathbb{R})$.

In the case of compact, locally symmetric spaces of real rank one, the best known error term in the prime geodesic theorem was given by Avdispahić-Gušić [2] (see also, [10], [13], [12]).

In this paper we pay our particular attention to the case of compact hyperbolic Riemann surfaces, and the corresponding prime geodesic theorem.

II. Preliminaries

Let Ω be a compact Riemann surface, regarded as a quotient of the upper half-plane H^+ by a discontinuous group Γ.

We assume that H^+ is equipped with the metric $y^{-2}\left((dx)^2 + (dy)^2\right)$.

Denote by A the volume of Ω.

As it is known, an element $\gamma \in \Gamma$, $\gamma \neq 1$ can be put into normal form $z \to N(\gamma)z$, with $N(\gamma) > 1$.

The number $N(\gamma)$ is the same within a conjugacy class, and is called the norm of the element.

Thus, $l(\gamma) = \log N(\gamma)$ is the length of the closed geodesic corresponding to the conjugacy class of γ.

Recall that a closed geodesic is called primitive if it is not a positive integral power of any geodesic other than itself.

Dž. Gušić is with the Department of Mathematics, Faculty of Sciences and Mathematics, University of Sarajevo, Zmaja od Bosne 33-35, 71000 Sarajevo, Bosnia and Herzegovina e-mail: dzenang@pmf.unsa.ba

Consequently, we define $\Lambda(\gamma)$ to be $\log N(\gamma_0)$, where $\gamma = \gamma_0^n$, and γ_0 is primitive.

Let g be the genus of Ω.

We assume that $g \geq 2$.

It is understood that $\Gamma \subseteq PSL(2, \mathbb{R})$.

The Gaussian curvature of Ω is -1.

Let Γ_h resp. $P\Gamma_h$ denote the set of the Γ-conjugacy classes of hyperbolic resp. primitive hyperbolic elements in Γ.

The Selberg zeta function is defined by (see, e.g., [20, p. 8], [14, p. 72])

$$Z(s) = \prod_{\gamma_0 \in P\Gamma_h} \prod_{k=0}^{+\infty} \left(1 - e^{-(s+k)l(\gamma_0)}\right),$$

$\mathrm{Re}(s) > 1$.

By [14, p. 72, Th. 4.10.]:

$Z(s)$ is an entire function,

$Z(s)$ has trivial zeros $s = -k$, $k \geq 1$ with multiplicities $(2g-2)(2k+1)$,

$s = 0$ is a zero of multiplicity $2g-1$,

$s = 1$ is a zero of multiplicity 1,

the non-trivial zeros of $Z(s)$ are located at $\frac{1}{2} \pm \mathrm{i}\, r_n$.

More precisely, the zeros in the critical strip $0 < \mathrm{Re}(s) < 1$ are located at points which are solutions of the equations $s(1-s) = \lambda_n$, where λ_n runs through the sequence of eigenvalues (omitting $\lambda_0 = 0$) for the problem $\Delta f + \lambda f = 0$ on Ω, where Δ is the Laplace operator on Ω.

Thus, the numbers r_n are normalized by the condition $\mathrm{Arg}(r_n) \in \left\{0, -\frac{\pi}{2}\right\}$, so the non-trivial zeros are located precisely at the points

$$\begin{aligned} s_n &= \frac{1}{2} + \mathrm{i}\, r_n, \\ \tilde{s}_n &= \frac{1}{2} - \mathrm{i}\, r_n. \end{aligned}$$

Hence, apart from a finite number of exceptional zeros s_0, $\tilde{s}_0$,..., s_M, $\tilde{s}_M$ concentrated along $[0,1]$, the non-trivial zeros of $Z(s)$ all lie on the line $\mathrm{Re}(s) = \frac{1}{2}$ (the Riemann hypothesis is true for $Z(s)$).

Let $\pi_\Gamma(x)$ be the number of prime geodesics over Ω, whose length is not larger then $\log x$.

In [19], the author derived a prime goedesic theorem for compact Riemann surfaces.

The main purpose of this paper is to give yet another proof of the same theorem (with the same error terms) by using different means.

Note that one may interpret $l(\gamma)$ as the period of a periodic orbit for the geodesic flow on Ω.

This suggests defining a zeta function by

$$Z_R(s) = \prod_{\gamma \in P} \left(1 - e^{-s\tau(\gamma)}\right)^{-1},$$

where P is the set of periodic orbits, and $\tau(\gamma)$ is the period of γ.

In our setting, this zeta function reads as

$$Z_R(s) = \prod_{\gamma_0 \in P\Gamma_h} \left(1 - e^{-sl(\gamma_0)}\right)^{-1},$$

$\mathrm{Re}(s) > 1$, and is called the Ruelle zeta function.

Define $N(t)$ to be the number of zeros of $Z(s)$ on the critical line on the interval $\frac{1}{2} + \mathrm{i}\, x$, with $0 < x \leq t$.

III. Preliminary results

The following results will be applied in the sequel.

Theorem 1. *[14, p. 81, Th. 4.25.] $Z(s)$ is an entire function of order 2.*

Theorem 2. *[6, p. 509, Prop. 7.] Suppose $Z(s)$ is the ratio of two nonzero entire functions of order at most n. Then, there is a $D > 0$ such that for arbitrarily large choices of r*

$$\int_r \left| \frac{Z'(s)}{Z(s)} \right| |ds| \leq D r^n \log r.$$

Theorem 3. *[14, p. 102, Th. 6.4.] Let $s = \sigma + \mathrm{i}T$, where $-1 \leq \sigma \leq 2$, and $T \neq r_n$ for all r_n. Then,*

$$\frac{Z'(s)}{Z(s)} = O(T) + \sum_{|r_n - T| \leq 1} \frac{1}{s - \frac{1}{2} - \mathrm{i}\, r_n}.$$

Theorem 4. *[14, p. 102, Prop. 6.6.] Suppose that $0 < \varepsilon < 1$, $s = \sigma + \mathrm{i}\, t$, $t \geq 1000$. Then,*

$$\frac{Z'(s)}{Z(s)} = O\left(\varepsilon^{-1} t\right)$$

for $\sigma \geq \frac{1}{2} + \varepsilon$.

IV. Main result

The following theorem is the main result of this paper.

Theorem 5. *Let Ω be as above. Then,*

$$\pi_\Gamma(x) = \mathrm{li}(x) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + O\left(x^{\frac{3}{4}} (\log x)^{-1}\right)$$

as $x \to \infty$, where $s_k \in [0,1]$, $k \in \{1, 2, ..., M\}$ is a zero of the corresponding Selberg zeta function $Z(s)$.

Proof: Suppose that $k \geq 2$ is an integer.

Let $x > 1$, and $c > 2 \cdot \frac{1}{2} = 1$.

We define,

$$\begin{aligned} \psi_0(x) &= \sum_{N(\gamma) \leq x} \Lambda(\gamma), \\ \psi_j(x) &= \int_0^x \psi_{j-1}(t)\, dt, \end{aligned}$$

where $j = 1, 2,...$.

As it is known,

$$\psi_j(x) = (j!)^{-1} \sum_{N(\gamma)\leq x} \Lambda(\gamma)(x - N(\gamma))^j .$$

Furthermore, by [19, p. 244],

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\infty}^{c+\mathrm{i}\,\infty}\frac{Z'(s)}{Z(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^s ds\\=&\frac{1}{k!}\sum_{N(\gamma)\leq x}\Lambda(\gamma)\left(1-\frac{N(\gamma)}{x}\right)^k+\\&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\infty}^{c+\mathrm{i}\,\infty}\frac{Z'(s+1)}{Z(s+1)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^s ds.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\infty}^{c+\mathrm{i}\,\infty}\frac{Z'(s)}{Z(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds-\\&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\infty}^{c+\mathrm{i}\,\infty}\frac{Z'(s+1)}{Z(s+1)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds.\\=&(k!)^{-1}\sum_{N(\gamma)\leq x}\Lambda(\gamma)(x-N(\gamma))^k\\=&\psi_k(x).\end{aligned}$$

As already noted (see, [20, p. 9]),

$$Z_R(s) = \frac{Z(s+1)}{Z(s)}. \tag{1}$$

Hence,

$$\frac{Z'(s)}{Z(s)} - \frac{Z'(s+1)}{Z(s+1)} = -\frac{Z_R'(s)}{Z_R(s)}. \tag{2}$$

Consequently,

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\infty}^{c+\mathrm{i}\,\infty}-\frac{Z_R'(s)}{Z_R(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds\\=&\psi_k(x).\end{aligned}$$

Let $T_1 \gg 0$ be arbitrary number.

Following [19, p. 243], we consider the segment of the critical line $\frac{1}{2} + \mathrm{i}\,t$, with $T_1 - 1 < t \leq T_1 + 1$.

Applying the Dirichlet principle, we obtain that there exists a $\frac{1}{2} + \mathrm{i}\,\tilde{T}$ in the segment whose distance from any root of $Z(s)$ is greater than $\frac{C_1}{\tilde{T}}$, for some fixed $C_1 > 0$.

Thus,

$$\left|\frac{1}{2} + \mathrm{i}\,\tilde{T} - \alpha\right| > \frac{C_1}{\tilde{T}} \tag{3}$$

for $\alpha \in S_{Sel}$, where S_{Sel} is the set of zeros of the Selberg zeta function $Z(s)$.

We put

$$\begin{aligned}&R(T)\\=&\left\{s\in\mathbb{C}\,:\,|s|\leq T, \mathrm{Re}(s)\leq\frac{1}{2}\right\}\cup\\&\left\{s\in\mathbb{C}\,:\,\frac{1}{2}\leq\mathrm{Re}(s)\leq c, -\tilde{T}\leq\mathrm{Im}(s)\leq\tilde{T}\right\},\end{aligned}$$

where $T = \sqrt{\tilde{T}^2 + \frac{1}{4}}$.

Bearing in mind the choice of the value $\tilde{T}$, and the location of the zeros of $Z(s)$, we conclude that no pole of

$$-\frac{Z_R'(s)}{Z_R(s)}s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}$$

occurs on the square part of the boundary of $R(T)$.

Furthermore, we may, without loss of generality, assume that no pole of

$$-\frac{Z_R'(s)}{Z_R(s)}s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}$$

occurs on the circular part of the boundary of $R(T)$ (see, e.g., [18, p. 98]).

Now, we are in position to apply the Cauchy integral formula to the integrand of $\psi_k(x)$ over $R(T)$, to obtain

$$\begin{aligned}&\int\limits_{R(T)^+}-\frac{Z_R'(s)}{Z_R(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds\\=&2\pi\,\mathrm{i}\sum_{z\in R(T)}\mathrm{Res}_{s=z}\Bigg(-\frac{Z_R'(s)}{Z_R(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}\Bigg),\end{aligned}$$

where $R(T)^+$ denotes the boundary of $R(T)$ with the anti-clockwise orientation.

Therefore,

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\tilde{T}}^{c+\mathrm{i}\,\tilde{T}}-\frac{Z_R'(s)}{Z_R(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds\end{aligned} \tag{4}$$

$$
\begin{aligned}
&= \sum_{z \in R(T)} \operatorname{Res}_{s=z} \left(-\frac{Z_R'(s)}{Z_R(s)} \times \right. \\
&\quad \left. \times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} \right) - \\
&\quad \frac{1}{2\pi \mathrm{i}} \int_{C_T} -\frac{Z_R'(s)}{Z_R(s)} \times \\
&\quad \times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\quad \frac{1}{2\pi \mathrm{i}} \int_{\frac{1}{2}+\mathrm{i}\tilde{T}}^{\frac{1}{2}+\delta+\mathrm{i}\tilde{T}} -\frac{Z_R'(s)}{Z_R(s)} \times \\
&\quad \times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\quad \frac{1}{2\pi \mathrm{i}} \int_{\frac{1}{2}+\delta-\mathrm{i}\tilde{T}}^{\frac{1}{2}-\mathrm{i}\tilde{T}} -\frac{Z_R'(s)}{Z_R(s)} \times \\
&\quad \times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\quad \frac{1}{2\pi \mathrm{i}} \int_{\frac{1}{2}+\delta+\mathrm{i}\tilde{T}}^{c+\mathrm{i}\tilde{T}} -\frac{Z_R'(s)}{Z_R(s)} \times \\
&\quad \times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\quad \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\tilde{T}}^{\frac{1}{2}+\delta-\mathrm{i}\tilde{T}} -\frac{Z_R'(s)}{Z_R(s)} \times \\
&\quad \times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds,
\end{aligned}
$$

where C_T is the circular part of $R(T)^+$.

By [19, p. 242, Lemma 1],

$$
\frac{Z'(s)}{Z(s)} = \sum_{\gamma \in \Gamma_h} \Lambda(\gamma) N(\gamma)^{-s} + \frac{Z'(s+1)}{Z(s+1)}
$$

for $\operatorname{Re}(s) > 1$.

Therefore, $\frac{Z_R'(s)}{Z_R(s)}$ is bounded in any half-plane of the form $\operatorname{Re}(s) > 1 + \varepsilon$.

Since

$$
\begin{aligned}
&\psi_k(x) \\
&= \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\tilde{T}}^{c+\mathrm{i}\tilde{T}} -\frac{Z_R'(s)}{Z_R(s)} \times \\
&\quad \times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\quad \frac{1}{2\pi \mathrm{i}} \int_{c+\mathrm{i}\tilde{T}}^{c+\mathrm{i}\infty} -\frac{Z_R'(s)}{Z_R(s)} \times \\
&\quad \times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\quad \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c-\mathrm{i}\tilde{T}} -\frac{Z_R'(s)}{Z_R(s)} \times \\
&\quad \times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds,
\end{aligned} \tag{5}
$$

we estimate

$$
\begin{aligned}
&\frac{1}{2\pi \mathrm{i}} \int_{c+\mathrm{i}\tilde{T}}^{c+\mathrm{i}\infty} -\frac{Z_R'(s)}{Z_R(s)} \times \\
&\times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds \\
=& O\left(x^{c+k} \int_{\tilde{T}}^{+\infty} \frac{dt}{t^{k+1}} \right) = O\left(x^{c+k} \tilde{T}^{-k} \right).
\end{aligned} \tag{6}
$$

Similarly,

$$
\begin{aligned}
&\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c-\mathrm{i}\tilde{T}} -\frac{Z_R'(s)}{Z_R(s)} \times \\
&\times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds \\
=& O\left(x^{c+k} \tilde{T}^{-k} \right).
\end{aligned} \tag{7}
$$

Combining (4)-(7), we obtain

$$
\begin{aligned}
&\psi_k(x) - O\left(x^{c+k} \tilde{T}^{-k} \right) \\
&= \sum_{z \in R(T)} \operatorname{Res}_{s=z} \left(-\frac{Z_R'(s)}{Z_R(s)} \times \right. \\
&\quad \left. \times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} \right) - \\
&\quad \frac{1}{2\pi \mathrm{i}} \int_{C_T} -\frac{Z_R'(s)}{Z_R(s)} \times \\
&\quad \times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\quad \frac{1}{2\pi \mathrm{i}} \int_{\frac{1}{2}+\mathrm{i}\tilde{T}}^{\frac{1}{2}+\delta+\mathrm{i}\tilde{T}} -\frac{Z_R'(s)}{Z_R(s)} \times \\
&\quad \times\, s^{-1} (s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\quad \frac{1}{2\pi \mathrm{i}} \int_{\frac{1}{2}+\delta-\mathrm{i}\tilde{T}}^{\frac{1}{2}-\mathrm{i}\tilde{T}} -\frac{Z_R'(s)}{Z_R(s)} \times
\end{aligned} \tag{8}
$$

$$\times s^{-1}(s+1)^{-1}\dots(s+k)^{-1}x^{s+k}ds+$$
$$\frac{1}{2\pi\,\mathrm{i}}\int_{\frac{1}{2}+\delta+\mathrm{i}\,\tilde{T}}^{c+\mathrm{i}\,\tilde{T}}-\frac{Z_R'(s)}{Z_R(s)}\times$$
$$\times s^{-1}(s+1)^{-1}\dots(s+k)^{-1}x^{s+k}ds+$$
$$\frac{1}{2\pi\,\mathrm{i}}\int_{c-\mathrm{i}\,\tilde{T}}^{\frac{1}{2}+\delta-\mathrm{i}\,\tilde{T}}-\frac{Z_R'(s)}{Z_R(s)}\times$$
$$\times s^{-1}(s+1)^{-1}\dots(s+k)^{-1}x^{s+k}ds.$$

Now, we estimate the integrals on the right hand side of (8).

By (1) and Theorem 1, a meromorphic extension over $\mathbb{C}$ of the Ruelle zeta function $Z_R(s)$ is a quotient of two entire functions of order 2 over $\mathbb{C}$.

Thus, by Theorem 2,

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int_{C_T}-\frac{Z_R'(s)}{Z_R(s)}\times\\&\times s^{-1}(s+1)^{-1}\dots(s+k)^{-1}x^{s+k}ds\\=&O\left(x^{\frac{1}{2}+k}T^{-k-1}\int_{C_T}\left|\frac{Z_R'(s)}{Z_R(s)}\right||ds|\right)\\=&O\left(x^{\frac{1}{2}+k}T^{-k-1}\int_{|s|=T}\left|\frac{Z_R'(s)}{Z_R(s)}\right||ds|\right)\\=&O\left(x^{\frac{1}{2}+k}T^{-k+1}\log T\right).\end{aligned}\tag{9}$$

By Theorem 3 and (2), we obtain that for $s=\sigma_1+\mathrm{i}\,\tilde{T}$, $\frac{1}{2}\leq\sigma_1\leq\frac{1}{2}+\delta$,

$$\begin{aligned}-\frac{Z_R'(s)}{Z_R(s)}=&\frac{Z'(s)}{Z(s)}-\frac{Z'(s+1)}{Z(s+1)}\\=&O\left(\tilde{T}\right)+\sum_{\left|\tilde{T}-r_n\right|\leq1}\frac{1}{s-\frac{1}{2}-\mathrm{i}\,r_n}-\\&\frac{Z'(s+1)}{Z(s+1)}.\end{aligned}$$

Since $\frac{Z'(s+1)}{Z(s+1)}$ is bounded for $s=\sigma_1+\mathrm{i}\,\tilde{T}$, $\frac{1}{2}\leq\sigma_1\leq\frac{1}{2}+\delta$, $N(t)=\frac{A}{4\pi}t^2+O(t)$, and (3) holds true, we estimate

$$\begin{aligned}&-\frac{Z_R'(s)}{Z_R(s)}\\=&O\left(\tilde{T}\right)+\sum_{\left|\tilde{T}-r_n\right|\leq1}\frac{1}{s-\frac{1}{2}-\mathrm{i}\,r_n}+O(1)\\=&O\left(\tilde{T}\right)+O\left(\sum_{\left|\tilde{T}-r_n\right|\leq1}\frac{1}{\left|\frac{1}{2}+\mathrm{i}\,\tilde{T}-\frac{1}{2}-\mathrm{i}\,r_n\right|}\right)\\=&O\left(\tilde{T}\right)+O\left(\tilde{T}\sum_{\left|\tilde{T}-r_n\right|\leq1}1\right)\\=&O\left(\tilde{T}\right)+O\left(\tilde{T}^2\right)=O\left(\tilde{T}^2\right)=O\left(T^2\right)\end{aligned}$$

for $s=\sigma_1+\mathrm{i}\,\tilde{T}$, $\frac{1}{2}\leq\sigma_1\leq\frac{1}{2}+\delta$.

Thus,

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int_{\frac{1}{2}+\mathrm{i}\,\tilde{T}}^{\frac{1}{2}+\delta+\mathrm{i}\,\tilde{T}}-\frac{Z_R'(s)}{Z_R(s)}\times\\&\times s^{-1}(s+1)^{-1}\dots(s+k)^{-1}x^{s+k}ds\\=&O\left(x^{\frac{1}{2}+\delta+k}T^{-k+1}\right).\end{aligned}\tag{10}$$

Similarly,

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int_{\frac{1}{2}+\delta-\mathrm{i}\,\tilde{T}}^{\frac{1}{2}-\mathrm{i}\,\tilde{T}}-\frac{Z_R'(s)}{Z_R(s)}\times\\&\times s^{-1}(s+1)^{-1}\dots(s+k)^{-1}x^{s+k}ds\\=&O\left(x^{\frac{1}{2}+\delta+k}T^{-k+1}\right).\end{aligned}\tag{11}$$

By Theorem 4 and (2), we obtain that for $s=\sigma_1+\mathrm{i}\,\tilde{T}$, $\frac{1}{2}+\delta\leq\sigma_1\leq c$,

$$\begin{aligned}-\frac{Z_R'(s)}{Z_R(s)}=&\frac{Z'(s)}{Z(s)}-\frac{Z'(s+1)}{Z(s+1)}\\=&O\left(\delta^{-1}\tilde{T}\right)-\frac{Z'(s+1)}{Z(s+1)}.\end{aligned}$$

Since $\frac{Z'(s+1)}{Z(s+1)}$ is bounded for $s=\sigma_1+\mathrm{i}\,\tilde{T}$, $\frac{1}{2}+\delta\leq\sigma_1\leq c$, it follows that

$$-\frac{Z_R'(s)}{Z_R(s)}=O\left(\delta^{-1}\tilde{T}\right)=O\left(\delta^{-1}T\right)$$

for $s=\sigma_1+\mathrm{i}\,\tilde{T}$, $\frac{1}{2}+\delta\leq\sigma_1\leq c$.

Hence,

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int_{\frac{1}{2}+\delta+\mathrm{i}\,\tilde{T}}^{c+\mathrm{i}\,\tilde{T}}-\frac{Z_R'(s)}{Z_R(s)}\times\\&\times s^{-1}(s+1)^{-1}\dots(s+k)^{-1}x^{s+k}ds\\=&O\left(x^{c+k}T^{-k-1}\int_{\frac{1}{2}+\delta+\mathrm{i}\,\tilde{T}}^{c+\mathrm{i}\,\tilde{T}}\left|\frac{Z_R'(s)}{Z_R(s)}\right||ds|\right)\\=&O\left(\delta^{-1}x^{c+k}T^{-k}\right).\end{aligned}\tag{12}$$

Similarly,

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int_{c-\mathrm{i}\,\tilde{T}}^{\frac{1}{2}+\delta-\mathrm{i}\,\tilde{T}}-\frac{Z_R^{'}(s)}{Z_R(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds\\=&O\left(\delta^{-1}x^{c+k}T^{-k}\right).\end{aligned} \tag{13}$$

Combining (8)-(13), taking into account (2), the fact that $k \geq 2$, and letting $T \to \infty$, we obtain that

$$\begin{aligned}&\psi_k(x)\\=&\sum_{z\in A_{0,k}}\mathrm{Res}_{s=z}\left(\frac{Z^{'}(s)}{Z(s)}\times\right.\\&\left.\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}\right)-\\&\sum_{z\in A_{1,k}}\mathrm{Res}_{s=z}\left(\frac{Z^{'}(s+1)}{Z(s+1)}\times\right.\\&\left.\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}\right)\\=&\sum_{z\in A_{0,k}}c_z(0,k)-\sum_{z\in A_{1,k}}c_z(1,k),\end{aligned}$$

where $A_{i,k}$, $i \in \{0,1\}$ denotes the set of poles of

$$\frac{Z^{'}(s+i)}{Z(s+i)}s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k},$$

and

$$\begin{aligned}&c_z(i,k)\\=&\mathrm{Res}_{s=z}\left(\frac{Z^{'}(s+i)}{Z(s+i)}\times\right.\\&\left.\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}\right).\end{aligned}$$

Putting $k=2$, we end up with

$$\begin{aligned}&\psi_2(x)\\=&\sum_{z\in A_{0,2}}c_z(0,2)-\sum_{z\in A_{1,2}}c_z(1,2).\end{aligned} \tag{14}$$

Note that the equality (14) represents the equality from Theorem $1^{'}$ in [19, p. 245] for $k=2$.

Now, proceeding in exactly the same way as in [19, pp. 245-246], one obtains the equality from Theorem 2 in [19, p. 245].

Taking into account our notation, the aforementioned equality may be written in the following, equivalent form

$$\psi_0(x)=\sum_{n=0}^{M}\frac{x^{s_n}}{s_n}+O\left(x^{\frac{3}{4}}\right),$$

or

$$\pi_\Gamma(x)=\mathrm{li}(x)+\sum_{n=1}^{M}\mathrm{li}(x^{s_n})+O\left(x^{\frac{3}{4}}(\log x)^{-1}\right).$$

This completes the proof. ■

V. Conclusion

The idea for this research comes from [18], where the author applied the contour integration over circular boundaries. Recently, in [10] and [13], the author derived general results, which are analogous to the present result, in the case of compact, even-dimensional and odd-dimensional locally symmetric Riemannian manifolds of strictly negative sectional curvature, respectively. For a weighted form of the even-dimensional result, we refer to [11]. In the future work the author plans to consider a weighted form as well as a logarithmic form of the corresponding prime geodesic theorem in the case of compact symmetric spaces formed as quotients of the Lie group $SL_4(\mathbb{R})$ (see, [5]).

References

[1] M. Avdispahić and Dž. Gušić, "On the error term in the prime geodesic theorem," *Bull. Korean Math. Soc.*, vol. 49, pp. 367–372, 2012.

[2] M. Avdispahić and Dž. Gušić, "On the length spectrum for compact locally symmetric spaces of real rank one," *WSEAS Trans. on Math.*, vol. 16, pp. 303–321, 2017.

[3] Y. Cai, "Prime geodesic theorem," *J. Theor. Nombres Bordeaux*, vol. 14, pp. 59–72, 2002.

[4] D. L. DeGeorge, "Length spectrum for compact locally symmetric spaces of strictly negative curvature," *Ann. Sci. Ec. Norm. Sup.*, vol. 10, pp. 133–152, 1977.

[5] A. Deitmar and M. Pavey, "A prime geodesic theorem for SL_4," *Ann. Glob. Anal. Geom.*, vol. 33, pp. 161–205, 2008.

[6] D. Fried, "The zeta functions of Ruelle and Selberg. I," *Ann. Sci. Ec. Norm. Sup.*, vol. 19, pp. 491–517, 1986.

[7] R. Gangolli, "Zeta functions of Selberg's type for compact space forms of symmetric spaces of rank one," *Illinois J. Math.*, vol. 21, pp. 1–41, 1977.

[8] R. Gangolli, "The length spectra of some compact manifolds of negative curvature," *J. Diff. Geom.*, vol. 12, pp. 403–424, 1977.

[9] R. Gangolli and G. Warner, "Zeta functions of Selberg's type for some noncompact quotients of symmetric spaces of rank one," *Nagoya Math.*, vol. 78, pp. 1–44, 1980.

[10] Dž. Gušić, "Prime geodesic theorem for compact even-dimensional locally symmetric Riemannian manifolds of strictly negative sectional curvature," *WSEAS Trans. on Math.*, vol. 17, pp. 188–196, 2018.

[11] Dž. Gušić, "A Weighted Generalized Prime Geodesic Theorem," *WSEAS Trans. on Math.*, vol. 17, pp. 237–251, 2018.

[12] Dž. Gušić, "On the logarithmic derivative of zeta functions for compact, odd-dimensional hyperbolic spaces," *WSEAS Trans. on Math.*, vol. 18, pp. 176–184, 2019.

[13] Dž. Gušić, "On the Length Spectrum for Compact, Odd-dimensional, Real Hyperbolic Spaces," *WSEAS Trans. on Math.*, vol. 18, pp. 211–222, 2019.

[14] D. Hejhal, *The Selberg trace formula for $PSL(2,\mathbb{R})$. Vol. I. Lecture Notes in Mathematics 548.* Berlin–Heidelberg: Springer–Verlag, 1976.

[15] H. Iwaniec, "Prime geodesic theorem," *J. Reine Angew. Math.*, vol. 349, pp. 136–159, 1984.

[16] W. Z. Luo and P. Sarnak, "Quantum ergodicity of eigenvalues on $PSL_2(\mathbb{Z}) \setminus H^2$," *Inst. Hautes Etudes Sci. Publ. Math.*, vol. 81, pp. 207–237, 1995.

[17] M. Nakasuji, "Prime geodesic theorem for higher-dimensional hyperbolic manifold," *Trans. Amer. Math. Soc.*, vol. 358, pp. 3285–3303, 2006.

[18] J. Park, "Ruelle zeta function and prime geodesic theorem for hyperbolic manifolds with cusps," in *Casimir force, Casimir operators and Riemann hypothesis*, Berlin 2010, pp. 89–104.

[19] B. Randol, "On the asymptotic distributon of closed geodesics on compact Riemann surfaces," *Trans. Amer. Math. Soc.*, vol. 233, pp. 241–247, 1977.

[20] D. Ruelle, *Dynamical Zeta Functions for Piecewise Monotone Maps of the Interval. CRM Monograph Series Vol. 4.* Providence: American Mathematical Society, 1993.
[21] K. Soundararajan and M. P. Young, ”The prime geodesic theorem,” *J. Reine Angew. Math.*, vol. 676, pp. 105-120, 2013.
[22] M. Wakayama, ”Zeta functions of Selberg’s type associated with homogeneous vector bundles,” *Hiroshima. Math. J.*, vol. 15, pp. 235–295, 1985.