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Approximate Formulas for Zeta Functions of Selberg's Type in Quotients of SL_4

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Abstract—The goal of the paper is to derive some approximate formulas for the logarithmic derivative of several zata functions of Selberg's type for compact symmetric spaces formed as quotients of the Lie group $SL_4(\mathbb{R})$. Such formulas, known in literature as Tutchmarsh-Landau style approximate formulas, are usually applied in order to obtain prime geodesic theorems in various settings of underlying locally symmetric spaces.

Keywords—Approximate formulas, logarithmic derivative, zeta functions, Selberg, Ruelle.

I. Preliminaries

OUR notation is based on [11] and [4]. We introduce it step by step.

Let $G = SL_4(\mathbb{R})$.

Suppose that K is the maximal compact subgroup of G.

Then, $K = SO(4)$.

Finally, let $\Gamma \subset G$ be discrete and co-compact.

Let

$$A = \left\{ \begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1} \end{pmatrix} : a > 0 \right\}$$

be the rank one torus, and

$$B = \begin{pmatrix} SO(2) & \\ & SO(2) \end{pmatrix}$$

a compact Cartan subgroup of

$$\begin{aligned} &M \\ &\cong \Big\{ (x, y) \in \mathrm{Mat}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{R}) : \\ &\det(x), \det y = \pm 1, \det x \det y = 1 \Big\}. \end{aligned}$$

Let

$$N = \begin{pmatrix} I_2 & \mathrm{Mat}_2(\mathbb{R}) \\ 0 & I_2 \end{pmatrix}$$

and $P = MAN$ be a parabolic subgroup of G with Levi component MA and unipotent radical N.

Put

$$A^- = \left\{ \begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1} \end{pmatrix} : 0 < a < 1 \right\}$$

to be the negative Weyl chamber in A with respect to the root system given by the choice of parabolic.

Let $\mathcal{E}_P(\Gamma)$ be the set of Γ-conjugacy classes of elements $\gamma \in \Gamma$ which are conjugate in G to an element of $A^- B$.

We say that $g \in G$ is regular if its centralizer is a torus and non-regular otherwise.

Let $[\gamma] \in \mathcal{E}_P(\Gamma)$.

We denote by G_γ the centralizer of γ in G.

The element γ is conjugate in G to an element $a_\gamma b_\gamma \in A^- B$, and we define the length l_γ of γ to be

$$l_\gamma = b(\log a_\gamma, \log b_\gamma)^{\frac{1}{2}},$$

where b is an invariant bilinear form on $\mathfrak{g} = sl_4(\mathbb{C})$ ($\mathfrak{g}$ is the complexified Lie algebra of G).

It follows that if γ is regular, then $G_\gamma \cong AB$.

In the first case, $K_\gamma = B$ is a maximal compact subgroup of G_γ.

The group B is then a Cartan subgroup of K_γ.

Furthermore, the product AB is a θ-stable Cartan subgroup of G_γ, and A is central in G_γ, where θ is the Cartan involution fixing K pointwise.

We put $\Gamma_\gamma = \Gamma \cap G_\gamma$ to be the centralizer of γ in Γ.

Then, Γ_γ is discrete and co-compact in G_γ.

Let Γ' be a torsion free subgroup of finite index in Γ, and let $\Gamma'_\gamma = \Gamma' \cap G_\gamma$.

Then, Γ'_γ is a torsion free subgroup of finite index in Γ_γ.

We define $\Gamma_{\gamma,A}$, $\Gamma'_{\gamma,A}$ in the same way as the author defined Γ_A, Γ'_A in [11, p. 6].

Thus, the higher Euler characteristics $\chi_1(\Gamma_\gamma)$ of Γ_γ is then defined by

$$\chi_1(\Gamma_\gamma) = \frac{\left[\Gamma_{\gamma,A} : \Gamma'_{\gamma,A}\right]}{\left[\Gamma_\gamma : \Gamma'_\gamma\right]} \chi_1\left(\Gamma'_\gamma\right),$$

where $\chi_1\left(\Gamma'_\gamma\right)$ is defined (since Γ'_γ is torsion free) via equality

$$\chi_1(\Gamma) = \chi_1(X_\Gamma) = \sum_{j=0}^{\dim X_\Gamma} (-1)^{j+1} j h^j(X_\Gamma),$$

where Γ is any torsion free subgroup of G, $X_\Gamma = \Gamma \setminus G / K$, $h^j(X_\Gamma)$ is the j-th Betti number of the symmetric space X_Γ.

In the second case, $K_\gamma \cong S(O(2) \times O(2))$ is a maximal compact subgroup of G_γ.

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Moreover, B is a Cartan subgroup of K_γ, the product AB is a θ-stable Cartan subgroup of G_γ, and A is central in G_γ.

The definition of $\chi_1(\Gamma_\gamma)$ then proceeds exactly as above.

Put

$$\bar{N} = \begin{pmatrix} I_2 & 0 \\ \mathrm{Mat}_2(\mathbb{R}) & I_2 \end{pmatrix}.$$

Let $\mathfrak{n}$, $\bar{\mathfrak{n}}$ be the complexified Lie algebras of N, $\bar{N}$, respectively.

For any finite-dimensional virtual representation σ of M (see, [4, p. 169]), we define, for $\mathrm{Re}(s)$ large, the generalized Selberg zeta function

$$Z_{P,\sigma}(s) = \exp\left(-\sum_{[\gamma]\in\mathcal{E}_P(\Gamma)} \frac{\mathrm{tr}\,\sigma(b_\gamma)\,\chi_1(\Gamma_\gamma)\,l_{\gamma_0}}{l_\gamma \det\left(1-(a_\gamma b_\gamma)^{-1}|_{\bar{\mathfrak{n}}}\right)} e^{-sl_\gamma}\right).$$

For any finite-dimensional virtual representation σ of M, and $\mathrm{Re}(s) \gg 0$, the generalized Ruelle zeta function $R_{\Gamma,\sigma}(s)$ is defined in [11, p. 43].

The following equality holds true (see, [4, p. 169], [5])

$$R_{\Gamma,\sigma}(s) = \prod_{q=0}^{4} Z_{P,(\bigwedge^q \bar{\mathfrak{n}}\otimes V_\sigma)}\left(s+\frac{q}{4}\right)^{(-1)^q}.$$

Here, we note that $\gamma \in \Gamma$ is called primitive if for $\delta \in \Gamma$ and $n \in \mathbb{N}$, the equation $\delta^n = \gamma$ yields that $n = 1$.

Thus, we write γ for an element of $\mathcal{E}_P(\Gamma)$ and γ_0 for a primitive element.

II. Main result

The main purpose of this paper is to prove the following theorem.

Theorem 1. *Let $\varepsilon > 0$, $\eta > 0$.*

(a) (i) *Suppose $t \gg 0$ is chosen so that $\frac{1}{2} + \mathrm{i}\,t$ is not a singularity of $Z_{P,1}(s)$. Then,*

$$\frac{Z'_{P,1}(s)}{Z_{P,1}(s)} = O\left(t^{J-1+\varepsilon}\right) + \sum_{|t-\gamma_{P,1}|\le 1} \frac{1}{s-\rho_{P,1}}$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} \le \sigma^1 < \frac{1}{4}t + \frac{1}{2}$, where $\rho_{P,1} = \frac{1}{2} + \mathrm{i}\,\gamma_{P,1}$ is a singularity of $Z_{P,1}(s)$ on the line $\mathrm{Re}(s) = \frac{1}{2}$.

(ii)

$$\frac{Z'_{P,1}(s)}{Z_{P,1}(s)} = O\left(\frac{1}{\eta} t^{J-1+\varepsilon}\right)$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} + \eta \le \sigma^1 < \frac{1}{4}t + \frac{1}{2}$.

(b) *Suppose $t \gg 0$ is chosen so that $\frac{1}{2} + \mathrm{i}\,t$ is not a singularity of $Z_{P,(\bigwedge^q \bar{\mathfrak{n}})}(s)$, $q \in \{0,1,...,4\}$. Then,*

(i)

$$\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} = O\left(t^{J-1+\varepsilon}\right) + \sum_{|t-\gamma_{P,1}|\le 1} \frac{1}{s-\rho_{P,1}}$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} \le \sigma^1 < \frac{1}{4}t - \frac{1}{2}$.

(ii)

$$\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} = O\left(\frac{1}{\eta} t^{J-1+\varepsilon}\right)$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} + \eta \le \sigma^1 < \frac{1}{4}t - \frac{1}{2}$.

Proof: (a) (i) We put $r = \frac{1}{2}t$.

Select c such that $1 < c < \frac{1}{8}t + \frac{1}{2}$, and define $s_0 = c + \mathrm{i}\,t$.

Since $c < \frac{1}{8}t + \frac{1}{2}$, we conclude that the circles $|s-s_0| \le \frac{1}{2}t$, $|s-s_0| \le \frac{1}{4}t$ and $|s-s_0| \le \frac{1}{8}t$ cross the line $\mathrm{Re}(s) = \frac{1}{2}$.

Recall Proposition 2.4.4 in [11, p. 46] and Theorem 2.2.1 in [11, p. 34].

By these results and the discussion given in [11, p. 46], $Z_{P,1}(s)$ has a double zero at $s = 1$, and the remaining poles and zeros of $Z_{P,1}(s)$ lie in $\left[0, \frac{3}{4}\right] \cup \left(\frac{1}{2} + \mathrm{i}\,\mathbb{R}\right)$.

Denote the set of poles of $Z_{P,1}(s)$ lying in the circle $|s-s_0| \le \frac{1}{2}t$ by P.

Now, the function

$$\mathcal{H}(s) = Z_{P,1}(s) \prod_{\rho_1\in P} (s-\rho_1)$$

is regular in the circle $|s-s_0| \le \frac{1}{2}t$.

$Z_{P,1}(s)$ extends to a meromorphic function on the whole $\mathbb{C}$.

Therefore, $Z_{P,1}(s)$ may be written as the quotient of two entire functions

$$Z_{P,1}(s) = \frac{Z_1(s)}{Z_2(s)},$$

where the zeros of $Z_1(s)$ correspond to the zeros of $Z_{P,1}(s)$ and the zeros of $Z_2(s)$ correspond to the poles of $Z_{P,1}(s)$.

The orders of the zeros of $Z_1(s)$ (resp. $Z_2(s)$) equal the orders of the corresponding zeros (resp. poles) of $Z_{P,1}(s)$.

According to proof of Lemma 2.3.2 in [11, pp. 38-40], there exists $J \in \mathbb{N}$ such that $Z_1(s)$ and $Z_2(s)$ are both of order at most J.

By definition of the order of a function, J is the infimum of numbers ω such that

$$|Z_1(s)| = O\left(e^{|s|^\omega}\right),$$

as $s \to \infty$.

Therefore,

$$|Z_1(s)| = O\left(e^{|s|^{J+\varepsilon}}\right),$$

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as $s \to \infty$, i.e.,

$$|Z_1(s)| \le Q_1 e^{|s|^{J+\varepsilon}},$$

as $s \to \infty$, where Q_1 is a constant.

Since $Q_1 \le e^{Q_2|s|^{J+\varepsilon}}$ for some constant Q_2, we obtain that

$$|Z_1(s)| \le e^{Q_3|s|^{J+\varepsilon}}, \tag{1}$$

as $s \to \infty$, for $Q_3 = Q_2 + 1$.

We consider the half-strip $c - \frac{1}{2}t \le \sigma^1 \le c + \frac{1}{2}t$, $t^1 \ge \alpha$, where $\alpha > 0$ is large.

Here, we assume that $s = \mathrm{Re}(s) + \mathrm{i}\,\mathrm{Im}(s) = \sigma^1 + \mathrm{i}\,t^1$, and that α is selected such that $\alpha \ll \frac{1}{2}t$.

Since $|s|$ is large for s belonging to given half-strip, we obtain from (1) that

$$|Z_1(s)| \le e^{Q_3\left|\sigma^1 + \mathrm{i}\,t^1\right|^{J+\varepsilon}}$$

for $s = \sigma^1 + \mathrm{i}\,t^1$, $c - \frac{1}{2}t \le \sigma^1 \le c + \frac{1}{2}t$, $t^1 \ge \alpha$.

Note that

$$\left|c - \frac{1}{2}t\right| \le c + \frac{1}{2}t,$$
$$\left|c + \frac{1}{2}t\right| = c + \frac{1}{2}t.$$

Hence,

$$\left|c - \frac{1}{2}t\right| \le \left|c + \frac{1}{2}t\right| = c + \frac{1}{2}t.$$

We conclude, $\left|\sigma^1\right| \le c + \frac{1}{2}t$ for $c - \frac{1}{2}t \le \sigma^1 \le c + \frac{1}{2}t$.

Hence,

$$\begin{aligned}\left|\sigma^1 + \mathrm{i}\,t^1\right| &\le \left|\sigma^1\right| + t^1 \le c + \frac{1}{2}t + t^1\\ &< \frac{1}{8}t + \frac{1}{2} + \frac{1}{2}t + t^1\\ &= \frac{5}{8}t + \frac{1}{2} + t^1\end{aligned}$$

for $c - \frac{1}{2}t \le \sigma^1 \le c + \frac{1}{2}t$.

Since t is large, it follows that $\frac{1}{2} < t$.

We may assume, without loss og generality, that $\frac{1}{2} < \frac{3}{8}t$.

Therefore,

$$\left|\sigma^1 + \mathrm{i}\,t^1\right| < t + t^1$$

for $c - \frac{1}{2}t \le \sigma^1 \le c + \frac{1}{2}t$.

We conclude,

$$\begin{aligned}|Z_1(s)| &\le e^{Q_3\left|\sigma^1 + \mathrm{i}\,t^1\right|^{J+\varepsilon}}\\ &< e^{Q_3\left(t+t^1\right)^{J+\varepsilon}}\\ &= e^{Q_3(t+\mathrm{Im}(s))^{J+\varepsilon}}\end{aligned}$$

for $s = \sigma^1 + \mathrm{i}\,t^1$, $c - \frac{1}{2}t \le \sigma^1 \le c + \frac{1}{2}t$, $t^1 \ge \alpha$, i.e.,

$$|Z_1(s)| = e^{O\left((t+\mathrm{Im}(s))^{J+\varepsilon}\right)}$$

for $s = \sigma^1 + \mathrm{i}\,t^1$, $c - \frac{1}{2}t \le \sigma^1 \le c + \frac{1}{2}t$, $t^1 \ge \alpha$.

Analogously,

$$|Z_2(s)| = e^{O\left((t+\mathrm{Im}(s))^{J+\varepsilon}\right)}$$

for $s = \sigma^1 + \mathrm{i}\,t^1$, $c - \frac{1}{2}t \le \sigma^1 \le c + \frac{1}{2}t$, $t^1 \ge \alpha$.

Hence,

$$\begin{aligned}|Z_{P,1}(s)| &= \frac{|Z_1(s)|}{|Z_2(s)|}\\ &= \frac{e^{O\left((t+\mathrm{Im}(s))^{J+\varepsilon}\right)}}{e^{O\left((t+\mathrm{Im}(s))^{J+\varepsilon}\right)}}\\ &= e^{O\left((t+\mathrm{Im}(s))^{J+\varepsilon}\right)}\end{aligned}$$

for $s = \sigma^1 + \mathrm{i}\,t^1$, $c - \frac{1}{2}t \le \sigma^1 \le c + \frac{1}{2}t$, $t^1 \ge \alpha$.

In particular,

$$|Z_{P,1}(s)| = e^{O\left((t+\mathrm{Im}(s))^{J+\varepsilon}\right)}$$

for $s = \sigma^1 + \mathrm{i}\,t^1$, $|s - s_0| \le \frac{1}{2}t$.

For $s = s_0$, we obtain

$$|Z_{P,1}(s_0)| = e^{O\left((t+t)^{J+\varepsilon}\right)} = e^{O\left(t^{J+\varepsilon}\right)}.$$

Since $t^1 \le \frac{3}{2}t$ for $s = \sigma^1 + \mathrm{i}\,t^1$, $|s - s_0| \le \frac{1}{2}t$, it follows that

$$\begin{aligned}|Z_{P,1}(s)| &\le e^{Q(t+\mathrm{Im}(s))^{J+\varepsilon}}\\ &\le e^{Q\left(t+\frac{3}{2}t\right)^{J+\varepsilon}} = e^{Q\left(\frac{5}{2}\right)^{J+\varepsilon}t^{J+\varepsilon}}\end{aligned}$$

for $s = \sigma^1 + \mathrm{i}\,t^1$, $|s - s_0| \le \frac{1}{2}t$, where Q is a constant, i.e.,

$$|Z_{P,1}(s)| = e^{O\left(t^{J+\varepsilon}\right)}$$

for $s = \sigma^1 + \mathrm{i}\,t^1$, $|s - s_0| \le \frac{1}{2}t$.

Therefore,

$$\left|\frac{Z_{P,1}(s)}{Z_{P,1}(s_0)}\right| = \frac{e^{O\left(t^{J+\varepsilon}\right)}}{e^{O\left(t^{J+\varepsilon}\right)}} = e^{O\left(t^{J+\varepsilon}\right)}$$

for $s = \sigma^1 + \mathrm{i}\,t^1$, $|s - s_0| \le \frac{1}{2}t$.

For $t > 0$, let $N(t)$ denote the number of poles and zeros of $Z_{P,1}(s)$ at points $s = \frac{1}{2} + \mathrm{i}\,x$, where $0 < x < t$.

By [11, p. 58, Lemma 3.1.2], $N(t) = O\left(t^D\right)$, where D is the degree of the polynomial $G(s)$ such that

$$Z_{P,1}(1-s) = e^{-G(s)} Z_{P,1}(s).$$

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Now, we estimate the number of the elements of the set P.

We intersect the curve $|s-s_0| = \frac{1}{2}t$ with the line $\mathrm{Re}\,(s) = \frac{1}{2}$.

As before, we assume that $s = \mathrm{Re}\,(s) + \mathrm{i}\,\mathrm{Im}\,(s) = \sigma^1 + \mathrm{i}\,t^1$.

We obtain,

$$\frac{1}{2}t = |s-s_0| = \left|\frac{1}{2} + \mathrm{i}\,t^1 - c - \mathrm{i}\,t\right|$$

if and only if

$$t^1 = t \pm \sqrt{\frac{1}{4}t^2 - \left(c-\frac{1}{2}\right)^2}.$$

Thus, the number of the elements of the set P is dominated by the difference

$$N\left(t + \sqrt{\frac{1}{4}t^2 - \left(c-\frac{1}{2}\right)^2}\right) - N\left(t - \sqrt{\frac{1}{4}t^2 - \left(c-\frac{1}{2}\right)^2}\right).$$

Since $N(t) = O\left(t^D\right)$, it follows that the difference is $O\left(t^D\right)$.

Note that for $s = \sigma^1 + \mathrm{i}\,t^1$, $|s-s_0| \leq \frac{1}{2}t$, we have that $|s-\rho_1| \leq t$ for all $\rho_1 \in P$.

Now, we are in position to estimate the product $\prod_{\rho_1\in P}|s-\rho_1|$ for $s = \sigma^1 + \mathrm{i}\,t^1$, $|s-s_0| \leq \frac{1}{2}t$.

Since

$$\log\prod_{\rho_1\in P}|s-\rho_1| = \sum_{\rho_1\in P}\log|s-\rho_1| \leq \sum_{\rho_1\in P}\log t = \log t\sum_{\rho_1\in P}1$$

it follows that

$$\prod_{\rho_1\in P}|s-\rho_1| \leq e^{\log t\sum_{\rho_1\in P}1},$$

i.e.,

$$\prod_{\rho_1\in P}|s-\rho_1| = e^{O\left(t^D\log t\right)},$$

for $s = \sigma^1 + \mathrm{i}\,t^1$, $|s-s_0| \leq \frac{1}{2}t$.

Also, note that $|s_0-\rho_1| > \frac{1}{2}$ for all $\rho_1 \in P$.

We obtain,

$$\left|\frac{\mathcal{H}(s)}{\mathcal{H}(s_0)}\right| = \left|\frac{Z_{P,1}(s)}{Z_{P,1}(s_0)}\right| \cdot \prod_{\rho_1\in P}\frac{|s-\rho_1|}{|s_0-\rho_1|} = e^{O\left(t^{J+\varepsilon}\right)} \cdot e^{O\left(t^D\log t\right)} = e^{O\left(t^{J+\varepsilon}\right)} \cdot e^{O\left(t^{D+\varepsilon}\right)}$$

for $s = \sigma^1 + \mathrm{i}\,t^1$, $|s-s_0| \leq \frac{1}{2}t$.

Note that

$$G(s) = g_1(s) - g_2(s) + h_2(s) - h_1(s),$$

where

$$Z_i(s) = s^{n_i}e^{g_i(s)}\prod_{\rho\in R_i\setminus\{0\}}\left(1-\frac{s}{\rho}\right) \times \exp\left(\frac{s}{\rho} + \frac{s^2}{2\rho^2} + \ldots + \frac{s^k}{k\rho^k}\right), \tag{2}$$

$i \in \{1,2\}$, n_i is the order of the zero of $Z_i(s)$ at $s=0$, R_i is the set of zeros of $Z_i(s)$ counted with multiplicities, $g_1(s)$ and $g_2(s)$ are polynomials, and $k \in \mathbb{N}$ is some explicitly determined number.

Furthermore,

$$Z_{P,1}(1-s) = \frac{W_1(s)}{W_2(s)},$$

where

$$W_i(s) = s^{n_i}e^{h_i(s)}\prod_{\rho\in R_i\setminus\{0\}}\left(1-\frac{s}{\rho}\right) \times \exp\left(\frac{s}{\rho} + \frac{s^2}{2\rho^2} + \ldots + \frac{s^k}{k\rho^k}\right), \tag{3}$$

and $h_1(s)$ and $h_2(s)$ are polynomials (see, [11, pp. 42-43]).

By [11, p. 40], $g_1(s)$ and $g_2(s)$ are both polynomials of degree at most $J-1$.

Furthermore, by [11, p. 39, (2.17)]

$$\sum_{\rho\in R_i\setminus\{0\}}\frac{1}{\rho^k} < \infty,$$

i.e., we may assume, without loss of generality, that $k-1$ is the rank of $Z_i(s)$.

This means that the Weierstrass factorization (2) is unique, except that $g_i(s)$ may be replaced by $g_i(s) + 2\,\pi\,m\,\mathrm{i}$ for any integer m, i.e., $g_i(s)$ is uniquely determined up to adding a multiple of $2\,\pi\,\mathrm{i}$ (see, e.g., [3, pp. 282-283] or [2]).

Hence, (3) yields that $h_1(s)$ and $h_2(s)$ are both polynomials of degree at most $J-1$.

Now, the fact that D is the degree of $G(s)$, and

$$G(s) = g_1(s) - g_2(s) + h_2(s) - h_1(s),$$

imply that $D \leq J-1$.

Consequently,

$$\left|\frac{\mathcal{H}(s)}{\mathcal{H}(s_0)}\right| = e^{O\left(t^{J+\varepsilon}\right)}$$

for $s = \sigma^1 + \mathrm{i}\,t^1$, $|s-s_0| \leq \frac{1}{2}t$.

In particular, there exists a constant C such that

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$$\left|\frac{\mathcal{H}(s)}{\mathcal{H}(s_0)}\right| < e^{Ct^{J+\varepsilon}}$$

for $s = \sigma^1 + \mathrm{i}\, t^1$, $|s - s_0| \le \frac{1}{2}t$.

Put $M = C\ t^{J+\varepsilon} > 1$.

Hence,

$$\left|\frac{\mathcal{H}(s)}{\mathcal{H}(s_0)}\right| < e^{M},$$

for $s = \sigma^1 + \mathrm{i}\, t^1$, $|s - s_0| \le \frac{1}{2}t$.

By Lemma α in [13, p. 56]

$$\left|\frac{\mathcal{H}'(s)}{\mathcal{H}(s)} - \sum_{\rho_2 \in Q} \frac{1}{s-\rho_2}\right| < A\frac{M}{r} = A\frac{Ct^{J+\varepsilon}}{\frac{1}{2}t} = 2ACt^{J-1+\varepsilon}$$

for $s = \sigma^1 + \mathrm{i}\, t^1$, $|s - s_0| \le \frac{1}{8}t$, where Q is the set of zeros ρ_2 of $\mathcal{H}(s)$ such that $|\rho_2 - s_0| \le \frac{1}{4}t$, and A is a constant.

Thus,

$$\frac{\mathcal{H}'(s)}{\mathcal{H}(s)} = O\left(t^{J-1+\varepsilon}\right) + \sum_{\rho_2 \in Q} \frac{1}{s-\rho_2}$$

for $s = \sigma^1 + \mathrm{i}\, t^1$, $|s - s_0| \le \frac{1}{8}t$.

Having in mind the definition of $\mathcal{H}(s)$, we obtain

$$\frac{Z'_{P,1}(s)}{Z_{P,1}(s)} = O\left(t^{J-1+\varepsilon}\right) + \sum_{\rho_2 \in Q} \frac{1}{s-\rho_2} - \sum_{\rho_1 \in P} \frac{1}{s-\rho_1}$$

for $s = \sigma^1 + \mathrm{i}\, t^1$, $|s - s_0| \le \frac{1}{8}t$.

Consequently, this equality holds true for $s = \sigma^1 + \mathrm{i}\, t$, $\frac{1}{2} \le \sigma^1 \le c + \frac{1}{8}t$.

Hence,

$$\frac{Z'_{P,1}(s)}{Z_{P,1}(s)} = O\left(t^{J-1+\varepsilon}\right) + \sum_{\rho_2 \in Q} \frac{1}{s-\rho_2} - \sum_{\rho_1 \in P} \frac{1}{s-\rho_1}$$

for $s = \sigma^1 + \mathrm{i}\, t$, $\frac{1}{2} \le \sigma^1 \le c + \frac{1}{8}t$.

Since Q is the set of zeros ρ_2 of $\mathcal{H}(s)$ lying in $|s - s_0| \le \frac{1}{4}t$, the definition of $\mathcal{H}(s)$ yields that Q is actually the set of zeros of $Z_{P,1}(s)$ lying in $|s - s_0| \le \frac{1}{4}t$.

Put $\rho_2 = \frac{1}{2} + \mathrm{i}\, \gamma_1$.

It follows that $|\rho_2 - s_0| \le \frac{1}{4}t$ if and only if

$$t - \sqrt{\frac{1}{16}t^2 - \left(c - \frac{1}{2}\right)^2} \le \gamma_1 \le t + \sqrt{\frac{1}{16}t^2 - \left(c - \frac{1}{2}\right)^2}.$$

Note that $1 < c < \frac{1}{8}t + \frac{1}{2}$.

Hence,

$$\frac{\sqrt{3}}{8}t < \sqrt{\frac{1}{16}t^2 - \left(c - \frac{1}{2}\right)^2} < \sqrt{\frac{1}{16}t^2 - \frac{1}{4}}.$$

Since $t \gg 0$, we may assume, without loss of generality, that $\frac{\sqrt{3}}{8}t > 1$.

Now, we may write

$$\sum_{\rho_2 \in Q} \frac{1}{s-\rho_2} = \sum_{t-1 \le \gamma_1 \le t+1} \frac{1}{s-\rho_2} + \sum_{t+1 < \gamma_1 \le t + \sqrt{\frac{1}{16}t^2 - \left(c-\frac{1}{2}\right)^2}} \frac{1}{s-\rho_2} + \sum_{t - \sqrt{\frac{1}{16}t^2 - \left(c-\frac{1}{2}\right)^2} \le \gamma_1 < t-1} \frac{1}{s-\rho_2}.$$

Put $\rho_1 = \frac{1}{2} + \mathrm{i}\, \gamma_2$ for $\rho_1 \in P$.

Reasoning in the same way as in the case $\rho_2 \in Q$, we conclude that $|\rho_1 - s_0| \le \frac{1}{2}t$ if and only if

$$t - \sqrt{\frac{1}{4}t^2 - \left(c - \frac{1}{2}\right)^2} \le \gamma_2 \le t + \sqrt{\frac{1}{4}t^2 - \left(c - \frac{1}{2}\right)^2}.$$

Obviously,

$$\sqrt{\frac{1}{4}t^2 - \left(c - \frac{1}{2}\right)^2} > \sqrt{\frac{1}{16}t^2 - \left(c - \frac{1}{2}\right)^2} > \frac{\sqrt{3}}{8}t > 1.$$

Hence, we may write

$$\sum_{\rho_1 \in P} \frac{1}{s-\rho_1} = \sum_{t-1 \le \gamma_2 \le t+1} \frac{1}{s-\rho_1} + \sum_{t+1 < \gamma_2 \le t + \sqrt{\frac{1}{4}t^2 - \left(c-\frac{1}{2}\right)^2}} \frac{1}{s-\rho_1} + \sum_{t - \sqrt{\frac{1}{4}t^2 - \left(c-\frac{1}{2}\right)^2} \le \gamma_2 < t-1} \frac{1}{s-\rho_1}.$$

We obtain,

$$\frac{Z'_{P,1}(s)}{Z_{P,1}(s)} = O\left(t^{J-1+\varepsilon}\right) + \sum_{t-1\le\gamma_1\le t+1}\frac{1}{s-\rho_2} - \sum_{t-1\le\gamma_2\le t+1}\frac{1}{s-\rho_1} + \sum_{t+1<\gamma_1\le t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{1}{2}\right)^2}}\frac{1}{s-\rho_2} - \sum_{t+1<\gamma_2\le t+\sqrt{\frac{1}{4}t^2-\left(c-\frac{1}{2}\right)^2}}\frac{1}{s-\rho_1} + \sum_{t-\sqrt{\frac{1}{16}t^2-\left(c-\frac{1}{2}\right)^2}\le\gamma_1<t-1}\frac{1}{s-\rho_2} - \sum_{t-\sqrt{\frac{1}{4}t^2-\left(c-\frac{1}{2}\right)^2}\le\gamma_2<t-1}\frac{1}{s-\rho_1}$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} \le \sigma^1 \le c + \frac{1}{8}t$.

We estimate,

$$\left|\sum_{t+1<\gamma_1\le t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{1}{2}\right)^2}}\frac{1}{s-\rho_2} - \sum_{t+1<\gamma_2\le t+\sqrt{\frac{1}{4}t^2-\left(c-\frac{1}{2}\right)^2}}\frac{1}{s-\rho_1}\right| \le \sum_{t+1<\gamma_1\le t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{1}{2}\right)^2}}\frac{1}{|s-\rho_2|} + \sum_{t+1<\gamma_2\le t+\sqrt{\frac{1}{4}t^2-\left(c-\frac{1}{2}\right)^2}}\frac{1}{|s-\rho_1|}$$
$$\le \sum_{t+1<\gamma_1\le t+\sqrt{\frac{1}{4}t^2-\left(c-\frac{1}{2}\right)^2}}\frac{1}{|s-\rho_2|} + \sum_{t+1<\gamma_2\le t+\sqrt{\frac{1}{4}t^2-\left(c-\frac{1}{2}\right)^2}}\frac{1}{|s-\rho_1|} \le \sum_{t+1<\gamma_1\le t+\sqrt{\frac{1}{4}t^2-\left(c-\frac{1}{2}\right)^2}}1 + \sum_{t+1<\gamma_2\le t+\sqrt{\frac{1}{4}t^2-\left(c-\frac{1}{2}\right)^2}}1$$
$$= N\left(t+\sqrt{\frac{1}{4}t^2-\left(c-\frac{1}{2}\right)^2}\right) - N(t+1)$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} \le \sigma^1 \le c + \frac{1}{8}t$.

Hence,

$$\sum_{t+1<\gamma_1\le t+\sqrt{\frac{1}{16}t^2-\left(c-\frac{1}{2}\right)^2}}\frac{1}{s-\rho_2} - \sum_{t+1<\gamma_2\le t+\sqrt{\frac{1}{4}t^2-\left(c-\frac{1}{2}\right)^2}}\frac{1}{s-\rho_1} = O\left(t^D\right)$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} \le \sigma^1 \le c + \frac{1}{8}t$.

Analogously,

$$\sum_{t-\sqrt{\frac{1}{16}t^2-\left(c-\frac{1}{2}\right)^2}\le\gamma_1<t-1}\frac{1}{s-\rho_2} - \sum_{t-\sqrt{\frac{1}{4}t^2-\left(c-\frac{1}{2}\right)^2}\le\gamma_2<t-1}\frac{1}{s-\rho_1} = O\left(t^D\right)$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} \le \sigma^1 \le c + \frac{1}{8}t$.

Since $D \le J - 1$, we obtain

$$\frac{Z'_{P,1}(s)}{Z_{P,1}(s)} = O\left(t^{J-1+\varepsilon}\right) + \sum_{|t-\gamma_{P,1}|\le 1}\frac{1}{s-\rho_{P,1}}$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} \le \sigma^1 \le c + \frac{1}{8}t$.

Having in mind that $c < \frac{1}{8}t + \frac{1}{2}$, it finally follows that

$$\frac{Z'_{P,1}(s)}{Z_{P,1}(s)} = O\left(t^{J-1+\varepsilon}\right) + \sum_{|t-\gamma_{P,1}|\le 1}\frac{1}{s-\rho_{P,1}} \tag{4}$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} \le \sigma^1 < \frac{1}{4}t + \frac{1}{2}$.

(ii) We estimate,

$$\left|\sum_{|t-\gamma_{P,1}|\le 1}\frac{1}{s-\rho_{P,1}}\right| \le \sum_{|t-\gamma_{P,1}|\le 1}\frac{1}{|s-\rho_{P,1}|} < \frac{1}{\eta}\sum_{|t-\gamma_{P,1}|\le 1}1 = \frac{1}{\eta}\left(N(t+1) - N(t-1)\right)$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} + \eta \le \sigma^1 < \frac{1}{4}t + \frac{1}{2}$.

Hence,

$$\sum_{|t-\gamma_{P,1}|\le 1}\frac{1}{s-\rho_{P,1}} = O\left(\frac{1}{\eta}t^D\right)$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} + \eta \le \sigma^1 < \frac{1}{4}t + \frac{1}{2}$.

This equality and (4) yield that

$$\frac{Z'_{P,1}(s)}{Z_{P,1}(s)} = O\left(\frac{1}{\eta}t^{J-1+\varepsilon}\right) \tag{5}$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} + \eta \le \sigma^1 < \frac{1}{4}t + \frac{1}{2}$.

(b) (i) We have,

$$R_{\Gamma,1}(s) = \prod_{q=0}^{4} Z_{P,(\bigwedge^q \bar{\mathfrak{n}})}\left(s+\frac{q}{4}\right)^{(-1)^q}$$
$$= Z_{P,1}(s) \cdot \prod_{q=1}^{4} Z_{P,(\bigwedge^q \bar{\mathfrak{n}})}\left(s+\frac{q}{4}\right)^{(-1)^q}.$$

Hence,

$$\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} = \frac{Z'_{P,1}(s)}{Z_{P,1}(s)} + \sum_{q=1}^{4}(-1)^q \frac{Z'_{P,(\bigwedge^q \bar{\mathfrak{n}})}\left(s+\frac{q}{4}\right)}{Z_{P,(\bigwedge^q \bar{\mathfrak{n}})}\left(s+\frac{q}{4}\right)}.$$

This equation, the equation (4), and the derivation of (5), yield

$$\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} = O\left(t^{J-1+\varepsilon}\right) + \sum_{|t-\gamma_{P,1}|\le 1} \frac{1}{s-\rho_{P,1}} + \sum_{q=1}^{4} O\left(\frac{1}{\frac{q}{4}} t^{J-1+\varepsilon}\right) \tag{6}$$
$$= O\left(t^{J-1+\varepsilon}\right) + \sum_{|t-\gamma_{P,1}|\le 1} \frac{1}{s-\rho_{P,1}}$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} \le \sigma^1 < \frac{1}{4}t - \frac{1}{2}$.

(ii) Reasoning in the same way as in the proof of (a) (i), we obtain that

$$\sum_{|t-\gamma_{P,1}|\le 1} \frac{1}{s-\rho_{P,1}} = O\left(\frac{1}{\eta} t^{D}\right)$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} + \eta \le \sigma^1 < \frac{1}{4}t - \frac{1}{2}$.

Now, (6) implies that

$$\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} = O\left(\frac{1}{\eta} t^{J-1+\varepsilon}\right)$$

for $s = \sigma^1 + \mathrm{i}\,t$, $\frac{1}{2} + \eta \le \sigma^1 < \frac{1}{4}t - \frac{1}{2}$.

This completes the proof. ■

III. Applications

Note that analogous formulas of the formulas derived in this paper are already very well applied in literature for various settings of locally symmetric spaces.

Such formula [9, p. 102, Th. 6.4.] is applied to obtain a prime geodesic theorem (see, [9, p. 113, Th. 6.19.]) in the case of compact Riemann surfaces.

In the same setting, it is used in the proof of Lemma 4 in [12, p. 213].

In the case of compact locally symmetric spaces of real rank one, such formulas are first derived in [1, p. 314, Th. 4.1.] (even-dimensional case), [7, pp. 177-178, Th. 1.] (odd-dimensional case), and then applied in proofs of the corresponding prime geodesic theorems [6, p. 190, Th. 1.] (even-dimensional case), [8, p. 216, Th. 1.] (odd-dimensional case), respectively.

In [10, p. 99], the author also applied a variant of the formula in the proof of the prime geodesic theorem for d-dimensional real hyperbolic manifolds with cusps.

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