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On the Prime Geodesic Theorem for SL_4

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Abstract—In 1949, A. Selberg discovered a real variable (an elementary) proof of the prime number theorem. A number of authors have adapted Selberg's method to achieve quite a good corresponding error term. The Riemann hypothesis has never been proved or disproved however. Any generalization of the prime number theorem to the more general situations is known in literature as a prime geodesic theorem. In this paper we derive yet another proof of the prime geodesic theorem for compact symmetric spaces formed as quotients of the Lie group $SL_4(\mathbb{R})$. While the first known proof in this setting applies contour integration over square boundaries, our proof relies on an application of modified circular boundaries. Recently, A. Deitmar and M. Pavey applied such prime geodesic theorem to derive an asymptotic formula for class numbers of orders in totally complex quartic fields with no real quadratic subfields.

Keywords—Prime geodesic theorem, virtual representations, Euler characteristics, conjugacy classes, Langlands decomposition.

I. Introduction

We introduce the notation following [13] and [5]. It will be introduced in the sequel.

Let $G = SL_4(\mathbb{R})$ and let K be the maximal compact subgroup $SO(4)$.

Let $\Gamma \subset G$ be discrete and co-compact.

Then, there exists a one to one correspondence between conjugacy classes in Γ and free homotopy classes of closed geodesics on he symmetric space $X_\Gamma = \Gamma \setminus G \,/\, K$.

Let

$$A^- = \left\{ \begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1} \end{pmatrix} : 0 < a < 1 \right\},$$

and

$$B = \begin{pmatrix} SO(2) & \\ & SO(2) \end{pmatrix}.$$

Let $\mathcal{E}(\Gamma)$ be the set of primitive conjugacy classes $[\gamma]$ in Γ such that γ is conjugate in G to an element $a_\gamma b_\gamma$ of $A^- B$. For $\gamma \in \Gamma$ we write a_γ also for the top left entry in the matrix a_γ and define the length l_γ of γ to be $8 \log a_\gamma$.

Suppose that G_γ, Γ_γ are the centralizers of γ in G and Γ, respectively, and $K_\gamma = K \cap G_\gamma$.

For $x > 0$ we define the function

$$\pi(x) = \sum_{\substack{[\gamma] \in \mathcal{E}(\Gamma) \\ e^{l_\gamma} \le x}} \chi_1(\Gamma_\gamma),$$

where $\chi_1(\Gamma_\gamma)$ is the first higher Euler characteristics of the symmetric space $X_{\Gamma_\gamma} = \Gamma_\gamma \setminus G_\gamma \,/\, K_\gamma$.

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In [13], the author proved (a prime geodesic theorem) that for $x \to \infty$

$$\pi(x) \sim \frac{2x}{\log x},$$

and more sharply, that

$$\pi(x) = 2 \operatorname{li}(x) + O\left(x^{\frac{3}{4}} (\log x)^{-1}\right)$$

as $x \to \infty$, where $\operatorname{li}(x) = \int_2^x \frac{dt}{\log t}$ is the integral logarithm.

The main purpose of this paper is to give yet another proof of the same prime geodesic theorem by applying different means.

II. Preliminaries

Note that if Γ is torsion-free, the first higher Euler characteristics of Γ is defined to be

$$\chi_1(X_\Gamma) = \chi_1(\Gamma) = \sum_{j=0}^{\dim X_\Gamma} (-1)^{j+1} j h^j(X_\Gamma),$$

where $h^j(X_\Gamma)$ is the j-th Betti number of X_Γ.

If Γ is not torsion-free, then, the fact that every arithmetic subgroup of G has a torsion-free subgroup of finite index (see, e.g., [3]), we may assume that $\Gamma' \subset \Gamma$ is such a group.

Let θ be the Cartan involution fixing K pointwise.

Then, there exists a θ-stable Cartan subgroup $H = AB$ of G, where A is a connected split torus and $B \subset K$ is a Cartan of K.

We assume that A is central in G and of dimension one.

Suppose that C is the center of G.

Then, $C \subset H$.

Let $B_C = B \cap C$, $\Gamma_C = \Gamma \cap C$.

Furthermore, let $\Gamma_A = A \cap \Gamma_C B_C$ be the projection of Γ_C to A.

By [17, Lemma 3.3], Γ_A is a discrete and cocompact in A.

Now, the first higher Euler characteristics of Γ is defined by (see, [13, p. 9, (1.4)])

$$\chi_1(X_\Gamma) = \chi_1(\Gamma) = \chi_1\left(\Gamma'\right) \cdot \frac{\left[\Gamma_A : \Gamma'_A\right]}{\left[\Gamma : \Gamma'\right]}.$$

In our particular case $G = SL_4(\mathbb{R})$, $K = SO(4)$, the rank one torus A is given by

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$$A=\left\{\begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1}\end{pmatrix} : a>0\right\},$$

and

$$B=\begin{pmatrix} SO(2) & \\ & SO(2)\end{pmatrix}$$

is a compact Cartan subgroup of

$$M=S\begin{pmatrix} SL_2^{\pm}(\mathbb{R}) & \\ & SL_2^{\pm}(\mathbb{R})\end{pmatrix}.$$

The first higher Euler characteristics $\chi_1(\Gamma_\gamma)$ of Γ_γ is then defined as before, i.e., by (1.7) in [13, p. 16].

Let $\mathfrak{g}_{\mathbb{R}}=sl_4(\mathbb{R})$ and $\mathfrak{g}=sl_4(\mathbb{C})$ be respectively the Lie algebra and complexified Lie algebra of G.

Let b be the following multiple of the trace form on $\mathfrak{g}$ (see, [4])

$$b(X,Y)=16\,\mathrm{tr}(XY).$$

Furthermore, let $\mathfrak{k}_{\mathbb{R}}\subset\mathfrak{g}_{\mathbb{R}}$ be the Lie algebra of K and $\mathfrak{p}_{\mathbb{R}}\subset\mathfrak{g}_{\mathbb{R}}$ the orthogonal space of $\mathfrak{k}_{\mathbb{R}}$ with respect to the form b.

Then, b is positive definite and $\mathrm{Ad}(K)$-invariant on $\mathfrak{p}_{\mathbb{R}}$, and thus defines a G-invariant metric on the symmetric space $X = G\,/\,K$ attached to G.

We put

$$N=\begin{pmatrix} I_2 & \mathrm{Mat}_2(\mathbb{R}) \\ 0 & I_2\end{pmatrix},$$

$$\bar{N}=\begin{pmatrix} I_2 & 0 \\ \mathrm{Mat}_2(\mathbb{R}) & I_2\end{pmatrix}.$$

Let $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ be the complexified Lie algebras of M, A, N and $\bar{N}$, respectively.

In particular, we put P to be the parabolic with Langlands decomposition $P=M\ A\ N$, ρ_P to be the half-sum of the positive roots of the system $(\mathfrak{g},\mathfrak{a})$ such that

$$\rho_P\begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1}\end{pmatrix}=4a.$$

Now, $A^-=\{\exp(tH_1) : t>0\}$ is the negative Weyl chamber in A, where

$$H_1=\frac{1}{8}\begin{pmatrix} a & & & \\ & a & & \\ & & a^{-1} & \\ & & & a^{-1}\end{pmatrix}\in\mathfrak{a}_{\mathbb{R}}.$$

In order to describe what we said above in a more detailed manner, we let $\mathcal{E}_P(\Gamma)$ to be the set of all conjugacy classes $[\gamma]$ in Γ such that γ is conjugate in G to an element $a_\gamma b_\gamma$ of A^-B.

An element $\gamma\in\Gamma$ is called primitive if for $\delta\in\Gamma$ and $n\in\mathbb{N}$ the equation $\delta^n=\gamma$ implies that $n=1$.

Let $\mathcal{E}_P^p(\Gamma)\subset\mathcal{E}_P(\Gamma)$ be the subset of primitive classes.

A virtual representation σ of a group is a formal difference of two representations $\sigma=\sigma^+-\sigma^-$, which is called finite dimensional if both σ^+ and σ^- are.

The trace, the determinant, the dimension of a virtual representation $\sigma=\sigma^+-\sigma^-$ are defined by

$$\mathrm{tr}\,\sigma=\mathrm{tr}\,\sigma^+-\mathrm{tr}\,\sigma^-,$$
$$\det\sigma=\frac{\det\sigma^+}{\det\sigma^-},$$
$$\dim\sigma=\dim\sigma^+-\dim\sigma^-,$$

respectively.

If V is a representation space with $\mathbb{Z}$ grading, then, we should consider it naturally as a virtual representations space by $V^+=V_{EVEN}$ and $V^-=V_{ODD}$.

In particular, if V is a subspace of $\mathfrak{g}$, we shall always consider the exterior product $\bigwedge^* V$ as a virtual representation $\bigwedge^* V=\bigwedge^{EVEN}V-\bigwedge^{ODD}V$ with respect to the adjoint reresentation.

For any finite-dimensional virtual representation σ of M, we define, for $\mathrm{Re}(s)$ large, the generalized Selberg zeta function

$$Z_{P,\sigma}(s)= \exp\left(-\sum_{[\gamma]\in\mathcal{E}_{\mathcal{P}}(\Gamma)}\frac{\mathrm{tr}\,\sigma(b_\gamma)\,\chi_1(\Gamma_\gamma)\,l_{\gamma_0}}{l_\gamma\det\left(1-(a_\gamma b_\gamma)^{-1}|_{\bar{\mathfrak{n}}}\right)}e^{-sl_\gamma}\right).$$

We define, for $\mathrm{Re}(s)$ large, the generalized Ruelle zeta function

$$R_\Gamma(s) =\exp\left(-\sum_{[\gamma]\in\mathcal{E}_{\mathcal{P}}(\Gamma)}\frac{\chi_1(\Gamma_\gamma)\,l_{\gamma_0}}{l_\gamma}e^{-sl_\gamma}\right).$$

For any finite-dimensional virtual representation σ of M, and $\mathrm{Re}(s)\gg 0$, the generalized Ruelle zeta function $R_{\Gamma,\sigma}(s)$ is defined in [13, p. 43].

Thus, $R_{\Gamma,\sigma}(s)$ extends to a meromorphic function on $\mathbb{C}$, and

$$R_{\Gamma,\sigma}(s)=\prod_{q=0}^{4}Z_{P,(\bigwedge^q\bar{\mathfrak{n}}\otimes V_\sigma)}\left(s+\frac{q}{4}\right)^{(-1)^q}.$$

For $\gamma\in\Gamma$, let $N(\gamma)=e^{l_\gamma}$.

Thus, for $x>0$,

$$\pi(x)=\sum_{\substack{[\gamma]\in\mathcal{E}_P^p(\Gamma)\\ N(\gamma)\le x}}\chi_1(\Gamma_\gamma).$$

III. Main result

Theorem 1. *(Prime Geodesic Theorem)*

$$\pi(x) = 2\,\mathrm{li}(x) + O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)$$

as $x \to \infty$*, where* $\mathrm{li}(x) = \int\limits_{2}^{x} \frac{dt}{\log t}$ *is the integral logarithm.*

Proof: Let $k \geq \max\{J, 2D\}$, $k \in \mathbb{N}$, where D is the degree of the polynomial $G(s)$ such that

$$Z_{P,\sigma}(1-s) = e^{-G(s)} Z_{P,\sigma}(s),$$

and $J \in \mathbb{N}$ is such that $Z_1(s)$ and $Z_2(s)$ are both of order at most J, where

$$Z_{P,\sigma}(s) = \frac{Z_1(s)}{Z_2(s)}.$$

Moreover, let $x > 1$, and $c > 1$.

We have for $\mathrm{Re}(s) > 1$,

$$\frac{R'_{\Gamma,\sigma}(s)}{R_{\Gamma,\sigma}(s)} = \sum_{\gamma} \chi_1(\Gamma_\gamma)\,\mathrm{tr}\,\sigma(b_\gamma)\, l_{\gamma_0} e^{-sl_\gamma}.$$

Hence,

$$\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} = \sum_{\gamma} \chi_1(\Gamma_\gamma)\, l_{\gamma_0} e^{-sl_\gamma},$$

$\mathrm{Re}(s) > 1$.

By [13, p. 62, (3.5)],

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}} \int\limits_{c-\mathrm{i}\,\infty}^{c+\mathrm{i}\,\infty} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \\ &\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1} x^s ds \\ =&\frac{1}{k!} \sum_{N(\gamma)\leq x} \chi_1(\Gamma_\gamma)\, l_{\gamma_0} \left(1 - \frac{N(\gamma)}{x}\right)^k.\end{aligned}$$

Since

$$\psi_j(x) = \frac{1}{j!} \sum_{N(\gamma)\leq x} \chi_1(\Gamma_\gamma)\, l_{\gamma_0} (x - N(\gamma))^j,$$

where

$$\psi_0(x) = \psi(x) = \sum_{[\gamma]\in\mathcal{E}_P(\Gamma)} \chi_1(\Gamma_\gamma)\, l_{\gamma_0},$$

$$\psi_j(x) = \int\limits_0^x \psi_{j-1}(t)\, dt,$$

$j \in \mathbb{N}$, it follows that

$$\begin{aligned}&\psi_k(x) \\ =&\frac{1}{2\pi\,\mathrm{i}} \int\limits_{c-\mathrm{i}\,\infty}^{c+\mathrm{i}\,\infty} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \\ &\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1} x^{s+k} ds.\end{aligned}$$

Let $A \gg 0$ be fixed and consider the segment of the line $\mathrm{Re}(s) = \frac{1}{2}$ given by $\frac{1}{2} + \mathrm{i}\,t$, $A - 1 < t \leq A + 1$.

Applying the Dirichlet principle (in eexactly the same way as in the proof of Lemma 3.1.3 in [13, p. 61]), we conclude that there exists a $\frac{1}{2} + \mathrm{i}\,\bar{A}$ in the segment whose distance from any pole or zero of the zeta functions $Z_{P,1}(s)$, $Z_{P,\wedge^q \bar{\mathfrak{n}}}(s)$, $q \in \{1,2,3,4\}$ is larger than $\frac{C}{\bar{A}^D}$ for some fixed $C > 0$.

Define

$$T = \sqrt{\bar{A}^2 + \frac{1}{4}}.$$

Furthermore, let

$$\begin{aligned}&C(T) \\ =&\left\{s \in \mathbb{C} : |s| \leq T, \mathrm{Re}(s) \leq \frac{1}{2}\right\} \cup \\ &\left\{s \in \mathbb{C} : \frac{1}{2} \leq \mathrm{Re}(s) \leq c, -\bar{A} \leq \mathrm{Im}(s) \leq \bar{A}\right\}.\end{aligned}$$

Having in mind the singularity pattern of the Ruelle zeta function $R_{\Gamma,1}(s)$ (given by Theorem 3.4.3 in [13]), and the fact that

$$\begin{aligned}&\left|\frac{1}{2} + \mathrm{i}\,\bar{A} - \frac{1}{2} - \mathrm{i}\,\gamma_{P,1}\right| \\ =&\left|\frac{1}{2} + \mathrm{i}\,\bar{A} - \rho_{P,1}\right| \\ >&\frac{C}{\bar{A}^D}\end{aligned}$$

for all singularities $\rho_{P,1}$ of the Selberg zeta function $Z_{P,1}(s)$, we conclude that no pole of $\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}$ is on the boundary of $C(T)$ for $\mathrm{Re}(s) \geq \frac{1}{2}$.

Hence, obviously, no pole of

$$\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} s^{-1}(s+1)^{-1}\ldots(s+k)^{-1} x^{s+k}$$

is on the boundary of $C(T)$ for $\mathrm{Re}(s) \geq \frac{1}{2}$.

Suppose that no pole of $\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} s^{-1}(s+1)^{-1}\ldots(s+k)^{-1} x^{s+k}$ is on the boundary of $C(T)$ for $\mathrm{Re}(s) \leq \frac{1}{2}$.

Applying the Cauchy integral formula to the integrand of $\psi_k(x)$ along the boundary of $C(T)$, we obtain

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$$\int\limits_{C(T)^{+}} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds = 2\pi \mathrm{i} \sum_{z \in C(T)} \mathrm{Res}_{s=z} \left(\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} \right),$$

where $C(T)^{+}$ denotes the boundary of $C(T)$ with the anti-clockwise orientation.

We have,

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\bar{A}}^{c+\mathrm{i}\bar{A}} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds \\
&= \sum_{z \in C(T)} \mathrm{Res}_{s=z} \left(\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} \right) - \\
&\frac{1}{2\pi \mathrm{i}} \int\limits_{C_T} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\frac{1}{2\pi \mathrm{i}} \int\limits_{\frac{1}{2}+\mathrm{i}\bar{A}}^{\frac{1}{2}+\delta+\mathrm{i}\bar{A}} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\frac{1}{2\pi \mathrm{i}} \int\limits_{\frac{1}{2}+\delta-\mathrm{i}\bar{A}}^{\frac{1}{2}-\mathrm{i}\bar{A}} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\frac{1}{2\pi \mathrm{i}} \int\limits_{\frac{1}{2}+\delta+\mathrm{i}\bar{A}}^{c+\mathrm{i}\bar{A}} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\frac{1}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\bar{A}}^{\frac{1}{2}+\delta-\mathrm{i}\bar{A}} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds,
\end{aligned} \tag{1}$$

where C_T is the circular part of $C(T)^{+}$, and $0 < \delta < c - \frac{1}{2}$.

We may write

$$\begin{aligned}
&\psi_k(x) \\
&= \frac{1}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\bar{A}}^{c+\mathrm{i}\bar{A}} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\frac{1}{2\pi \mathrm{i}} \int\limits_{c+\mathrm{i}\bar{A}}^{c+\mathrm{i}\infty} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds + \\
&\frac{1}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\infty}^{c-\mathrm{i}\bar{A}} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds.
\end{aligned} \tag{2}$$

Since

$$\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} = \sum_{q=0}^{4} (-1)^q \frac{Z'_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s+\frac{q}{4}\right)}{Z_{P,\bigwedge^q \bar{\mathfrak{n}}}\left(s+\frac{q}{4}\right)},$$

it follows that $\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}$ is bounded in any half-plane of the form $\mathrm{Re}(s) > 1 + \varepsilon$.

We estimate,

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}} \int\limits_{c+\mathrm{i}\bar{A}}^{c+\mathrm{i}\infty} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds \\
&= O\left(x^{c+k} \int\limits_{\bar{A}}^{+\infty} \frac{dt}{t^{k+1}} \right) = O\left(x^{c+k} \bar{A}^{-k} \right).
\end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}} \int\limits_{c-\mathrm{i}\infty}^{c-\mathrm{i}\bar{A}} \frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} ds \\
&= O\left(x^{c+k} \bar{A}^{-k} \right).
\end{aligned} \tag{4}$$

Combining (1)-(4), it follows that

$$\begin{aligned}
&\psi_k(x) - O\left(x^{c+k} \bar{A}^{-k} \right) \\
&= \sum_{z \in C(T)} \mathrm{Res}_{s=z} \left(\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)} \times \times s^{-1}(s+1)^{-1} \ldots (s+k)^{-1} x^{s+k} \right) -
\end{aligned}$$

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{C_T}\frac{R^{'}_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds+\\&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{\frac{1}{2}+\mathrm{i}\,\bar{A}}^{\frac{1}{2}+\delta+\mathrm{i}\,\bar{A}}\frac{R^{'}_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds+\\&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{\frac{1}{2}+\delta-\mathrm{i}\,\bar{A}}^{\frac{1}{2}-\mathrm{i}\,\bar{A}}\frac{R^{'}_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds+\\&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{\frac{1}{2}+\delta+\mathrm{i}\,\bar{A}}^{c+\mathrm{i}\,\bar{A}}\frac{R^{'}_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds+\\&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\bar{A}}^{\frac{1}{2}+\delta-\mathrm{i}\,\bar{A}}\frac{R^{'}_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds.\end{aligned}\tag{5}$$

By [13, p. 38, Lemma 2.3.2],

$$Z_{P,\sigma}(s)=\frac{Z_1(s)}{Z_2(s)},$$

where the zeros of $Z_1(s)$ correspond to the zeros of $Z_{P,\sigma}(s)$, the zeros of $Z_2(s)$ correspond to the poles of $Z_{P,\sigma}(s)$, and σ is a finite-dimensional virtual representation of M.

The orders of the zeros of $Z_1(s)$ (resp. $Z_2(s)$) equal the orders of the corresponding zeros (resp. poles) of $Z_{P,\sigma}(s)$.

Functions $Z_1(s)$ and $Z_2(s)$ are both of finite order, i.e., they are both of order at most J.

Hence, the fact that for a finite-dimensional virtual representation σ of M

$$R_{\Gamma,\sigma}(s)=\prod_{q=0}^{4}Z_{P,(\bigwedge^q\bar{\mathfrak{n}}\otimes V_\sigma)}\left(s+\frac{q}{4}\right)^{(-1)^q},$$

it follows immediately that a meromorphic extension over $\mathbb{C}$ of the Ruelle zeta function $R_{\Gamma,\sigma}(s)$ can be expressed as

$$R_{\Gamma,\sigma}(s)=\frac{Z^1_{R,\sigma}(s)}{Z^2_{R,\sigma}(s)},$$

where $Z^1_{R,\sigma}(s)$, $Z^2_{R,\sigma}(s)$ are entire functions of order at most J over $\mathbb{C}$.

In particular,

$$R_{\Gamma,1}(s)=\frac{Z^1_{R,1}(s)}{Z^2_{R,1}(s)},\tag{6}$$

where $Z^1_{R,1}(s)$, $Z^2_{R,1}(s)$ are entire functions of order at most J over $\mathbb{C}$.

Now, we first estimate the integral over C_T on the right hand side of (5).

By applying Proposition 7 in [6, p. 509], and (6), we obtain that

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{C_T}\frac{R^{'}_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds\\&=O\left(x^{\frac{1}{2}+k}T^{-k-1}\int\limits_{C_T}\left|\frac{R^{'}_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\right||ds|\right)\\&=O\left(x^{\frac{1}{2}+k}T^{-k-1}\int\limits_{|s|=T}\left|\frac{R^{'}_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\right||ds|\right)\\&=O\left(x^{\frac{1}{2}+k}T^{-k-1+J}\log T\right).\end{aligned}\tag{7}$$

Consider the integral over $\left[\frac{1}{2}+\mathrm{i}\,\bar{A},\frac{1}{2}+\delta+\mathrm{i}\,\bar{A}\right]$ on the right hand side of (5).

By Theorem 1, (b) (i), in [10],

$$\frac{R^{'}_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}=O\left(\bar{A}^{J-1+\varepsilon}\right)+\sum_{\left|\bar{A}-\gamma_{P,1}\right|\leq1}\frac{1}{s-\rho_{P,1}}$$

for $s=\sigma^1+\mathrm{i}\,\bar{A}$, $\frac{1}{2}\leq\sigma^1\leq\frac{1}{2}+\delta$.

Since $\left|\bar{A}-\gamma_{P,1}\right|>\frac{C}{\bar{A}^D}$ for any singularity $\rho_{P,1}=\frac{1}{2}+\mathrm{i}\,\gamma_{P,1}$ of $Z_{P,1}(s)$, we obtain that

$$\begin{aligned}&\frac{R^{'}_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\\&=O\left(\bar{A}^{J-1+\varepsilon}\right)+O\left(\sum_{\left|\bar{A}-\gamma_{P,1}\right|\leq1}\frac{1}{|s-\rho_{P,1}|}\right)\\&=O\left(\bar{A}^{J-1+\varepsilon}\right)+O\left(\sum_{\left|\bar{A}-\gamma_{P,1}\right|\leq1}\frac{1}{\left|\bar{A}-\gamma_{P,1}\right|}\right)\\&=O\left(\bar{A}^{J-1+\varepsilon}\right)+O\left(\bar{A}^D\sum_{\left|\bar{A}-\gamma_{P,1}\right|\leq1}1\right)\\&=O\left(\bar{A}^{J-1+\varepsilon}\right)+O\left(\bar{A}^{2D}\right)\\&=O\left(T^{J-1+\varepsilon}\right)+O\left(T^{2D}\right).\end{aligned}$$

for $s=\sigma^1+\mathrm{i}\,\bar{A}$, $\frac{1}{2}\leq\sigma^1\leq\frac{1}{2}+\delta$.

We estimate,

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{\frac{1}{2}+\mathrm{i}\,\bar{A}}^{\frac{1}{2}+\delta+\mathrm{i}\,\bar{A}}\frac{R^{'}_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\times\\&\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds\end{aligned}$$

$$\begin{aligned}&=O\left(x^{\frac{1}{2}+\delta+k}T^{-k-1}\int\limits_{\frac{1}{2}+\mathrm{i}\,\bar{A}}^{\frac{1}{2}+\delta+\mathrm{i}\,\bar{A}}\left|\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\right||s|\right)\\&=O\left(x^{\frac{1}{2}+\delta+k}T^{-k-2+J+\varepsilon}\right)+\\&\quad O\left(x^{\frac{1}{2}+\delta+k}T^{-k-1+2D}\right).\end{aligned}\tag{8}$$

Analogously,

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{\frac{1}{2}+\delta-\mathrm{i}\,\bar{A}}^{\frac{1}{2}-\mathrm{i}\,\bar{A}}\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\times\\&\quad\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds\\&=O\left(x^{\frac{1}{2}+\delta+k}T^{-k-2+J+\varepsilon}\right)+\\&\quad O\left(x^{\frac{1}{2}+\delta+k}T^{-k-1+2D}\right).\end{aligned}\tag{9}$$

Finally, we estimate the integrals over $\left[\frac{1}{2}+\delta+\mathrm{i}\,\bar{A},c+\mathrm{i}\,\bar{A}\right]$ and $\left[c-\mathrm{i}\,\bar{A},\frac{1}{2}+\delta-\mathrm{i}\,\bar{A}\right]$ in (5).

By Theorem 1, (b) (ii), in [10],

$$\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}=O\left(\delta^{-1}\bar{A}^{J-1+\varepsilon}\right)$$

for $s=\sigma^1+\mathrm{i}\,\bar{A}$, $\frac{1}{2}+\delta\le\sigma^1\le c$.

We obtain,

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{\frac{1}{2}+\delta+\mathrm{i}\,\bar{A}}^{c+\mathrm{i}\,\bar{A}}\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\times\\&\quad\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds\\&=O\left(x^{c+k}T^{-k-1}\int\limits_{\frac{1}{2}+\delta+\mathrm{i}\,\bar{A}}^{c+\mathrm{i}\,\bar{A}}\left|\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\right||s|\right)\\&=O\left(\delta^{-1}x^{c+k}T^{-k-1}\,\bar{A}^{J-1+\varepsilon}\right)\\&=O\left(\delta^{-1}x^{c+k}T^{-k-1}T^{J-1+\varepsilon}\right)\\&=O\left(\delta^{-1}x^{c+k}T^{-k-2+J+\varepsilon}\right).\end{aligned}\tag{10}$$

Similarly,

$$\begin{aligned}&\frac{1}{2\pi\,\mathrm{i}}\int\limits_{c-\mathrm{i}\,\bar{A}}^{\frac{1}{2}+\delta-\mathrm{i}\,\bar{A}}\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\times\\&\quad\times s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}ds\\&=O\left(\delta^{-1}x^{c+k}T^{-k-2+J+\varepsilon}\right).\end{aligned}\tag{11}$$

Combining (5), (7)-(11), we obtain

$$\begin{aligned}&\psi_k(x)-O\left(x^{c+k}\bar{A}^{-k}\right)\\&=\sum_{z\in C(T)}\mathrm{Res}_{s=z}\left(\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}\times\right.\\&\quad\left.\times\,s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}x^{s+k}\right)+\\&\quad O\left(x^{\frac{1}{2}+k}T^{-k-1+J}\log T\right)+\\&\quad O\left(x^{\frac{1}{2}+\delta+k}T^{-k-2+J+\varepsilon}\right)+\\&\quad O\left(x^{\frac{1}{2}+\delta+k}T^{-k-1+2D}\right)+\\&\quad O\left(\delta^{-1}x^{c+k}T^{-k-2+J+\varepsilon}\right).\end{aligned}$$

Here, letting $T\to\infty$ (note that $T\to\infty$ if and only if $\bar{A}\to\infty$), we conclude that

$$\psi_k(x)=\sum_{\alpha\in S_k}c_k(\alpha)\,x^{\alpha+k},\tag{12}$$

where S_k denotes the set of poles of $\frac{R'_{\Gamma,1}(s)}{R_{\Gamma,1}(s)}s^{-1}(s+1)^{-1}\ldots(s+k)^{-1}$, and $c_k(\alpha)$ denotes the residue at α.

Since (12) actually represents the equality (3.7) in [13, p. 63], the equality

$$\psi(x)=2x+O\left(x^{\frac{3}{4}}\right)\tag{13}$$

as $x\to\infty$, follows in exactly the same way as in [13, pp. 63-65].

Hence, the prime geodesic theorem

$$\pi(x)=2\,\mathrm{li}(x)+O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)$$

as $x\to\infty$, follows from Proposition 3.7.1 in [13, p. 82].

This completes the proof. ■

IV. Remarks

By [13, p. 65],

$$\begin{aligned}&d^{-2D}\triangle\left(\sum_{\alpha\in S_{2D}^{\frac{p}{4}}}c_{2D}(\alpha)\,x^{\alpha+2D}\right)\\&=O\left(K^{D-1}x^{\frac{p}{4}}\right)+\\&\quad O\left(K^{-D-1}x^{2D+\frac{p}{4}}d^{-2D}\right),\end{aligned}\tag{14}$$

where $p\in\{-2,-1,0,1,2\}$, $S_{2D}^{\frac{p}{4}}=S_{2D}\cap(q+\mathrm{i}\,(\mathbb{R}\setminus\{0\}))$, $q\in\left\{-\frac{1}{2},-\frac{1}{4},0,\frac{1}{4},\frac{1}{2}\right\}$, $d>0$, $K>0$, and the operator Δ is defined by

$$\Delta f(x)=\sum_{i=0}^{2D}(-1)^i\binom{2D}{i}f(x+(2D-i)\,d)$$

for a function $f : \mathbb{R} \to \mathbb{R}$.

Note that the error term in (14) is dominated by

$$O\left(K^{D-1}x^{\frac{1}{2}}\right) + O\left(K^{-D-1}x^{2D+\frac{1}{2}}d^{-2D}\right).$$

Putting $K = x^{\alpha}$, $d = x^{\beta}$, and requiring that

$$\alpha D - \alpha + \frac{1}{2} = \frac{3}{4},$$
$$-\alpha D - \alpha + 2D + \frac{1}{2} - 2D\beta = \frac{3}{4},$$

we find that $\alpha = \frac{1}{4(D-1)}$, $\beta = \frac{4D-5}{4D-4}$, i.e., that $K = x^{\frac{1}{4(D-1)}}$, $d = x^{\frac{4D-5}{4D-4}}$.

Thus, the error term $O\left(x^{\frac{3}{4}}\right)$ in (13) follows as required.

In [13] and [5], the authors applied the method developed in [14] and [15] for compact Riemann surfaces.

The method described in this paper is very well applied in [12], [1], [9], and [7] in the case of real hyperbolic manifolds with cusps, compact, odd-dimensional, real hyperbolic spaces, and compact, even-dimensional locally symmetric Riemannian manifolds of strictly negative curvature, respectively.

In order to derive their results, the authors usually apply approximate formulas for the logarithmic derivative of the corresponding zeta function, such as the Riemann or the Selberg, or the Ruelle zeta function (see, [16], [11], [8], [2]).

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