A New Two-Step Single Tone Frequency Estimation Algorithm

Aldo De Sabata, Liviu Toma, and Septimiu Mischie

***Abstract*—**We propose a two-step procedure to estimate the frequency of a deterministic sinusoid, with unknown parameters, corrupted by additive, white, zero-mean noise, based on the Pisarenko Harmonic Decomposition. A rough PHD estimation is performed in the first step, and a multiple of the unknown frequency is estimated in the second step. The variance of the PHD estimator is significantly reduced.

***Keywords*—**Frequency estimation**,** Noisy real sinusoid, Pisarenko's method, Variance analysis.

I. INTRODUCTION

WE consider the problem of the frequency estimation of a signal that consists of a sine wave with additive, white noise, from a finite number of consecutive samples. This problem has a long history [1], and it is relevant for a wide field of applications such as communications, radar, sonar, speech processing, measurement, etc [2].

In many applications, complex valued exponentials are prefered to real valued sinusoids, especially when both in-phase and quadrature components of real signals are available, so that complex valued signals can be readily constructed inside computers [2]. The interest in complex versions of signals in frequency estimation originated in the discovery of the fact that the maximum likelihood (ML) frequency estimator for a complex exponential is the maximum of the signal's periodogram [3]. As ML estimators are not computationally efficient, other more efficient suboptimal estimators have been proposed and studied [4−9].

The problem of the ML estimator for a real sinusoid has also been addressed [10], and the solution turned out to be less simple than in the complex case. The ML estimator suffers again of the problem of computational complexity, so that suboptimal but more computationally efficient estimators exist in this case too. Some of these estimators can be derived from the more general spectrum estimators presented in the review [1]. The PHD is such a suboptimal estimator that relies on the eigenvalues of the covariance matrix of the signal. Its statistical properties has been extensively studied, and implementation in the case of a real sinusoid turned out to be very simple (see [11], [12] and the references cited therein). Also for the case of a single real sinusoid, a Reformed PHD (RPHD) estimator, which exhibits better statistical properties than the PHD, has been introduced recently [13], [14].

In order to decrease the estimator variance, we propose in this paper a new frequency estimation algorithm. Namely, we propose to extend the PHD to a k-PHD, through which we compute the signal's (digital) frequency from the estimated cosine of its k'th multiple. In order to raise the inherent ambiguity, a rough estimate of the frequency must be known a priori, or it must be estimated by other, less computationally complex means. Here we will use the PHD for the initial estimate too, in order to show that the variance of the newly introduced estimator is significantly reduced with respect to the variance of the known one. We will also compare the variance of our estimator to the Cramer-Rao lower bound (CRLB).

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II. DESCRIPTION OF THE METHOD

We consider the following discrete-time signal model

$$x(n) = s(n) + q(n) = \alpha\cos(\omega n + \varphi) + q(n), \quad n = 1..N \tag{1}$$

where the amplitude $\alpha > 0$, (angular) frequency $\omega \in (0,\pi)$ and phase φ are deterministic but unknown quantities, and $q(n)$ is a Gaussian, white noise, uncorrelated with the signal, with zero-mean and variance σ^2. The signal-to-noise ratio is $SNR = \alpha^2/(2\sigma^2)$. Following [1], we review here briefly the derivation of the PHD estimator in order to introduce its extension to the estimation of a multiple of ω. The sinusoid obeys the following k'th order linear prediction equation for any fixed integer k:

$$s(n) - 2\cos(k\omega)s(n-k) + s(n-2k) = 0 \tag{2}$$

(n denotes the discrete time). It is possible to derive from (1) and (2) the following vector equation:

$$\mathbf{x}_n^T\mathbf{a} = \mathbf{q}_n^T\mathbf{a} \tag{3}$$

where $\mathbf{x}_n = [x(n) \quad x(n-k) \quad x(n-2k)]^T$, $\mathbf{q}_n = [q(n) \quad q(n-k) \quad q(n-2k)]^T$ and

$\mathbf{a} = [1 \quad -2\cos(k\omega) \quad 1]^T$. If both sides of (3) are premultiplied by $\mathbf{x}_n$, the expected value is taken, and the fact that s and q are uncorrelated is taken into account, the following eigenequation results

$$\mathbf{R}_{xx}\mathbf{a} = \sigma^2\mathbf{a}. \tag{4}$$

In (4), $\mathbf{R}_{xx}$ is the signal autocorrelation matrix, which has the shape

$$\mathbf{R}_{xx} = \begin{bmatrix} r_0 & r_k & r_{2k} \\ r_k & r_0 & r_k \\ r_{2k} & r_k & r_0 \end{bmatrix} \tag{5}$$

where, for any integer p, $r_p = E\{x(n)x(n-p)\} = E\{x(n+p)x(n)\}$ is the value of the signal's autocorrelation function at lag p.

Solving (4) for $\cos(k\omega)$ yields

$$\cos(k\omega) = \frac{r_{2k} + \sqrt{r_{2k}^2 + 8r_k^2}}{4r_k}. \tag{6}$$

As the values of the autocorrelation function are not known, the following estimate at lag m must be used [15]

$$r_m = \frac{1}{N-m}\sum_{n=m+1}^{N} x(n)x(n-m) \tag{7}$$

(by a slight abuse, we have used the same notation for the true values of the autocorrelation function and for its estimates).

The PHD method results for $k = 1$ in the above argument. From (6) we get the following estimate for ω, denoted $\hat{\omega}_1$

$$\hat{\omega}_1 = \cos^{-1}\left(\frac{r_2 + \sqrt{r_2^2 + 8r_1^2}}{4r_1}\right). \tag{8}$$

For $k > 1$, (6) and (7) provide an estimate $\hat{\omega}_k$ of ω as follows. Let $\rho_k = \cos(k\hat{\omega}_k)$ denote the value of the LHS of (6) when in the RHS the estimates (7) are substituted. The roots of (6) are

$$\omega_p = \frac{1}{k}\left((-1)^{p-1}\cos^{-1}(\rho_k) + \left[\frac{p}{2}\right]2\pi\right),\ p = 1, 2, \ldots, k \tag{9}$$

where we have denoted by $[x]$ the largest integer smaller than x. If we dispose of a rough estimate $\hat{\omega}_1$ for ω, then the above defined $\hat{\omega}_k$ is equal to the ω_p in (9) calculated for $p = 1 + \left[\frac{k\hat{\omega}_1}{\pi}\right]$.

We propose the following *two-step* frequency estimation procedure based on the PHD and theory presented above, which can be referred to as k-PHD: 1. Get a rough estimate of the unknown frequency using a low complexity method; in particular, the PHD (1-PHD) can be applied, perhaps with a small value of N.

2. Use (6) and (9), perhaps with a larger value of N than in the first step and $k > 1$ in order to obtain a better frequency estimate. The way to choose an appropriate value for k will be presented bellow.

We will show now that theory and computer experiments confirm that the estimator variance is significantly reduced in the case of the k-PHD with respect to the 1-PHD, the price paid being an increase in computational complexity that can be acceptable in many practical situations.

III. Estimator Variance

The 1-PHD is an asymptotically unbiased frequency estimator, whose variance has been calculated and reported in [12]. Following the same procedure, we have calculated the variance for the k-PHD, and the result is as follows

$$\mathrm{var}(\hat{\omega}_k) = \frac{A + B + C + D}{F^2\sin^2(k\omega)}, \tag{10}$$

where

$$A = \frac{\sigma^4\cos^2(2k\omega)}{N-k} + \frac{\sigma^4\cos^2(k\omega)}{N-2k}, \tag{11}$$

$$\begin{aligned} B = {} & \frac{\alpha^2\sigma^2(N-2k)\cos^3(2k\omega)}{(N-k)^2} + \frac{\alpha^2\sigma^2\cos^2(k\omega)}{N-2k} \\ & + \frac{\alpha^2\sigma^2(N-4k)\cos(4k\omega)\cos^2(k\omega)}{(N-2k)^2} - \frac{\alpha^2\sigma^2\cos(2k\omega)}{N-k} \\ & - \frac{2\alpha^2\sigma^2(N-3k)\cos(k\omega)\cos(2k\omega)\cos(3k\omega)}{(N-k)(N-2k)}, \end{aligned} \tag{12}$$

$$C = \left(\frac{\alpha^2\beta(N-k,k,\omega,\varphi)\cos(2k\omega)}{2(N-k)} - \frac{\alpha^2\beta(N-2k,2k,\omega,\varphi)\cos(k\omega)}{2(N-2k)}\right)^2, \tag{13}$$

$$D = \frac{\alpha^2\sigma^2\beta(N-k,0,\omega,\varphi)\cos^2(2k\omega)}{2(N-k)^2} + \frac{\alpha^2\sigma^2\beta(N-2k,2k,\omega,\varphi)\cos^2(2k\omega)}{(N-k)^2}$$

$$
\begin{aligned}
&+\frac{\alpha^2\sigma^2\beta(N-k,2k,\omega,\varphi)\cos^2(2k\omega)}{2(N-k)^2}\\
&+\frac{\alpha^2\sigma^2\beta(N-2k,0,\omega,\varphi)\cos^2(k\omega)}{2(N-2k)^2}\\
&+\frac{\alpha^2\sigma^2\beta(N-4k,4k,\omega,\varphi)\cos^2(k\omega)}{(N-2k)^2}\\
&+\frac{\alpha^2\sigma^2\beta(N-2k,4k,\omega,\varphi)\cos^2(k\omega)}{2(N-2k)^2} \qquad (14)\\
&-\frac{\alpha^2\sigma^2\beta(N-2k,k,\omega,\varphi)\cos(k\omega)\cos(2k\omega)}{(N-k)(N-2k)}\\
&-\frac{2\alpha^2\sigma^2\beta(N-3k,3k,\omega,\varphi)\cos(k\omega)\cos(2k\omega)}{(N-k)(N-2k)}\\
&-\frac{\alpha^2\sigma^2\beta(N-2k,3k,\omega,\varphi)\cos(k\omega)\cos(2k\omega)}{(N-k)(N-2k)},
\end{aligned}
$$

$$
\begin{aligned}
F=&\frac{\alpha^2(2+\cos(2k\omega))}{2}\\
&+\frac{2\alpha^2\cos(k\omega)\beta(N-k,k,\omega,\varphi)}{N-k} \qquad (15)\\
&-\frac{\alpha^2\beta(N-2k,2k,\omega,\varphi)}{2(N-2k)},
\end{aligned}
$$

and

$$
\begin{aligned}
\beta(N,M,\omega,\varphi)&=\sum_{n=1}^{N}\cos\big((2n+M)\omega+2\varphi\big)\\
&=\frac{\sin(\omega N)\cos(\omega(N+M+1)+2\varphi)}{\sin(\omega)}. \qquad (16)
\end{aligned}
$$

The derivation of the variance of the k-PHD estimator parallels the derivation for the variance of the 1-PHD estimator from [12] and therefore will be skipped. The results in [12] can be obtained from (10)...(16) by making k=1.

IV. Computer Simulation Results

In order to demonstrate the effectiveness of the k-PHD estimator we have performed computer experiments with a sinusoid of amplitude $\sqrt{2}$ with white, additive noise. The results presented in Figs. 1..3 are averages of 1000 runs.

In Fig. 1, the variance of the frequency estimate in function of frequency is presented. The calculated variances and the CRLB [15] are also shown in Fig.1. We have used the same signal length N for both the initial estimate (k=1) and the second estimate (k=3) in order to show the decrease of the variance, which is between 5 and 15 dB, except for frequencies around $\frac{\pi}{3}$ and $2\frac{\pi}{3}$. The reason of this behaviour results from the following.

We have

$$
\mathrm{var}(\rho_k)=E\{\left(\cos(k\omega)-\cos(k\hat{\omega}_k)\right)^2\}=
E\left\{4\sin^2\left(\frac{k(\omega+\hat{\omega}_k)}{2}\right)\sin^2\left(\frac{k(\omega-\hat{\omega}_k)}{2}\right)\right\}.
$$

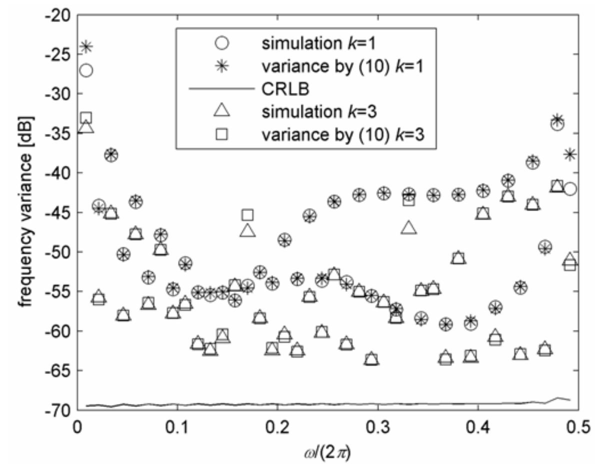

Fig. 1 Variances versus frequency at SNR=20 dB, N=100, and $\varphi=\pi/8$

For small k and a good estimation (small $|\omega-\omega_k|$), this equation implies

$$
\mathrm{var}\{\hat{\omega}_k\}\approx\frac{\mathrm{var}(\rho_k)}{k^2\sin^2(k\omega)}. \qquad (17)
$$

The ρ_k are calculated in terms of the signal's autocorrelation function at lags k and 2k. Therefore the numerator in (17) increases much less with k than the denominator. Furthermore, the variance becomes unbounded at frequencies $m\frac{\pi}{k},\ m=0..k$. In a practical situation, we dispose of a first frequency estimate, so that k can be chosen such that the case when the frequency variance becomes unbounded can be avoided.

In Fig. 2, the frequency variance versus SNR, and in Fig. 3 the frequency variance versus signal length N are presented for a fixed frequency. Both figures clearly indicate the decrease of the frequency variance for k>1 with respect to k=1.

We will now tackle the problem of computational complexity. For the calculation of the r_m in (7), $N-m+1$ real multiplications and $N-m-1$ additions are necessary. For k=1, the estimator is given by (8), and it requires the calculation of the r_1 and r_2. This can be implemented by $2N-1$ real multiplications, $2N-5$ additions and a ROM access. For $k>1$, the estimator requires the calculation of r_k and r_{2k}. This can be implemented by $2N-3k+2$ real multiplications, $2N-3k-2$ additions and two ROM accesses. The first one is necessary in order to calculate the value of p to be substituted in (9) and the second for (9) itself. This would give a total of $4N-3k+1$ real multiplications, $4N-3k-7$ additions and three ROM accesses. Thus, the number of operations remains $O(N)$ in the two-step method (like in the case of the PHD).

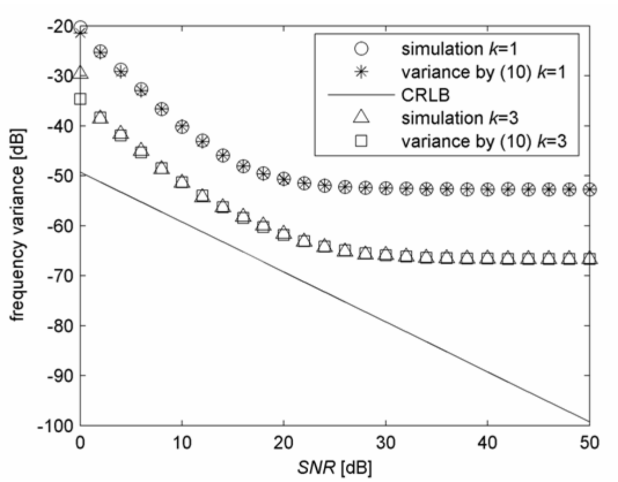

Fig. 2 Variances versus SNR at ω=0.417π, N=100, and φ=π/8

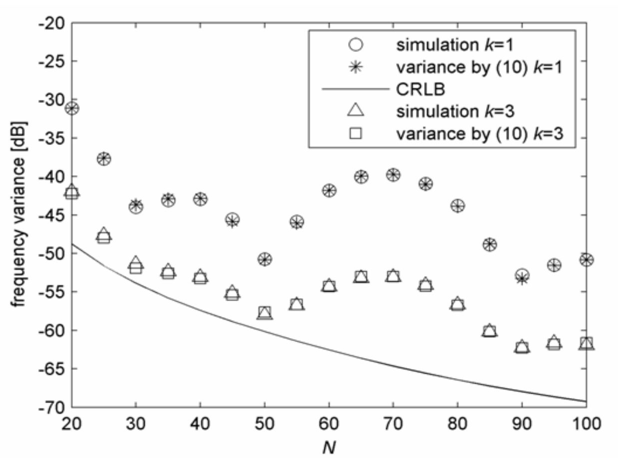

Fig. 3 Variances versus N at ω=0.417π, SNR=20 dB, and φ=π/8

V. Conclusion

We have proposed a two-step procedure to estimate the frequency of a sinusoid with deterministic but unknown amplitude, frequency and phase, corrupted by Gaussian, additive, zero-mean white noise, namely the k-PHD. In the first step, a rough estimate of the frequency is obtained, and it is used in the second step to raise the ambiguity introduced by the fact that the cosine of a multiple of the frequency is estimated. We have selected the Pisarenko Harmonic Decomposition for illustration of the two-step method, due to its low computational complexity. We have calculated the variance of the proposed frequency estimator and showed, both analitically and through computer experiments, that it resulted significantly lower than in the case of the one-step, classical PHD method.

Application of the two-step procedure is not restricted to the PHD. We have demonstrated its effectiveness when applied to the RPHD [16]. Future work will be dedicated to the application of the two-step method to the complex case.

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