Controller Reduction of Discrete Linear Closed Loop Systems in a Certain Frequency Domain

R. Sadeghian, P. Karimaghaee, and A. Khayatian

***Abstract*—**In this paper, a novel controller reduction method for discrete linear time invariant system is presented. The reduction method is based on defining new controllability and observability grammians which are calculated from input to state and state to output charactrestics of the controller in a certain frequency domain. These grammians are defined for the closed loop system to keep the performance of original controller. After defining these new grammians, Moore balance truncation method is used in a certain frequency domain to reduce the order of controller. The stability property of the new method is investigated. It is shown by forming Lyapunov equations. Simulation results on a typical example show the effectiveness of the method.

***Keywords*—** Controller reduction, Model reduction, Discrete time systems, Grammians, Stability, Frequency domain.

I. Introduction

MODERN controller design techniques such as *LQR*, H_2, H_∞,... often lead to high order and complex controllers. It is obvious that these high order controllers are more complicated in implementing and also for debugging. Hence, there is a real need for reliable reduction methods which allow a low order controller to be extracted from a high order controller without incurring too much error.

Commonly, model reduction techniques are used for controller reduction but without loop considerations. Therefore, for reviewing the controller reduction methods, it is useful to review the model reduction techniques. One of the main model reduction methods is Moore [2] model reduction which is based on balancing controllability and observability grammians. Another method is frequency weighted method which was first introduced by Enns [3]. Enns shows that these weights can improve the accuracy of model reduction. Enns' weights are one sided and he shows that two sided weights may make the system unstable. In [7] for a balanced controller system, Generalized Singular Perturbation is used for controller reduction. In [8] by balancing new impulse response gramminas a new controller reduction method is defined. Although most of these methods are presented for continuous time systems but in [9] it is shown by a comparison that the results for the discrete time systems are the same. This method was developed for H_∞ designed controllers in [12].

In this paper, we use the extension of Moore balanced truncation [4] for discrete controller reduction in a closed loop system. This method was first introduced for continues systems and then extended for discrete systems in [6] and [11] and the idea wass developed in [10]. In this new method, reduction will be done by considering the energy distribution between closed loop system input and controller states and also controller states to the output of the closed loop system. Based on this consideration, by defining the new controllability and observability grammians, the balanced truncation method will be used to reduce the order of the controller. This procedure is performed in a certain frequency domain because most systems are designed to work properly in a certain frequency bound.

This paper is organized as follows. In section 2, balanced truncation model reduction for discrete time systemls will be discussed. In section 3, the new controllability and observability grammians for the controller with closed loop consideration will be proposed and the algorithm for controller reduction with these new grammians will be discussed and also the stability of reduced order controller will be shown. The benefits of new method are demonstrated by an example in section 4. Finally in section 5, the conclusion will be presented.

II. Balanced Realization and Model Reduction

Let us consider an nth order linear time invariant asymptotically stable discrete system (A,B,C) as:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where $u \in R^p$, $y \in R^q$, $x \in R^n$ are the input, output, and state (1) respectively. Also, $A \in R^{n\times n}$, $B \in R^{n\times p}$, $C \in R^{q\times n}$ are real valued matrices.

The controllability and observability grammians of system (1) can be defined respectively as:

$$W_c = \sum_{k=0}^{\infty} A^k BB^T (A^T)^k \tag{2}$$

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$$W_o = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k \tag{3}$$

The grammians of equations (2) and (3) are the solutions of the Lyapunov equations of (4) and (5) respectively:

$$A W_c A^T - W_c = -BB^T \tag{4}$$

$$A^T W_o A - W_o = -C^T C \tag{5}$$

It has been shown that similarty transformation can be found such that the system (1) is internally balanced, that is, the matrices W_c and W_o are equal and diagonal:

$$W_c = W_o = \Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_n) \tag{6}$$

where $\sigma_i \geq \sigma_{i+1}$, *i=1,2,...,n-1* are the grammians singular values and are invariant under similarty transformation.

Based on the order of magnitude of singular values, this balanced system and grammian can be partitioned as below:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$
$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{7}$$

And it can be shown that if $\sigma_i >> \sigma_{i+1}$, the subsystem (A_{11}, B_1, C_1) is a good reduced order approximation of the main full order system (A, B, C). technique is called Balance Truncation (BT).

III. New Controller Reduction Approach

In this section, the new controllability and observability frequency based grammians from the input/output energy distribution viewpoint will be proposed. These grammianns will be used in next step for controller reduction. Consider the closed loop system of figure 1.

Where in this figure, the transfer function of the plant, G(z)

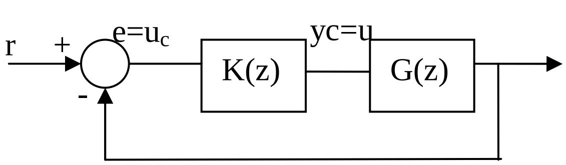

Fig. 1 The closed loop system

has a state space realization as equation (1). K(z) is a high order stable controllable and observable controller with state space realization as (8).

$$\begin{cases} x_c(k+1) = A_c x_c(k) + B_c u_c(k) \\ y_c(k) = C_c x_c(k) \end{cases} \tag{8}$$

Where $x_c \in R^{n_c}$ is a state vector, $u_c \in R^q$ represents the input vector of the controller, and $y_c \in R^p$ is the output of the controller with matrices A_c, B_c, and C_c in the appropriate dimensions. The order of the plant is n and the order of controller is n_c

For the closed loop system of fig. 1, consider the following equations:

$$X_c(e^{j\omega}) = (e^{j\omega} I - A_c)^{-1} B_c U_c(e^{j\omega}) \tag{9}$$

$$U_c = R - Y = R - GC_c X_c \tag{10}$$

This can be simplified to result the relation between controller states and system input as:

$$\begin{aligned} X_c &= (e^{j\omega} I - A_c + B_c G C_c)^{-1} B_c R \\ &= (e^{j\omega} I - A_c)^{-1} B_c (I + GK)^{-1} R \end{aligned} \tag{11}$$

where the dependence on $e^{j\omega}$ has been dropped for simplicity.

By forming the following energy related quantity due to input R as in [5] and using Parsaval theorem we have:

$$\begin{aligned} E_i &= \sum_{k=-\infty}^{\infty} x_c(k) x_c^*(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_c(e^{j\omega}) X_c^*(e^{j\omega}) d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{j\omega} I - A_c)^{-1} B_c (I + GK)^{-1} R R^* (I + GK)^{-*} B_c^* (e^{j\omega} I - A_c)^{-*} d\omega \end{aligned} \tag{12}$$

By considering the input as white noise, the frequency domain closed loop controllability gammian of controller W_{cc} is defined as:

$$W_{cc} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{j\omega} I - A_c)^{-1} B_c (I + GK)^{-1} R R^* (I + GK)^{-*} B_c^* (e^{j\omega} I - A_c)^{-*} d\omega \tag{13}$$

A similar interpretation for the output of the closed loop system can hold. This time the output energy of the system due to controller state will be considered. Because $Y = GY_c = GC_c X_c$ then using relation (10) to (12) and Parsaval theorem, we have:

$$\begin{aligned} E_o &= \sum_{k=-\infty}^{\infty} y^*(k) y(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y^*(e^{j\omega}) Y(e^{j\omega}) d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{-j\omega} I - A_c^*)^{-1} C_c^* (I + GK)^{-*} G^* R^* R G_c (I + GK)^{-1} C_c (e^{j\omega} I - A_c)^{-1} d\omega \end{aligned} \tag{14}$$

Relation (14) was obtained by considering the states as white noise and eliminating the input of the closed loop system. The frequency domain closed loop observability gammian of controller W_{oc} is defined as (15).

$$W_{oc} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{-j\omega} I - A_c^*)^{-1} C_c^* (I + GK)^{-*} G^* R^* R G_c (I + GK)^{-1} C_c (e^{j\omega} I - A_c)^{-1} d\omega \tag{15}$$

Because most systems work in a certain frequency domain it is desirable to tight the frequency domain. In this case, consider the input signal $r(e^{j\omega})$ which its energy density spectrum is unity in frequency range $[\omega_0, \omega_1]$ and zero elsewhere, i. e:

$$\left|R(e^{j\omega})\right| = \begin{cases} 1, & \omega \in [\omega_0, \omega_1] \\ 0, & Otherwise \end{cases} \tag{16}$$

So the grammians in (13) and (15) for finite frequency range will be:

$$W_{cc} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{j\omega}I - A_c)^{-1}B_c(I+GK)^{-1}RR^*(I+GK)^{-*}B_c^*(e^{j\omega}I - A_c)^{-*}d\omega \tag{17}$$

$$W_{oc} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{-j\omega}I - A_c^*)^{-1}C_c^*(I+GK)^{-*}G^*R^*RG_c(I+GK)^{-1}C_c(e^{j\omega}I - A_c)^{-1}d\omega \tag{18}$$

The following lemma shows that by using a transformation on these grammians the eingenvalues of the grammians will be invariant so these closed loop grammians can be used for controller reduction:

Lemma 3.1. For a discrete linear time-invariant controller (A_c, B_c, C_c) the following properties hold:

A. If a coordinate transformation such as $x(k) = T\hat{x}(k)$ is considered, the transformed closed loop controllability and observability grammians of the controller can be calculated as: $\widetilde{W}_{cc} = T^{-1}W_{cc}T^{-*}$, $\widetilde{W}_{co} = T^*W_{co}T$

B. Under the selected transformation T, the singular values of product W_{cc}, W_{co} hat is, $\sigma_i = \sqrt{\lambda_i(W_{cc}W_{oo})}$ are invariant.

C. There exists a special transformation $\overline{T}$ such that the closed loop controllability and observability grammians of controller can be diagnolized and equal

Proof. The result follows using direct substitutions. □

Based on lemma 3.1, the singular values of the closed loop system are invariant so one can find a similarty transformation which can balance the controllability and observability grammians of the closed loop system (17),(18). This transformation can be calculated by the procedure presented in [1]. After using this transformation, these grammians which are diagnolized and equal can be represented in the new coordinate can be as $\sum_f$, where

$$\sum_f = diag(\sigma_1, \sigma_2, ..., \sigma_{n_c}), \quad \sigma_i \geq \sigma_{i+1}, \quad i = 1,2,.., n_c - 1$$

Partitioning the balanced controller and Grammian gives that

$$A_c = \begin{bmatrix} A_{c11} & A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix}, \quad B_c = \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix}, \quad C_c = \begin{bmatrix} C_{c1} & C_{c2} \end{bmatrix}$$
$$\Sigma_f = \begin{bmatrix} \Sigma_{f1} & 0 \\ 0 & \Sigma_{f2} \end{bmatrix}, \quad x_c(k) = \begin{bmatrix} x_{c1}(k) \\ x_{c2}(k) \end{bmatrix} \tag{19}$$

where A_{c11} and $\sum_{f1}$ are $r \times r$ ($r<n_c$) matrices. The system $(A_{c11}, B_{c1}, C_{c1})$ is the reduced-order controller.

The following steps described an algorithm for the proposed controller reduction method:

A. Calculate closed loop controllability and observability grammians W_{cc} and W_{co} in the given frequency range. They can be obtained by (17), (18).

B. Find a similarity transformation T that makes the controller balanced, that is, $W_{cc} = W_{co} = \Sigma_f$.

C. Partition the transformed controller as (19) based on the grammians singular values. The subsystem $(A_{c11}, B_{c1}, C_{c1})$ is the reduced controller

Now we show the stability of the reduced order controller. Consider equation (17) where the finite frequency domain can be considered with a window $W(e^{i\omega})$ where

$$W(e^{j\omega}) = \begin{cases} 1, & \omega_0 < \omega < \omega_1 \\ 0, & Elsewhere \end{cases} \tag{20}$$

So we have

$$W_{cc} = \frac{1}{2\pi}\int_{-\pi}^{\pi}W(e^{j\omega}I - A_c)^{-1}B_cW_iW_i^*B_c^*(e^{j\omega}I - A_c^*)^{-1}W^*d\omega \tag{21}$$

where $W = (I + GK)^{-1}$

Using Parsaval theorem, the equation (21) in discrete time domain can be written as:

$$W_{cc} = \sum_{k=0}^{\infty} A_c^k B_k * W_2 W_2^* * B_k^*(A_c^*)^k = \sum_{k=0}^{\infty} A_c^k z_k z_k^*(A_c^*)^k \tag{22}$$

where * denotes convolution and $W_2 = W * W_i$ and $z = B_k * W_2$. By forming the Lyapunov equation for controller we have:

$$A_c W_{cc} A_c^* - W_{cc} = A_c^{\infty} z_k(\infty) z_k^*(\infty)(A_c^*)^{\infty} - z_k(0) z_k^*(0) \tag{23}$$

If the original controller is stable, in infinite time the states of the controller converge to zero, so we have:

$$\begin{aligned} & A_c W_{cc} A_c^* - W_{cc} = -z_k(0) z_k^*(0) \leq 0 \quad \Rightarrow \\ & A_c W_{cc} A_c^* - W_{cc} = -NN^* \end{aligned} \tag{24}$$

By the same procedure for the observability grammian, we have:

$$A_c^* W_{co} A_c - W_{co} = -LL^* \qquad (25)$$

So the controllability and observability grammians are negative semidefinite and based on [6] we can say that the reduced order controller is stable.

IV. Example

In [2] a plant is considered as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -50 & -79 & -33 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 50 & 15 & 1 & 0 \end{bmatrix}$$

We design a controller for this plant as:

$$C(z) = \frac{1.228 - 1.075 z^{-1} + .3323 z^{-2}}{1 - 2.207 z^{-1} + 1.777 z^{-2} - 0.5122 z^{-3}}$$

We reduce this controller by the new method in frequency range of $\left[0, \frac{\pi}{4}\right]$ and Moore method. The result of the error between the reduced order controller and the full order controller $\|K(z) - K_r(z)\|_\infty$ are summarized in Table 1. Moore method does not consider the loop. So the results show that this method cannot be so much accurate.

Method	Reduced Order Controller of Degree 2	Reduced Order Controller of Degree 1
BT	0.3177	0.3760
New Method	0.1857	0.1937

Table 1 The error of the closed loop system

Figure 2 shows the magnitude of the reduced order controller of degree 2 and 1 with the new method and original controller. It is obvious that the reduced order controller has the performance similar to original controller in the defined frequency range.

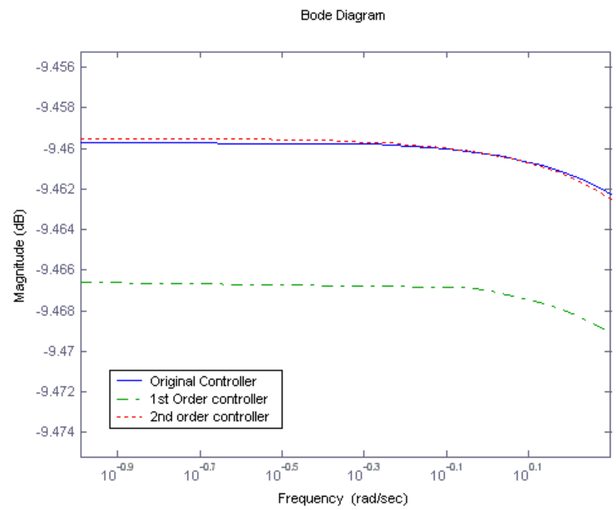

Fig. 2 Original controller and controller of degree 2 (--) and 1 (-.)

V. Conclusion

In this paper a new frequency based controller reduction method for discrete linear time invariant system was proposed. The method is based on new frequency-domain controllability and observability grammians. The stability of the reduced order controller was discussed and the simulation results show the effectiveness of this method.

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