Maximal Invariant Sets of Multiple Valued Iterative Dynamics in Disturbed Control Systems

Byungik Kahng

***Abstract*—** Invariant set theory is an important tool in the control theory. It has rich history that goes back over a century, yet it is still an active research topic both in pure mathematics and theoretical engineering. It is easy to reduce a traditional discrete-time control dynamical system to an iterative dynamics of one endomorphism in the phase space. It is not easy to do the same in the presence of the disturbance. The purpose of this paper is to show that we can overcome this difficulty using the dynamics of multiple valued maps. First, we show that the dynamics of disturbed control systems can be modeled by the multiple valued iterative dynamics. Second, we define and study the invariant sets, the maximal invariant sets, and the positively maximal invariant sets of the multiple valued iterative dynamical systems. Finally, as an application, we study the reachability problem of the maximal positively invariant sets of the multiple valued iterative dynamical systems.

***Keywords*—** Multiple valued dynamics, Control dynamical system with disturbance, Maximal invariant set, Reachability, Controlability.

I. Introduction

THE invariant set theory of iterative dynamical systems is an important topic in both pure and applied mathematics. It is a classical topic that is as old as the dynamical systems theory itself. See, for instance, [1] for the pure-mathematical treatment of the invariant set theory in conjunction to classical Lyapunov theory, which goes back to late 1800s. The history of its applications to engineering is also rich. See, for example, [10] for the survey of the invariant set theory written in engineering perspective.

Recently, the invariant set theory came back as an active research topic again, particularly in control and automation theory [3], [9], [10], [14], [16], [18], [20], [21], [22], [23], [26], [27], [28], [29], [31], [30], [32], and also, to a certain degree, in robotics[1] [8], [11], [12].

The specific problem that this article is most interested in is the study of the maximal invariant sets of discrete-time control dynamical systems. This is an important branch of the invariant set theory with lots of applications [10]. Not only does it appear in relatively dated problems like Model Predictive Control [21], [23], but also, in more up to date topics such as Hybrid Systems [9], [22], [30], [31], [32] and Control Systems with Disturbance [16], [20], [26], [28], [29].

The resurgence of the maximal invariance is also visible in pure mathematics. The invariant sets and the invariant properties have been (and still are) considered important topics all over the pure mathematics. However, most of the pure mathematical research had been concentrated upon the study of specific invariant sets related to specific invariant properties, rather than as a whole. It is relatively recent development that one began to study the latter aspect of the invariant set theory in pure mathematics.

The classical results about the maximal invariance in conjunction to Lyapunov theory depend heavily upon the topological structure of the dynamical system (Theorem VII.1, for instance) [1]. In recent years, however, a number of significant progresses were made for the invariant set theory of discontinuous dynamical systems, which is still in primitive state by and large [4], [5], [6], [7], [13], [17], [18], [17], [19], [24], [25].

The purpose of this article is the continuation of [19], which was the author's first attempt to connect the invariant set theory of pure mathematics and that of control theory. [19] did not consider disturbed control systems. Consequently, the control dynamical systems discusses in [19] were somewhat dated, in the viewpoint of engineering. We aim to improve this aspect by including the disturbance.

The dynamical system that we are particularly interested in is the discrete-time control dynamical system with disturbance. As we will see in Section II, it is not difficult to reduce a classical discrete-time time-invariant control dynamical system (II.1) to an iterative dynamical system of one endomorphism (II.2). It is not easy to do the same when there is disturbance. We will overcome this difficulty by re-expressing the control dynamical systems with disturbance as the multiple valued iterative dynamical systems (Section IV). And then, we define the maximal invariant sets (Definition V.1) and the locally maximal positively invariant sets (Definition VI.3) for the multiple valued iterative dynamical systems.

As an application, we study a reachability problem (aka controlability problem) of the locally maximal positively invariant set (Section VIII). This is slightly different from the classical reachability/controlability problems, although closely related. Instead of looking for the set of reachable states from a given set of initial states, or searching for the set of initial states that produce the desired final states, we study the conditions to design the control dynamical systems that the locally maximal positively invariant sets are reachable. Also,

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[1]Due to the author's limited knowledge in engineering, the list of references in robotics are neither comprehensive nor well-represented. Only the papers that the author was fortunate enough to encounter in person were listed.

we will rely heavily upon the infinite-step reachabillity, rather than the traditional N-step reachability. Because we are trying to reach sets instead of points, the discrete time dynamical path from the starting set to the desired set (the locally maximal positively invariant set) involve the iterative images and the set operations. Also, by concentrating upon the infinite-step reachabillity, we will investigate the conditions under which the traditional N-step reachability problems and the resulting approximation schemes are theoretically valid.

II. The Invariant Set Theory of Discrete Time Control Dynamical Systems with No Disturbance

Let X be a set, which we will call the **phase space**. Let us call the elements of X, the **states**. Let U be a set, whose elements we will call the **control variables**. A **discrete-time time-invariant control dynamical system** is an iterative dynamical system given by a pair of maps,

$$\begin{cases} F : (x_k, u_k) \mapsto x_{k+1}, \\ G : x_k \mapsto u_k, \end{cases} \tag{II.1}$$

where $x_k \in X$ (the k-th **state** of the system) and $u_k \in U$ (the **control input**) [21]. The map $F : X \times U \to X$ represents the dynamics of this system. The map $G : X \to U$ is called the **feedback control law** [10], [21]. Given initial state $x_0 \in X$, the sequence $(x_0, x_1, x_2, \cdots)$ is called the **(forward) orbit** of x_0.

Due to the time-invariance, the input variable u_k depends only on the current state x_k. Therefore, the feedback control map $G : x_k \to u_k$ is well defined, in this case. Hence, we can reduce the control dynamical system (II.1), as the iterative dynamics of one endomorphism, $f : X \to X$, where

$$f : x \to F(x, G(x)). \tag{II.2}$$

Consequently, the control invariant sets and the positive control invariant sets of the system (II.1) can be expressed in terms of the iterated function system of the endomorphism (II.2) as follows.

Definition II.1. *Let X be a non-empty set (phase space) and $f : X \to X$ be an endomorphism. We say $S \subset X$ is* **invariant** *under the iteration of f, if $f(S) = S$. We say $S \subset X$ is* **positively invariant** *under the iteration of f, if $f(S) \subset S$.*

The importance of the invariant sets and the positively invariant sets is well documented. See, for instance, [10] for an extensive survey of the invariant set theory and its applications to the control and automation theory.

Let us now turn our attention to the maximal invariant set. The definition of the maximal invariant set differs from one author to another. It had been defined as the intersection of all the intersection of all the iterates of the whole space ([6], [13], for instance), the smallest (in terms of the set-inclusion) set that includes all the invariant sets ([24], for example), and the largest (again, in terms of the set-inclusion) invariant set [3], [10], [16], [20], [21], [23], [26], [27], [28], [29]. Each method has its advantages and disadvantages. The first method is problematic in that the intersection of all the iterates of the whole space is not invariant in general [17], [19], [24]. The second method is difficult handle, because it requires the axiom of choice (via Zorn's Lemma) [15]. Also, it is not obvious that the maximal invariant set defined as such is indeed invariant. The third method is most popular in engineering, but it is not always clear that such a set exists. This issue becomes even more problematic if the control system gets more complicated. In this paper, we follow the method developed in the author's earlier articles, [17], [18], [19].

Definition II.2. *Let X be a non-empty set (phase space) and $f : X \to X$ be an endomorphism. We define the* **maximal invariant set** *$\mathcal{M}(X)$ of f as the union of all invariant subset of X. That is,*

$$\mathcal{M}(X) = \bigcup \{S \subset X : f(S) = S\}.$$

The existence of $\mathcal{M}(X)$ and its maximality are obvious from the definition. Whether $\mathcal{M}(X)$ is indeed invariant or not is a potential problem. We will resolve this issue by proving stronger theorem in Section V (Theorem V.2).

Note that we did not define the *maximal positively invariant set*. The reason is because the maximal positively invariant set is always the whole space X, and consequently not worthy of new name. It makes sense to study the maximal positively invariant sets, however, if appropriate *localization* is done.

A positively invariant set acts like a whole space of a sub-dynamical system. Indeed, it is but a trivial observation to see that the iterative dynamical system given by the endomorphism $f : X \to X$ has a sub-dynamical-system $f|_Y : Y \to Y$ if and only if Y is positively invariant set. However, we must often consider the dynamics within a set of all *allowed states* $Y \subset X$, which is not positively invariant. In a flood-control system, for instance, it is necessary that the water-level must stay below the flood level all the time. In such a case, we must find the orbits that stay within Y all the time ($x_k \in Y$ for all $k = 0, 1, 2, \cdots$), and those that do not. The invariant set theory in this case must also reflect this adjustment, which we will call **localization**.

Definition II.3. *Let X be a non-empty set (phase space) and $f : X \to X$ be an endomorphism. Given non-empty subset $Y \subset X$, we define the* **locally maximal invariant subset** *$\mathcal{M}(Y)$ of $f|_Y$ as the union of all invariant subsets of Y. That is,*

$$\mathcal{M}(Y) = \bigcup \{S \subset Y : f(S) = S\}.$$

Similarly, we define **locally maximal positively invariant subset** *$\mathcal{M}^+(Y)$ of $f|_Y$ as the union of all positively invariant subsets of Y. That is,*

$$\mathcal{M}^+(Y) = \bigcup \{S \subset Y : f(S) \subset S\}.$$

Definition II.3 has the same problem as Definition II.2. It is obvious that $\mathcal{M}(Y)$ and $\mathcal{M}^+(Y)$ contain all the invariant subsets and all the positively invariant subsets of Y, respectively. It is not so obvious, however, whether $\mathcal{M}(Y)$ is invariant and $\mathcal{M}^+(Y)$ is positively invariant, respectively. Again, we will resolve this difficulty by proving stronger statement (Proposition VI.4).

The final topic of this section is to establish the equivalence between our definition of the locally maximal positively invariant set (Definition II.2) and another well known definition [14], [21].

Proposition II.4. *Let X be a non-empty set and $f : X \to X$ be an endomorphism. Let Y be a non-empty subset of X. Then, $x_0 \in \mathcal{M}^+(Y)$ if and only if $x_k = f^k(x_0) \in Y$ for all $k = 0, 1, 2, \cdots$.*

Proof. A stronger version of Proposition II.4 will be presented in Section VI (Theorem VI.5). □

III. Discrete Time Control Dynamical Systems with Disturbance

When there is disturbance, the control dynamical system is not time-invariant in general. The disturbance to a given state does not apply the same way all the time, or it is not a disturbance at all. Consequently, in the control dynamical system of form (II.1), neither

$$G : x_k \mapsto u_k, \qquad G : X \to U$$

nor

$$F : (x_k, u_k) \mapsto x_{k+1}, \qquad F : X \times U \to U$$

is well defined under the presence of the disturbance. That is, x_k does not always determine u_k uniquely, and also (x_k, u_k) does not determine x_{k+1} uniquely in general.

One way to overcome this difficulty is to bring in some extra independent variables to perturb the dynamics F and the feedback control G of the control system (II.1). As a result, we get

$$\begin{cases} F_w : (x_k, u_k, w_k) \mapsto x_{k+1}, \\ G_v : (x_k, v_k) \mapsto u_k, \end{cases} \tag{III.1}$$

where $u_k \in U$ and $x_k \in X$ are as in the time-invariant control system (II.1), while $v_k \in V$ and $w_k \in W$ are the **disturbance variables**. This is slightly more expansive approach than the traditional method considered in, say [20], [29], in that the feedback control map G is also disturbed. Due to the presence of the disturbance v_k and w_k, the system (III.1) is no longer time-invariant. It still remains sequential, however.

Another well known method is to regard all the maps and variables time-dependent. That is,

$$\begin{cases} F_t : (x(t), u(t), t) \mapsto x(t+1), \\ G_t : (x(t), t) \mapsto u(t). \end{cases} \tag{III.2}$$

See, for example, [26] for the approach similar to (III.2). Again, we followed slightly more expansive approach than the method considered in [26]. A part of the reason is because we wish to present more general form of the time-dependent control system. Another part is because we are not aiming to do any optimization or computation, and thus no simplification is necessary. The method given by (III.2) can be readily extended to the continuous-time dynamics, which results in the flow [2] in the phase space. We will not consider continuous-time dynamics in this paper, however.

In this paper, we choose somewhat different path. We will allow F, G, and consequently, f to be *multiple valued*, much like the approach discussed in [3]. The precise definition of the multiple valued map will be given in the next section (Definition IV.1).

One of the main advantages of this approach is that we can separate the dynamics part and the control part. The orbit of the multiple valued dynamical system gives us the set of all possible outcomes, for given initial state(s) and given iteration. Naturally, we must consider the dynamics of the sets rather than that of the states. The decision and control functions will change the states, but only within the image sets of the multiple valued dynamics.

IV. Multiple Valued Iterative Dynamical Systems

We begin this section with the precise definition of the multiple valued map.

Definition IV.1. *Let X, Y be non-empty sets and $\mathscr{P}(X)$, $\mathscr{P}(Y)$ be their power sets. We say a set function $f : \mathscr{P}(X) \to \mathscr{P}(Y)$ is a* **multiple valued map (function) from X to Y** *if*

$$f(S) = \bigcup \{f(x) : x \in S\}, \tag{IV.1}$$

for all $S \subset X$. In particular, if $X = Y$, we will call $f : \mathscr{P}(X) \to \mathscr{P}(X)$ that satisfies (IV.1), a **multiple valued endomorphism** *in X.*

Note that we used the abbreviated notation $f(x)$ instead of $f(\{x\})$ in the equality (IV.1). Indeed, it is customary to identify the singletons and points, when dealing with the set functions.

The idea of multiple valued map allows us to express a discrete-time control dynamical system with disturbance (III.1) to an iterative dynamics of a multiple valued endomorphism, $f : \mathscr{P}(X) \to \mathscr{P}(X)$ in X, as we did in (II.2) for a time-invariant system (II.1). Here, $f(x)$ represents the set of *all possible outcomes*, when the initial state is x. Let us elaborate this idea as the following proposition.

Proposition IV.2. *Let U, V, W and X be non-empty sets. Then the set function $f : \mathscr{P}(X) \to \mathscr{P}(X)$ given by*

$$f(S) = \{F(x, G(x, v), w) : v \in V, w \in W, x \in S\},$$

where $F : X \times U \times W \to X$ and $G : X \times V \to X$, is a multiple valued endomorphism in X.

Proof. Given $S \subset X$, we have

$$\begin{aligned} f(S) &= \{F(x, G(x, v), w) : x \in S, w \in W\} \\ &= \bigcup_{x \in S} \{F(x, G(x, v), w) : w \in W\} \\ &= \bigcup_{x \in S} f(x). \end{aligned}$$

Hence, the equality (IV.1) follows. □

Proposition IV.2 tells us that any control system given by (III.1) can be expressed as an iterative dynamics of a multiple valued endomorphism $f : \mathscr{P}(X) \to \mathscr{P}(X)$ in the phase space X, consequently making the latter more general way to express a control dynamical system with disturbance.

We can do the same to the disturbed control dynamical system modeled by (III.2).

Proposition IV.3. *Let T, U and X be non-empty sets. Then the set function $f : \mathscr{P}(X) \to \mathscr{P}(X)$ given by*

$$f(S) = \{F(x, G(x,t), t) : t \in T, x \in S\},$$

where $F : X \times U \times T \to X$ and $G : X \times T \to X$, is a multiple valued map from X to itself.

Proof. Given $S \subset X$, we have

$$\begin{aligned} f(S) &= \{F(x, G(x,t), t) : x \in S, t \in T\} \\ &= \bigcup_{x \in S} \{F(x, G(x,t), t) : t \in T\} \\ &= \bigcup_{x \in S} f(x). \end{aligned}$$

Hence, the equality (IV.1) follows as before. □

From Proposition IV.2 and Proposition IV.3, we conclude that the iterative dynamics of a multiple valued endomorphism $f : \mathscr{P}(X) \to \mathscr{P}(X)$ provides more general method to model the discrete-time control dynamical systems with disturbance. Let us call it, the **multiple valued iterative dynamical system (MVIDS)**.

Furthermore, note that the MVIDS is not *too general* for our purpose. Because each element of $f(x)$ is a possible outcome that we must account for, in a sense that each $y \in f(x)$ can be attained by a certain disturbance ($v \in V$ and $w \in W$, or the disturbance at a given time $t \in T$). For the rest of this paper, therefore, let us study the MVIDS as the model of the discrete-time control dynamical systems with disturbance.

V. The Maximal Invariant Sets of Multiple Valued Iterative Dynamical Systems

FROM this point on, we concentrate ourselves only upon the discrete-time control dynamical systems with disturbance, expressed as the multiple valued iterative dynamical systems (MVIDS). First, we define the invariant set and the maximal invariant set.

Definition V.1. *Let X be a non-empty set and $\mathscr{P}(X)$ be its power set. Let $f : \mathscr{P}(X) \to \mathscr{P}(X)$ be a multiple valued endomorphism in X. We say $S \subset X$ is* **invariant** *under f if $f(S) = S$. We define the* **maximal invariant set** *$\mathcal{M}(X)$ of $f : X \to X$ as the union of all invariant subsets of X. That is,*

$$\mathcal{M}(X) = \bigcup \{S \subset X : f(S) = S\}. \tag{V.1}$$

The following theorem supplements Definition V.1. It proves that $\mathcal{M}(X)$ defined in Definition V.1 is indeed *invariant* and *maximal*.

Theorem V.2. *Let X, $\mathscr{P}(X)$, f and $\mathcal{M}(X)$ be as in Definition V.1. Then, $\mathcal{M}(X)$ is the largest (in terms of the inclusion) invariant set under f.*

Proof. It is clear from the definition that $\mathcal{M}(X)$ contains all the invariant subsets of X. We need only to prove that $\mathcal{M}(X)$ is indeed invariant. That is,

$$f(\mathcal{M}(X)) = \mathcal{M}(X). \tag{V.2}$$

From IV.1, we must have

$$f(\mathcal{M}(X)) = \bigcup \{f(x) : x \in \mathcal{M}(X)\}.$$

Therefore, for all $y \in f(\mathcal{M}(X))$, there must be a certain $x \in \mathcal{M}(X)$ such that $y \in f(x)$. Now, from the equality (V.1), we get

$$x \in S, \quad f(S) = S,$$

for some $S \subset \mathcal{M}(X)$. Consequently,

$$y \in f(x) \subset f(S) = S \subset \mathcal{M}(X).$$

This holds for all $y \in f(\mathcal{M}(X))$. Hence,

$$f(\mathcal{M}(X)) \subset \mathcal{M}(X). \tag{V.3}$$

This proves a half of the set equality (V.2).

We now turn our attention to the other half of the equality (V.2). Choose any $x \in \mathcal{M}(X)$. Then, from the equality (V.1),

$$x \in S = f(S),$$

for some $S \subset X$. Now, from the equality (IV.1), $x \in f(z)$, for some $z \in S \subset \mathcal{M}(X)$. Consequently, $x \in f(\mathcal{M}(X))$. This holds for every $x \in \mathcal{M}(X)$. Hence,

$$\mathcal{M}(X) \subset f(\mathcal{M}(X)). \tag{V.4}$$

This proves the other half of the equality (V.2).

Combining the equalities (V.3) and (V.4), we get the invariance condition (V.2). □

Note that Theorem V.2 supplements Definition II.2 as well. Note also that Theorem V.2 justifies the *naive* definition of the maximal invariant set, *the invariant subset of X that includes all other invariant sets*, which appears in many engineering papers [10], [16], [21], [27], [29].

VI. The Locally Maximal Positively Invariant Sets of Multiple Valued Iterative Dynamics

IN the control theory, positively invariant sets are often more useful than the invariant sets, and consequently more widely studied. In a positively invariant set $S \subset X$, the iterative dynamics of the endomorphism $f : X \to X$ can be regarded as the whole space of a sub-dynamical-system given by the restricted map $f|_S : S \to S$. Let us establish the same with respect to the MVIDS.

Definition VI.1. *Let X, $\mathscr{P}(X)$ and f be as in Definition V.1. We say $S \subset X$ is* **positively invariant** *under the iteration of f if $f(S) \subset S$.*

In the viewpoint of the discrete-time control dynamical systems with disturbance, one can express the positive invariance given by Definition VI.1 as follows.

Remark VI.2. *S is positively invariant set of a discrete-time control dynamical system with disturbance if and only if* **every possible outcome** *of each state of S is contained in S.*

This approach is useful in *predictive control*, where one wished to ensure particular outcome(s) must be attained.

The locally maximal invariant sets and the locally maximal positively invariant sets of the MVIDS can be defined as follows.

Definition VI.3. *Let X, $\mathscr{P}(X)$ and f be as in Definition V.1. Given non-empty subset Y of X, we define the* **locally maximal invariant subset** *$\mathcal{M}(Y)$ of Y as the union of all invariant subsets of Y. That is,*

$$\mathcal{M}(Y) = \bigcup \{S \subset Y : f(S) = S\}. \qquad \text{(VI.1)}$$

Similarly, we define the **locally maximal positively invariant subset** *$\mathcal{M}^+(Y)$ of Y as the union of all positively invariant subset of Y. That is,*

$$\mathcal{M}^+(Y) = \bigcup \{S \subset Y : f(S) \subset S\}. \qquad \text{(VI.2)}$$

It is easy to see that $\mathcal{M}^+(Y) = Y$ if and only if $f(Y) \subset Y$. In particular, $\mathcal{M}^+(X) = X$. It is when Y is not positively invariant, the characterization of $\mathcal{M}^+(Y)$ gets complicated and meaningful.

The following proposition generalizes Theorem V.2 for $\mathcal{M}(Y)$ and $\mathcal{M}^+(Y)$, respectively.

Proposition VI.4. *Let X, $\mathscr{P}(X)$, f and Y be as in Definition II.3. Then, $\mathcal{M}(Y)$ is the largest invariant set contained in Y, and $\mathcal{M}^+(Y)$ is the largest positively invariant set contained in Y.*

Clearly, $\mathcal{M}(Y)$ and $\mathcal{M}^+(Y)$ include all the invariant subsets and all the positively invariant subsets of Y, respectively. One needs to prove the invariance and the positive invariance only.

The Proof of $f(\mathcal{M}(Y)) = \mathcal{M}(Y)$. This proof of analogous to that of Theorem V.2. We have only to make sure $x \in S \subset Y$. We leave the detail to the readers. ☐

The Proof of $f(\mathcal{M}^+(Y)) \subset \mathcal{M}^+(Y)$. This proof is also analogous to that of Theorem V.2. We need to prove only one direction, using slightly weaker condition. That is,

$$\begin{aligned} x \in \mathcal{M}^+(Y) &\Longrightarrow x \in S \subset Y, \;\; f(S) \subset S \subset \mathcal{M}^+(Y) \\ &\Longrightarrow \forall y \in f(x) \subset f(S) \subset S \subset \mathcal{M}^+(Y) \\ &\Longrightarrow f(x) \in \mathcal{M}^+(Y). \end{aligned}$$

Therefore, $\mathcal{M}^+(Y) \subset \mathcal{M}^+(Y)$. Note that the weakened condition, $f(S) \subset S$ was used in the second line, in place of $f(S) = S$ of the proof of Theorem V.2. ☐

The maximal positively invariant set can be characterized alternatively as follows.

Theorem VI.5. *Let X, $\mathscr{P}(X)$, f and Y be as in Definition II.3. Then, $x \in \mathcal{M}^+(Y)$ if and only if $f^k(x) \subset Y$ for all $k = 0, 1, 2, \cdots$.*

Proof. First, suppose that $x \in \mathcal{M}^+(Y)$. That is, there exists a certain $S \subset Y$ such that

$$x \in S \subset Y, \qquad f(S) \subset S. \qquad \text{(VI.3)}$$

Applying f to (VI.3), we get

$$f(x) \subset f(S) \subset S \subset Y. \qquad \text{(VI.4)}$$

In particular, $f(x) \subset S$. Applying f again, therefore, we get

$$f^2(x) \subset f(S) \subset S \subset Y. \qquad \text{(VI.5)}$$

Continuing this process, we get

$$\begin{aligned} f^3(x) &\subset f(S) \subset S \subset Y, \qquad \text{(VI.6)} \\ f^4(x) &\subset f(S) \subset S \subset Y, \\ \vdots \;\; &\subset \;\; \vdots \end{aligned}$$

Hence, from (VI.3) – (VI.6), we conclude that $f^k(x) \subset Y$ for all $k = 0, 1, 2, \cdots$.

Conversely, let $x \in Y$ such that $f^k(x) \subset Y$ for all $k = 0, 1, 2, \cdots$. Define the subset S of Y by

$$S = \bigcup_{k=0}^{\infty} f^k(x).$$

The elements of S consists of the elements of each $f^k(x)$. That is,

$$y \in S \Longleftrightarrow y \in f^k(x), \;\; \exists k \in \{0, 1, 2, \cdots\}.$$

Therefore, for each $y \in S$,

$$f(y) \in f(f^k(x)) = f^{k+1}(x) \subset S,$$

for some $k = 0, 1, 2, \cdots$. Therefore, $f(S) \subset S$. Consequently, $S \subset \mathcal{M}^+(Y)$. Hence, we must have

$$x \in f^0(x) \subset S \subset \mathcal{M}^+(Y).$$

That is, the converse is also true. ☐

Note that Theorem VI.5 is the multi-variable version of Proposition II.4, and that the latter follows immediately from the former as a corollary.

The definition of the positive invariant set and the maximal positively invariant set in Definition VI.3 are sometimes a little problematic for the multiple valued iterative dynamical systems. We can have completely different results if we use different notion of positive invariance. In this paper, we defined the positive invariant set of the multiple valued map through $f(S) \subset S$. That is, every state x in S *must* produce the output in S, as we emphasized in Remark VI.2. It is possible to consider an alternative notion of the positive invariance. That is, every state $x \in S$ *can* yield the outcome in S. That is, $f(S) \cap S \neq \emptyset$. In this paper, we consider only the first case, primarily because it is easier. We will come back to this discussion in Section IX.

VII. The Reachability Problem of the Maximal and the Locally Maximal Invariant Sets

As the applications of the MVIDS of the discrete-time control dynamical system with disturbance, we study two types of reachability problems. These reachability problems are slightly different from the traditional reachablity problems. Instead of looking for the set of reachable states from a given set of initial states, or searching for the set of initial states that produce the desired final states, we study the conditions to design the control dynamical systems that the maximal, the locally maximal, and/or the locally maximal positively invariant sets are reachable, possibly in infinite steps.

In this section, we provide a survey of the reachability problems of the maximal and the locally maximal invariant sets. The reachability problems of the locally maximal positively invariant sets will be discussed in the next section (Section VIII).

The answers to the infinite-step reachability problems of the maximal and the locally maximal invariant sets of a single valued iterative dynamical system are known, if the phase space X is compact and the map $f : X \to X$ is continuous. First, let us state the classical result about the maximal invariance.

Theorem VII.1. *Let X be a non-empty compact topological space and $f : X \to X$ be a continuous endomorphism. Then,*

$$\mathcal{M}(X) = \bigcap_{k=0}^{\infty} f^k(X). \tag{VII.1}$$

Proof. See [1]. See, also, [13], [19], [24]. □

The following theorem extends Theorem VII.1.

Theorem VII.2. *Let X be a non-empty topological space and let $f : X \to X$ be an endomorphism. Suppose that Y is a non-empty compact subspace of X and $f|_Y : Y \to X$ is continuous. Then,*

$$\mathcal{M}(Y) = \bigcap_{k=0}^{\infty} f^k(Y). \tag{VII.2}$$

Proof. See [19]. □

Theorem VII.1 and Theorem VII.2 provide the theoretical background of the finite-step approximate control problems and their optimization, for the maximal, and the locally maximal invariant sets. Because $f(X) \subset X$, the equality (VII.1) can be rewritten in the following form,

$$X \supset f(X) \supset f^2(X) \supset f^3(X) \supset \cdots \to \mathcal{M}(X), \tag{VII.3}$$

which allows us to consider the finite-step approximation problem,

$$f^N(X) \approx \lim_{n\to\infty} f^n(X) = \mathcal{M}(X). \tag{VII.4}$$

Similarly, we can re-express (VII.2) as,

$$Y^0 \supset Y^1 \supset Y^2 \supset Y^3 \supset \cdots \to \mathcal{M}(Y), \tag{VII.5}$$

where

$$Y^0 = Y, \quad Y^k = Y^{k-1} \cap f(Y^{k-1}). \tag{VII.6}$$

Consequently, we get the following N-step approximation problem.

$$Y^N \approx \lim_{n\to\infty} Y^n = \mathcal{M}(X). \tag{VII.7}$$

The second approximate control problem (VII.5) – (VII.7) and its variations are important research problems that are being actively studied nowadays. See, for instance, [16], [20], [27], [28], [29] and the references therein.

If we relax the continuity condition, however, the reachability problems of the maximal, and the locally maximal invariant sets get extremely complicated. It was discovered first in [24] that the equality (VII.1) does not hold in general if the dynamics is allowed to be discontinuous. Later, it was discovered that it can take uncountable ordinal number of iterations to reach the maximal invariant set, if the dynamics is almost continuous (but discontinuous nonetheless) [17].

Theorem VII.3. *Let X be a non-empty set and $f : X \to X$ be an endomorphism. Then, there exists a unique ordinal number ξ such that*

$$X_0^+ \supsetneq X_1^+ \supsetneq \cdots \supsetneq X_\xi^+ = \mathcal{M}(X),$$

where

$$\begin{cases} X_0^+ = X, & \\ X_\xi^+ = \bigcap_{k=0}^{\infty} f^k(X_{\xi-1}^+), & \text{if } \xi \text{ is a successor ordinal}, \\ X_\xi^+ = \bigcap_{\eta<\xi} X_\eta^+, & \text{if } \xi \text{ is a limit ordinal}. \end{cases}$$

We call the ordinal number ξ, the **maximal invariance order** *of $f : X \to X$.*

Proof. See [17]. □

Theorem VII.4. *Given ordinal number ξ, we can construct a compact metric space X and a piecewise continuous endomorphism $f : X \to X$ with the maximal invariance order ξ, which is almost continuous with respect to every Baire measure on X.*

Proof. See [17]. □

Theorem VII.3 tells us that the reachability problem of any maximal invariant set is *well posed*, provided that we are allowed to take trans-finite steps. Theorem VII.4, on the other hand, tells us that the reachability problem of discontinuous dynamics can be made to *as complicated as* one wishes to. These theorems effectively block out any attempt to expand the reachablity problems of the maximal and the locally maximal invariant sets, by asserting that even the simpler, single valued cases cannot be resolved in countable steps of iterations.

VIII. The Reachability Problem of the Locally Maximal Positively Invariant Sets

In this section, we study the reachability problem of the locally maximal positively invariant sets of the MVIDS. This problem is notably simpler than that of the maximal and the locally maximal invariant sets, at least for the single valued iterative dynamical systems. It is known that any locally maximal positively invariant set $\mathcal{M}^+(Y)$ is reachable from Y, possibly in infinite steps, if the dynamics is single valued. This result does not depend upon the topological structure of X and/or f, in contrast to Theorem VII.1 and Theorem VII.2. See, for example, [19]. See also, [10] for essentially the same results stated in terms of control dynamical systems of form (II.1).

In this paper, we establish an analogous result for the multiple valued iterative dynamical systems.

Theorem VIII.1. *Let X, $\mathscr{P}(X)$, $f : \mathscr{P}(X) \to \mathscr{P}(X)$ and $Y \subset X$ be as in Definition VI.3. Then,*

$$\mathcal{M}^+(Y) = \bigcap_{k=0}^{\infty} f^{-k}(Y), \tag{VIII.1}$$

where

$$f^{-k}(Y) = \bigcup \{x \in X : f^k(x) \subset Y\}. \tag{VIII.2}$$

Before proving Theorem VIII.1, let us examine its importance in control theory first. We can use the argument similar to (VII.5) – (VII.7) to re-express Theorem VIII.1 in the following fashion.

$$Y^0 \supset Y^{-1} \supset Y^{-2} \supset Y^{-3} \supset \cdots \to \mathcal{M}^+(Y), \tag{VIII.3}$$

where

$$Y^0 = Y, \quad Y^{-k} = Y^{-(k-1)} \cap f^{-1}(Y^{-(k-1)}), \tag{VIII.4}$$

for $k = 1, 2, 3, \cdots$. Consequently, we get the N-step approximation problem,

$$Y^{-N} \approx \lim_{n\to\infty} Y^{-n} = \mathcal{M}^+(Y). \tag{VIII.5}$$

Hence, we conclude that Theorem VIII.1 provides the theoretical background behind the finite-step approximate control problems (VIII.3) – (VIII.5) and their optimization, for the locally maximal positively invariant sets of the discrete-time control dynamical systems with disturbance.

Proof of Theorem VIII.1. First, we show

$$\mathcal{M}^+(Y) \subset \bigcap_{k=0}^{\infty} f^{-k}(Y). \tag{VIII.6}$$

Choose any $y \in \mathcal{M}^+(Y)$. That is, $y \in S$ for some $S \subset Y$ such that

$$f(S) \subset S \subset Y. \tag{VIII.7}$$

Taking the pre-image of f, we get

$$S \subset f^{-1}(S) \subset f^{-1}(Y).$$

Hence, from (VIII.7), we get

$$f(S) \subset S \subset f^{-1}(Y).$$

Taking the pre-image again, we get

$$S \subset f^{-1}(S) \subset f^{-2}(Y).$$

Therefore, from (VIII.7) again, we get

$$f(S) \subset S \subset f^{-2}(Y).$$

Repeating the same argument, we get

$$S \subset f^{-k}(Y), \qquad k = 1, 2, 3, \cdots.$$

Consequently,

$$y \in S \subset \bigcap_{k=0}^{\infty} f^{-k}(Y).$$

This holds for all $y \in \mathcal{M}^+(Y)$. Hence, the set inequality (VIII.6) follows.

Now, lets us prove the other direction,

$$\bigcap_{k=0}^{\infty} f^{-k}(Y) \subset \mathcal{M}^+(Y). \tag{VIII.8}$$

It is sufficient to prove that the former set is positively invariant, because the latter set includes all positively invariant subsets of Y. Choose any element x such that

$$x \in \bigcap_{k=0}^{\infty} f^{-k}(Y). \tag{VIII.9}$$

Then,

$$f^k(x) \subset Y, \qquad k = 0, 1, 2, \cdots,$$

and thus,

$$f^{k-1}(f(x)) \subset Y, \qquad k = 1, 2, 3, \cdots.$$

Consequently, we get

$$f(x) \subset f^{-k}(Y), \qquad k = 0, 1, 2, \cdots.$$

This holds for every x that satisfies (VIII.9). Hence,

$$f\left(\bigcap_{k=0}^{\infty} f^{-k}(Y)\right) \subset \bigcap_{k=0}^{\infty} f^{-k}(Y).$$

This proves (VIII.8). □

IX. Conclusion and Discussion

THE main purpose of this paper was to describe how we can reduce the invariant set theory of the control dynamical systems with disturbance to that of the multiple valued iterative dynamical systems (MVIDS). This was done in Section II - Section VI. As an application, we studied the reachability problem of the locally maximal positively invariant sets in Section VIII.

The reachability problem we studied in the last section is still an on-going and constantly changing topic. The biggest problem to the definitions we selected is that it is difficult to track down the previous state that does not always yield the desired outcome. That is, if we take the definition of the pre-image (predecessor) set as

$$f^{-1}(S) = \{x \in X : f(x) \cap S \neq \emptyset\}, \tag{IX.1}$$

instead of

$$f^{-1}(S) = \{x \in X : f(x) \subset S\}, \tag{IX.2}$$

as we did in Theorem VIII.1, the outcome of Theorem VIII.1 can become different.

The pre-image set defined by the equality (IX.1) represents the set of all states x that *may* cause a certain output $y \in S$ ($y \in f(x) \cap S$). The pre-image set given by the equality (IX.2), on the other hand, stands for the set of all states x that *must* yield the output in S ($y \in f(x) \subset S$). It is somewhat unclear, at this moment, how to deal with the reachability problems for the pre-image operation (predecessor operation), given by the equality (IX.1). We conjecture, though, it is closely related to the selection of the positive invariance, which we discussed at the end of Section VI. We conclude this paper leaving this as a challenge for the future research of the invariant set theory for the control dynamical systems with disturbance.

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