Resonant Transmission Line Modeling Of Holey Photonic Crystal Fibers

E. Georgantzos, T. E. Raptis, A. C. Boucouvalas, C. D. Papageorgiou

Abstract—A holey optical fiber with hexagonal modulation of the refractive index is studied with the aid of the Resonant Transmission Line (RTL) method. We show how to obtain an average approximation of the radial refractive index profile that helps to simplify the final lay out of the transmission line model for all three spatial components of the electric and magnetic fields respectively.

Keywords—holey optical fiber, EM modes, transmission lines.

I. Introduction

THE field of fiber optics is a rapidly expanding sector in both academia and industry with a growing body of applications. Starting with the pioneering work of Kapany [1] who first coined the term, as well as Kao [2], [3] on the single-mode fibers during the 20th century there is now a great variety of different realizations for both single-mode and multi-mode fibers. Apart from fiber optics wide applicability in communication technologies, there is also a decades long attempt to use this particular technology in another promising area which is that of tabletop laser based accelerators using the effect of plasma “wakefield” also called Laser Wakefield Accelerators (LWA) [4-7].

While the present technology allows at least three types of possible photonic crystal fibers classified as band-gap fibers, Bragg fibers and holey fibers, the particular application in LWAs requires multi-mode fibers with a hole replacing the usual inner metal core in order to be able to accommodate an amount of injected plasma which is then excited by a laser source. This method presents certain experimental challenges including computational problems especially in the area of very strong fields that can cause severe deviations from linearity due to the inherent dependence of the plasma refractive index to higher orders of the propagating electric field intensity.

In order to simplify the calculation of the propagating modes as well as to facilitate the extraction of so called “leaky” modes from surface modes known as “cladding” modes around the interior core we use the previously introduced Resonant Transmission Line method (RTL) [8-10] which presents certain appealing characteristics due to its simplicity and speed in comparison with other more complex methodologies like FDTD and FEM methods. In section 2, we provide details for the general equations of the model for arbitrary refractive index. In section 3, we analyze the particulars of the holey fiber with layered hexagonal grid of holes. In section 4, we present the details for obtaining reduced field equations in the RTL method and in section 5, the obtained average refractive index is presented in the truncated model.

II. Truncation Of Maxwell Equations

We start from a conventional representation of a common circular cylindrical optical fiber separated in hexagonal layers with a set of an increasing number of air holes per layer. For each circular cylindrical layer the associated Maxwell equations (for a constant wavelength i.e. constant frequency ‘ω’) can be written as

$$\begin{cases} \nabla X\vec{E} = -j\omega\mu_0\vec{H} \\ \nabla X\vec{H} = j\omega\varepsilon_0 n(\varphi)^2\vec{E} \end{cases} \quad (1)$$

We assume standard dispersion characteristics as $\omega\mu_0 = k_0 z_0$ and $\omega\varepsilon_0 = \frac{k_0}{z_0}$ where $k_0 = \frac{\omega}{c}$ and $z_0 = 120\pi$. We can then use the replacement $z_0\vec{H} \rightarrow \vec{H}$. We normalize (1) to have the same *MKSA* units of *V*/*m* as

$$\begin{cases} \nabla X\vec{E} = -jk_0\vec{H} \\ \nabla X\vec{H} = jk_0 n(\varphi)^2\vec{E} \end{cases} \quad (2)$$

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Standard variable separation in cylindrical coordinates immediately obtains from the first vector Maxwell equation the following set of three partial differential equations (PDE) as

$$\begin{cases} \frac{1}{r}\frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z} = -jk_0 H_r \\ \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -jk_0 H_\varphi \\ \frac{1}{r}\frac{\partial (rE_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial \varphi} = -jk_0 H_z \end{cases} \quad (3)$$

We are interested in the radial field distribution so we take a full Fourier transform along the z and φ directions with their associated wave numbers given as β and l respectively. Across the angular direction all modes are taken (*mod* 2π) so that l is integer.

$$\begin{cases} \frac{jl}{r}\overline{E_z} - j\beta\overline{E_\varphi} = -jk_0\overline{H_r} \\ j\beta\overline{E_r} - \frac{\partial \overline{E_z}}{\partial r} = -jk_0\overline{H_\varphi} \\ \frac{1}{r}\frac{\partial (r\overline{E_\varphi})}{\partial r} - \frac{jl}{r}\overline{E_r} = -jk_0\overline{H_z} \end{cases} \quad (4)$$

In (5), we introduced $\overline{E_r}, \overline{E_\varphi}, \overline{E_z}, \overline{H_r}, \overline{H_\varphi}, \overline{H_z}$ for the associated Fourier components. We can now replace β and r with their reduced components as $\frac{\beta}{k_0} \rightarrow \beta$ and $rk_0 \rightarrow r$ so that (4) takes the form

$$\begin{cases} \frac{jl}{r}\overline{E_z} - j\beta\overline{E_\varphi} = -j\overline{H_r} \\ j\beta\overline{E_r} - \frac{\partial \overline{E_z}}{\partial r} = -j\overline{H_\varphi} \\ \frac{1}{r}\frac{\partial (r\overline{E_\varphi})}{\partial r} - \frac{jl}{r}\overline{E_r} = -j\overline{H_z} \end{cases} \quad (5)$$

With exactly the same procedure we also obtain from the second vector Maxwell equation the complementary PDE system

$$\begin{cases} \frac{jl}{r}\overline{H_z} - j\beta\overline{H_\varphi} = jn(r,l)^2 \otimes \overline{E_r} \\ j\beta\overline{H_r} - \frac{\partial \overline{H_z}}{\partial r} = jn(r,l)^2 \otimes \overline{E_\varphi} \\ \frac{1}{r}\frac{\partial (r\overline{H_\varphi})}{\partial r} - \frac{jl}{r}\overline{H_r} = jn(r,l)^2 \otimes \overline{E_z} \end{cases} \quad (6)$$

In (6), the symbol $\otimes$ is used to denote the conventional convolution operator after Fourier decomposition with the arbitrary function $n(r,l)$ being the Fourier transform of the original refractive index $n(r,\varphi)$. In the next section we describe an approximate scheme for analyzing the influence of the particular type of refractive index composed of a superposition of square steps as a cut-off filter effect for the resulting harmonics.

III. REFRACTIVE INDEX MODULATION AS FILTERING

In the particular case of a hexagonal holey fiber, one can analyze mode propagation, as a succession of thin cylindrical layers. To this aim, we separate the whole fiber circular cross-section into a set of thin cylindrical layers allowed to extend beyond the cladding to take into account the surrounding air with n_0=1. Each layer's thickness δr = $r_1 + r_2$ is defined as

$$\frac{r_2 - r_1}{r_2 + r_1} = \frac{c}{2} \Rightarrow \frac{r_2}{r_1} = \frac{2+c}{2-c} \quad (7)$$

We can then approximate $n(r,\varphi)$ as $n(\varphi)$ for each $\langle r \rangle = \delta r/2$. One can write a series approximation for the refractive index as $n(\varphi)^2 = \langle n \rangle^2 + \sum_{-\infty}^{+\infty} N_k \exp(jl\varphi)$. Taking into account the symmetry of the Fourier transform we see that $\exp(jl\varphi) \otimes f(l) = \exp(jl\varphi) \otimes f(l+l')$ so that the expressions in the *rhs* of (6) spread around a spectrum of harmonics.

This is also to be understood as a result of successive scatterings from the bored air holes. We can now use the natural geometry of the hexagonal lattice to see that for each set of holes we can have either $6k$ or $6(k+1)$ harmonics. As each thin cylinder crosses this alternating numbers it sees a different set of periodic rectangle functions that will be shown rigorously to contribute a different number of harmonics. For a common harmonic to pass through one must then take an integer product which leads to higher and higher harmonics thus cutting out the entire spectrum apart from the last highest frequency.

We conclude that for holey optical fibers the approximation $n(r,l)^2 \approx n^2(r)$ suffices for further analysis of the resulting equations. The original system (6) becomes

$$\begin{cases} \frac{jl}{r}\overline{H_z} - j\beta\overline{H_\varphi} = jn^2(r)\overline{E_r} \\ j\beta\overline{H_r} - \frac{\partial \overline{H_z}}{\partial r} = jn^2\overline{(r)E_\varphi} \\ \frac{1}{r}\frac{\partial (r\overline{H_\varphi})}{\partial r} - \frac{jl}{r}\overline{H_r} = jn^2\overline{(r)\ E_z} \end{cases} \quad (7)$$

For a hexagonal pattern of holes we may utilize elementary analytical geometry to derive the two separate regions where the refractive index alternates between the vacuum value and the slightly higher value of the crystal material. We assume that along each separate layer a large circle corresponding to each cylindrical shell of radius r from the center of the fiber to the center of a smaller hole of radius $r \ll r0$ is cut while moving clockwise along the large circle.

We prescribe a set of circles of successive radii *r*. For each small radius *r* we can find the air holes (in 1/6 angle of the PCF) which are cut by the r and we can calculate its arcs inside the air holes. The sum of these arcs divided by $\pi/3$ can give the average squared refractive index. In fact the square of the refractive index in this sum is equal to one, while the refractive index in the rest arc is the square of the silica refractive index. Hence the average refractive index can be easily calculated along r. In figures 1-4, the average refractive indices of a hexagonal PCF of six layers as functions of the fraction r/d are shown. The figures were generated by the MATLAB code for various values of the fraction r0/d=0.49,0.40,0.30 and 0.20. Codes are given in Appendix A. We will proceed with the exact analysis of these in terms of the previously introduced general model of Resonant Transmission Lines (RTL) for one dimensional Sturm-Liouville problems.

IV. TRANSMISSION LINE REDUCTION METHOD

In order to utilize RTL, we need to find appropriate correspondence of the equations in (7) with a particular version of the Telegrapher's equation for an ideal tuned transmission line (without ohmic losses) via the introduction of an equivalent set of potentials and currents. For this reason we take as defining relations the following equations

$$V_M = \frac{l\overline{H_\varphi} + \beta r\overline{H_z}}{jF}$$
$$I_M = \frac{r\overline{H_r}}{j} = \frac{\beta r\overline{E_\varphi} - l\overline{E_z}}{j}$$
$$V_E = \frac{l\overline{E_\varphi} + \beta r\overline{E_z}}{F}$$
$$I_E = n^2 r\overline{E_r} = l\overline{H_z} - \beta r\overline{H_\varphi}$$

In (8), we defined a new auxiliary function as$F(r) = \frac{(\beta r)^2 + l^2}{r}$. The Fourier transforms of the original field components can also be given from the inverse of (8) as

$$\overline{H_r} = jI_M/r, \qquad \overline{E_r} = \frac{I_E}{n^2 r}$$
$$\overline{H_\varphi} = jlV_M - \frac{\beta r^2}{F} I_E$$
$$\overline{H_\varphi} = lV_E + j\frac{\beta r^2}{F} I_M$$
$$\overline{H_z} = \frac{l}{F} I_E + j\beta V_M$$
$$\overline{E_z} = -j\frac{l}{F} I_M + \beta V_E$$

The original equations take the equivalent form

$$\begin{cases} \frac{\partial V_M}{\partial r} = -\frac{\gamma^2}{jF} I_M - jMI_E \\ \frac{\partial I_M}{\partial r} = -jFV_M \\ \frac{\partial V_E}{\partial r} = -\frac{\gamma^2}{jn^2F} I_E - jMI_M \\ \frac{\partial I_E}{\partial r} = -jn^2FV_E \end{cases} \tag{10}$$

In (10), we have again defined the new auxiliary functions $\gamma(r)^2 = \frac{l^2}{r^2} + \beta^2 - n^2$ as well as $(r) = \frac{2l\beta}{[(\beta r)^2 + l^2]F} = \frac{2l\beta r}{[(\beta r)^2 + l^2]^2}$. Regarding boundary conditions, we also notice that because of the continuity of $\overrightarrow{H_\varphi}\ \overrightarrow{H_z}$ and $\overrightarrow{E_\varphi}\ \overrightarrow{E_z}$ on the cylindrical surface of any layer, the same is also true for the new variables V_M, I_M, V_E, I_E. The resulting system of coupled ordinary differential equations (ODEs) is equivalent with a pair of coupled ideal lossless transmission lines.

The system in (10) can be turned into an eigenvalue problem with the ensatz of exponential dependence of all variables as

$$V_M = V_{M0}e^{\xi r}, I_{M0} = I_{M0}e^{\xi r}, V_E = V_{E0}e^{\xi r}, I_E = I_{E0}e^{\xi r}$$

where the zero indices denote constants, in which case we may transform (10) into an algebraic system as

$$\begin{cases} \xi V_M = -\frac{\gamma^2}{jF} I_M - jMI_E \\ \xi I_M = -jFV_M \\ \xi V_E = -\frac{\gamma^2}{jn^2F} I_E - jMI_M \\ \xi I_E = -jn^2FV_E \end{cases} \tag{11}$$

Replacing $I_M = -\frac{jF}{\xi} V_M$, $I_E = -\frac{jn^2F}{\xi} V_E$, we obtain a set of two homogeneous equations

$$\begin{cases} \xi^2 V_M = \gamma^2 V_M - n^2 MFV_E \\ \xi^2 V_E = \gamma^2 V_E - MF \end{cases} \tag{12}$$

This is equivalent to the condition

$$Det\begin{pmatrix} \xi^2 - \gamma^2 & n^2 MF \\ MF & \xi^2 - \gamma^2 \end{pmatrix} = 0$$

(13)

We immediately obtain the eigenvalues for the original problem in (10) as $\xi^2 = \gamma^2 \pm nMF$. Hence the system has two eigenvalues and two mutually excluded or "normal" eigenvectors. Replacing the value of ξ in (12) gives the eigenvector elements as $V_M = nV_E$, $V_S = V_M + nV_E$ for the negative branch and $V_M = -nV_E$ $V_d = V_M - nV_E$ for the positive branch. Their respective "current" eigenvectors are then given by $\frac{I_M}{I_E} = \frac{V_M}{n^2 V_E} = \frac{1}{n}$, $I_M = \frac{I_E}{n}$ and $I_s = I_{M+}\frac{I_E}{n}$, $I_d = I_{M-}\frac{I_E}{n}$.

We notice that $M(r)$ is linear in l and exactly at $-l$ we have $(V_s, I_s) = (V_d, I_d)$. Thus we can consider as a unique solution, the set (V_s, I_s) with the integer 'l ' varying from $-\infty$ to $+\infty$, the first decoupled equation from (10) being

$$\begin{cases} \frac{\partial V_s}{\partial r} = -\frac{\xi^2}{jF} I_s \\ \frac{\partial I_s}{\partial r} = -jFI_s \end{cases} \tag{14}$$

Furthermore V_s, I_s are continuous functions at their boundaries although n(r) is varying from layer to layer because exactly at the boundaries we have $V_s = 2V_M$ and $I_s = 2I_M$. We then take the second decoupled equations as

$$\begin{cases} \frac{\partial V_{ss}}{\partial r} = -\frac{\xi^2}{jn^2F} I_{ss} \\ \frac{\partial I_{ss}}{\partial r} = -jFn^2 I_{ss} \end{cases} \tag{15}$$

Thus the set of two originally coupled transmission lines (10) is equivalent to two independent transmission lines (14) and (15). These two independent wave equations represent separately the two normal (***E.H*** = 0) electric and magnetic components. This is an inherent demand of optical fiber propagation due to birefringence effects.

From standard transmission line theory, we know that we can approximate each thin cylindrical layer with an equivalent T-quadrupole circuit having the same dynamics as that described by (14) or (15).

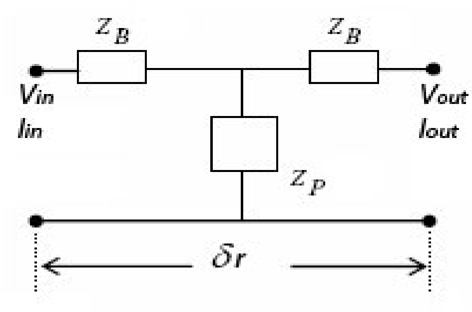

Fig. 1, Equivalent T-quarupole circuit for a thin cylindrical layer.

With recourse to figure 1, for the pair of (V_s, I_s) we have the defining relations

$$\begin{cases} Z_B = \frac{s}{jF}\tanh\left[{}^{(\xi\delta r)}/_2\right] \\ Z_p = \frac{\xi}{jF\sinh(\xi\delta r)} \end{cases}$$

For $\xi\Delta r \ll 1$ the impedances can be approximated to match (11) as

$$\begin{cases} Z_B = \frac{\xi^2\left({}^{\delta r}/_2\right)}{jF} \\ Z_p = \frac{1}{jF\delta r} \end{cases} \tag{16}$$

If $\xi^2 > 0$, both Z_B, Z_p are "capacitive" reactances, however for $\xi^2 < 0$, Z_B becomes "inductive". By the same token, For (V_{ss}, I_{ss}) the approximate respective impedances of the T-circuit are given by (12) as

$$\begin{cases} Z'_B = \frac{\xi^2\left({}^{\delta r}/_2\right)}{jn^2F} \\ Z'_p = \frac{1}{jn^2F\delta r} \end{cases} \tag{17}$$

We may now consider the succession of all cylindrical layers as a lumped circuit formed by a succession of equivalent T-circuits with only reactive elements. For given 'l', the 'β'' values that lead to the resonance of the overall transmission line are the eigenvalues of the whole optical fiber. When a transmission line is in resonance at any point r_0 of the line, the sum of reactive impedances arising from the successive T-circuits on the left and right sides of r_0 should be equal to zero. Hence, the overall resonance condition is given as

$$\overleftarrow{Z}_{L,r_0} + \overleftarrow{Z}_{R,r_0} = 0 \tag{18}$$

Equation (18) can be used for building a fast RTL numerical method for finding the β propagation eigenvalues, with $\overleftarrow{Z}_{L,r_0}$, $\overleftarrow{Z}_{R,r_0}$ being the overall reactive impedances of successive T-circuits on the left and right of r_0, for either (14) or (15). Each of them gives in general a different set of values which may only come close to each other in cases of strong birefringence. The latter case does not pose a problem for perfect cylindrical symmetry where this effect is considered negligible. We also notice that relations $I_M = \pm\frac{I_E}{n}, V_M = \pm nV_E$ result in $\bar{H}_r = jn\bar{E}_r$ and similarly for the other field components so that we always get $z_0\vec{H} = \pm jn\vec{E}$ the proportionality being a generic characteristic of all optical fibers.
In order to calculate the overall reactive impedances on the left and right of r_0 we have find the impedances for $r_0 \to 0$ and $r_0 \to \infty$. As we proceed to 0 or to ∞ the remaining piece of transmission line becomes "homogeneous" i.e. its overall reactive impedance is equal to its characteristic impedance given as $= \frac{\xi}{jF}\left(or\ \frac{\xi}{jn^2F}\right)$. For $r \to \infty\ F \to \beta^2 r, MF \to 0\ and\ \xi \to \sqrt{\beta^2 - n^2}$ thus $Z_{r\to\infty} = 0$. Similarly, for $r \to 0\ F \to \frac{l^2}{r}\ and\ \xi \to \frac{l}{r}$ thus $Z_{r\to 0} = \frac{1}{jl}\left(or\ \frac{1}{jn^2 l}\right)$. For *l=0* $Z_{r\to 0} = \infty$ (open circuit at the center of the equivalent transmission line).
It is useful to notice that in general we have the following equivalence between our formulation and the classic formulation for modes of optical fibers.

1. For *l=0* the modes (V_S, I_S) are the TM modes ,while the modes (V_{SS}, I_{SS}) are the TE modes.
2. For *l>0* the modes (V_S, I_S) are the HE modes ,while the modes (V_{SS}, I_{SS}) are their HE birefringence modes.
3. For *l<0* the modes (V_S, I_S) are the EH modes ,while the modes (V_{SS}, I_{SS}) are their EH birefringence modes.

V. Conclusions

In the paper it was shown the analysis by which the holey fibers can be approximated by a set of two, independent and non homogeneous, Resonant Transmission Lines (RTLs), each one representing one mode of the birefringence. The simulation of the holey fibers with RTLs, gives a new simple and effective method for the calculation of the eigenvalues of the RTLs representing the various modes of the holey fibers. Furthermore for each eigenvalue, the average values of E.M. fields for every thin cylinder (of radius r) of the holey fiber can be calculated by the related eigenfunctions of the RTLs.

APPENDIX

Average refractive index calculation method

```
function ref=eur_refp(r)
%PCF hexagonal
global n1 d r0 R m rt ff crin

% d=distance of the centers of the
hexagonal lattice % m=number of lattice
rows of air holes (m=6 in the figure
below)
% ro=radius of air holes r0<0.5*d
% R=external radius of the fiber gladding
R> m*d
% If not given R=1.2*m*R
if r<=d-r0;
  ref=n1;
elseif r>d-r0 && r<m*d+r0;
 for nn=1:m ;
 for
n=1:nn;rr(nn,n)=nn*d*exp(j*2*pi/3)+(n-
1)*d;
         rt(nn,n)=abs(rr(nn,n));
 end
 end
 f=0;
 for nn=1:m;
 for n=1:nn;rrr=rt(nn,n);drt=abs(rrr-r);
 if drt<r0; ff=2*acos((r^2+rrr^2-
r0^2)/2/r/rrr);f=ff+f;end
 end
 end
 ref=sqrt((f+(pi/3-f)*n1^2)/(pi/3));
else ref=n1;
end
% if r<rin; ref=1;end
if r>R;ref=1;
End
```

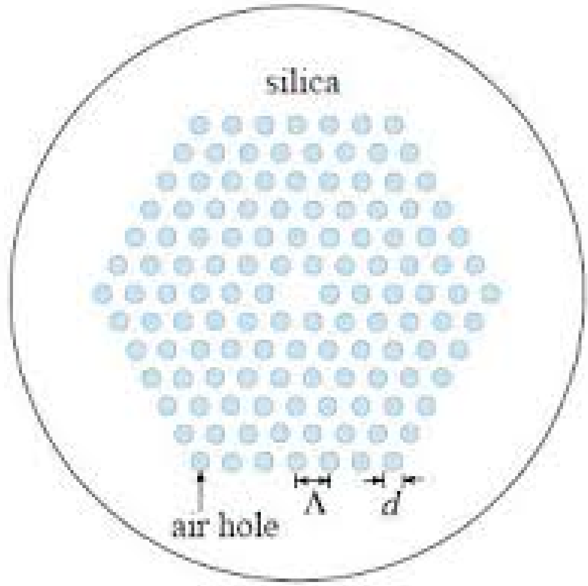

Fig. 2 Holey Fiber of m=6

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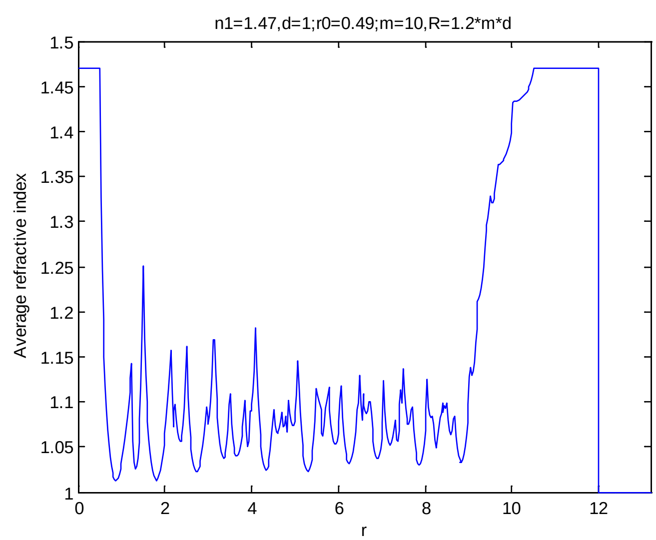

Fig. 3 Equivalent average refractive index for $r_0/d=0.49$ and m=10

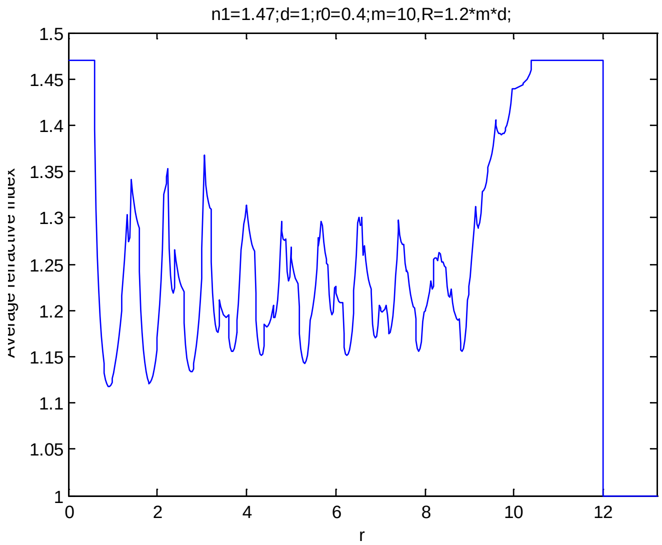

Fig. 4 Equivalent average refractive index for $r_0/d=0.40$ and m=10

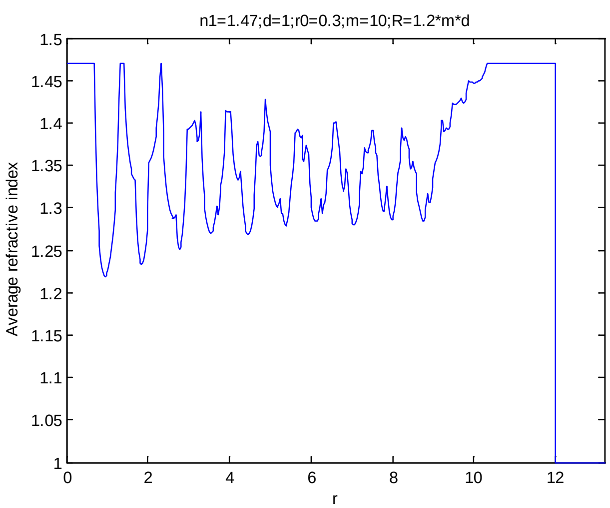

Fig. 5 Equivalent average refractive index for $r_0/d=0.30$ and m=10

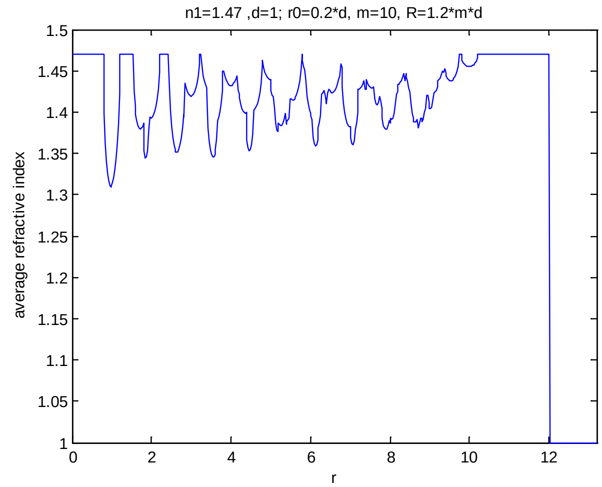

Fig. 6 Equivalent average refractive index for $r_0/d=0.20$ and m=10