Sampling - Reconstruction Algorithms of Gaussian Narrow- Band Process Realizations with an Arbitrary Number of Samples

V. Kazakov[1], F. Mendoza[2]

Abstract**—**In contrast to the well-known method of describing the sampling-recovery algorithm (SRA) of narrow-band processes, the method based on the conditional mean rule is used. This method provides the possibility of obtaining the optimal SRA when the number and location of samples are arbitrary. The basic functions (non-*SINC* functions) and the error recovery functions of the SRA realizations of some narrow-band Gaussian processes are investigated.

Keywords**—**Narrow-Band Gaussian Process, Sampling Reconstruction Algorithm.

I. Introduction

There is the well-known Balakrishnan theorem (BT) [1], which is devoted to the description of the sampling-recovery algorithm (SRA) of random realizations of a stationary process with a limited spectrum. The sampling process is fully described by one parameter: its frequency limit ω_b. The sampling interval is periodic and is determined by the Nyquist formula $\Delta T = \pi/\omega_b$. BT has some drawbacks: information about the probability density function (pdf) of the process is missing, both its covariance function $K(\tau)$ and the power spectrum $S(\omega)$ are not determined, the number of samples N is infinite, the basic recovery function SINC is the same for all types of processes and the recovery error is zero.

The situation with the description of a narrowband process is almost the same as in BT: the process being sampled is determined by the center frequency ω_0 and the bandwidth of the spectrum $\Delta\omega = \omega_2 - \omega_1$ (where the frequencies ω_2, ω_1 are limited). There are no other statistical characteristics. The sampling procedure is constructed in a special way: the SRA is performed on the two low-frequency components that define the envelope of a sampled process. The first low-frequency component is given, and the second is determined by the Hilbert transform of the first component. It is necessary to sample both components at the same time. The sampling interval is determined by the high-frequency bandwidth of the envelope. In other words, there is no sampling of the realization of the narrow-band process itself. The number of samples is infinite, the recovery error is zero. The option of an arbitrary number of samples is not analyzed. A set of formulas illustrating the described algorithm for sampling and recovery of both deterministic and random oscillations is described in detail in the literature (see, for example, the book [2], p. 228, 229, 240, 241).

This article proposes a different way of describing SRA of narrow-band processes, based on the application of the conditional mean rule (CMR) (see, for example, [3]). In this case, we analyze the SRA realization of the narrow-band process, but not its envelope. This method was used to study SRA realizations of various random processes (both Gaussian and non-Gaussian) with low-frequency spectra (see, for example, [5] - [7]). This method allows us to overcome the aforementioned drawbacks inherent in BT and its generalization for narrow-band processes. With the help of CMR, SRA realizations of random processes were found optimal by the criterion of the minimum mean-square error. Namely, for an arbitrary number of samples and for arbitrary intervals between them, the recovery functions were found and the minimum recovery errors were calculated (see [5] - [7]). In this case, the recovery function is a function of the conditional mean, and the function of the minimum recovery error is a function of the conditional variance. Studying the SRA realizations of Gaussian processes is facilitated by the fact that the general formulas for these conditional characteristics of random processes are known (see, for example, [4]). It is only necessary to write them in connection with the description of CMR random processes.

A comparative study of BT and suggested SRA with the same sampling parameters and a fixed number of samples was carried out in [8]. Sampling intervals were chosen by Nyquist formula. The basic recovery functions were the *SINC* function and the functions derived from the CMR. The results of calculations of the reconstruction error showed the advantage of algorithms based on CMR.

This work is a continuation of [8]. The difference is that random process models are Gaussian narrow-band processes. In this case, the appearance of the center frequency parameter

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ω_0 is an additional factor in the description of the SRA. Required calculation formulas are given below and three popular models of narrow-band processes are considered: a process with a rectangular spectrum, a process at the output of a resonant filter driven by white noise, and a process characterized by the covariance function with a Gaussian envelope. The calculation results with a fixed number of samples (with equal other parameters) show that taking into account the type of the covariance function significantly affects the characteristics of SRA. Revealed that the duration of the sampling interval is limited to $\Delta T \le 0.5T_0 = \pi/\omega_0$. If this value is exceeded, the recovery error reaches the maximum value.

Then, models of the process at the output of a resonant filter are considered, whose spectrum is sharply limited by frequencies ω_2, ω_1 that are symmetrical with respect to the center frequency ω_0. This is necessary for a comparative analysis of SRA based on the BT principle and CMR algorithms.

II. The Conditional Mean Rule for Gaussian Processes

Briefly discuss the features of CMR. The estimation of a random variable based on CMR automatically provides a minimum of the mean square error of the estimate, and this error coincides with the value of the conditional variance. Obviously, to use CMR when describing SRA realizations of a random process, it is necessary to know the set of fixed realization values (i.e. the set of samples $X,T = \{x(T_1), x(T_2),..., x(T_N)\}$) and the corresponding conditional probability density function $w(x,t|X,T)$, on the basis of which the conditional expectation function (the recovery function $\tilde{m}(t|X,T)$) and the conditional variance function $\tilde{\sigma}^2(t|X,T)$ characterizing a recovery quality. We emphasize that with reference to the description of SRA-realization of random processes, the current time should satisfy the following conditions: for the extrapolation mode $t \ge T_N$; for interpolation mode $T_k \le t < T_{k+1}, k = 1,2,..., N-1$.

Below we will compare the SRA based on the generalization of BT and using the CMR method. Obviously, it is necessary to consider options with equal sampling parameters for process models that have some of the same characteristics. Namely, the processes must be stationary with the same mean m , variance σ^2 , the center frequency and the bandwidth. We put $m = 0, \sigma^2 = 1$. Then the required formulas are as follows.

$$\tilde{m}(t) = \sum_{i}^{N}\sum_{j}^{N} K(t-T_i) a_{ij} x(T_j) = \sum_{j}^{N} x(T_j) b_j(t) \quad (1)$$

$$b_j(t) = \sum_{i=1}^{N} K(t-T_i) a_{ij} \quad (2)$$

$$A = K^{-1} = \begin{pmatrix} K(T_1-T_1)... \; K(T_1-T_N) \\ ... \quad ... \quad ... \\ K(T_N-T_1)... K(T_N-T_N) \end{pmatrix}^{-1} \quad (3)$$

$$\tilde{\sigma}^2(t) = 1 - \sum_{i=1}^{N}\sum_{j=1}^{N} K(t-T_i) a_{ij} K(T_j - t) \quad (4)$$

It is necessary to emphasize some features of the formulas (1) - (4):

1) Formula (2) defines the basис function $b_j(t)$ for each sample.

2) The above formulas are calculated.

3) When studying the characteristics of SRA for processes with limited spectrum, it is necessary to perform an operation to normalize the variance, so that the comparison of recovery errors is adequate.

4) Formulas (1) - (4) and their generalizations allow us to describe many variants of Gaussian SRA processes, including narrow-band processes considered here.

III. Models of Sampled Processes

The models of Gaussian narrow-band random processes, which are formed by various linear filters driven by white noise, are considered. Below are expressions for the normalized covariance functions $K(\tau)$ and energy spectra $S(\omega)$ that characterize the models used.

The process at the output of a single resonant filter (the first process):

$$K(\tau) = \exp(-\alpha|\tau|)\cos\omega_0\tau \quad (5)$$

$$S(\omega) = \frac{\alpha}{\alpha^2 + (\omega-\omega_0)^2} + \frac{\alpha}{\alpha^2 + (\omega+\omega_0)^2} \quad (6)$$

Process at the output of an ideal single resonant filter with a Gaussian frequency response (the second process):

$$K(\tau) = \exp(-\alpha\tau^2)\cos\omega_0\tau \quad (7)$$

$$S(\omega) = \sqrt{\frac{\pi}{4\alpha}}\left[\exp\left(-\frac{(\omega-\omega_0)^2}{4\alpha}\right) + \exp\left(-\frac{(\omega+\omega_0)^2}{4\alpha}\right)\right] \quad (8)$$

3) Process at the output of an ideal single resonant filter with a rectangular frequency response (the third process)

$$K(\tau) = \frac{\sin(\Delta\omega\tau/2)}{\Delta\omega\tau/2}\cos\omega_0\tau \quad (9)$$

$$S(\omega) = \begin{cases} \dfrac{\pi}{\Delta\omega} & when \; |\omega \pm \omega_0| \le \Delta\omega/2 \\ 0 & when \; |\omega \pm \omega_0| > \Delta\omega/2 \end{cases} \quad (10)$$

In order to properly compare the results of the SRA descriptions based on CMR and known variants, it is also necessary to consider models of processes with limited spectra (see below).

IV. EXAMPLES

A. *Example 1.*

Sampling of three processes; The spectra of the first and second process are not limited.

Case 1. $\Delta T < 0.5T_0$.

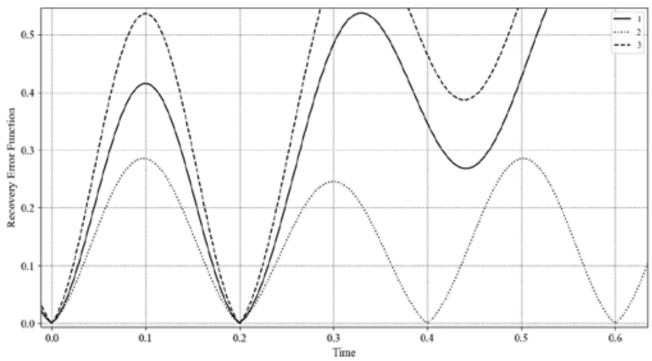

Fig.1. Errors of the first process: Curve 1- $N = 2, \Delta\omega = 1, \alpha = 2/\pi,$; Curve 2- $N = 4, \Delta\omega = 1, \alpha = 2/\pi,$; Curve 3- $N = 2, \Delta\omega = \pi/2, \alpha = 1,\ \Delta T = 0.2; T_0 = 0.5$.

Fig. 1 shows the recovery error of the first process with a different number of samples *N*. Curves 1 and 2 are calculated for $\Delta\omega = 1, \alpha = 2/\pi$. These curves demonstrates that the values of the recovery error in all midpoints of the sampling intervals ($N = 4$) are less than the error at $N = 2$. As one can see, the error in the average interval ($0.2 < t < 0.4$) is less than in the neighboring intervals. The reason for this phenomenon is the following: with small sampling interval and with a small number *N*, the samples affect the error function in each interval. This effect decreases with increasing interval $T - t$ in the argument of the covariance function. There is a symmetrical effect in the reconstruction operation. Curve 2 refers to the case.

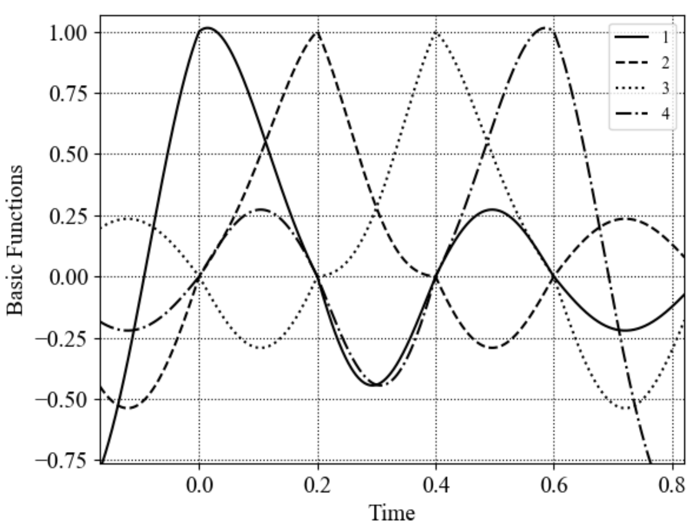

Fig.2. Basic functions for the first process $\Delta\omega = 1; \alpha = 2/\pi; \Delta T = 0.2; T_0 = 0.5; N = 4$

There are two types of errors in curves 1 and 3: the interpolation error function with $0 < t < 0.2$ and the extrapolation error function with $t > 0.2$. As you can see, the interpolation error is greater with increasing parameter $\Delta\omega$. The specific form of the extrapolation error functions is explained by the periodic component in the covariance function (5).

In Fig. 2 there are the basic functions $b_j(t)$, $j = 1, 2, 3, 4$. Note the following: 1) each sample has its basic function; 2) the forms of all basic functions are different when comparing the function *SINC*; 3) all basic functions in the middle interval ($0.2 < t < 0.4$) are symmetrical, therefore the error in the middle interval has a minimum value (see Fig. 1).

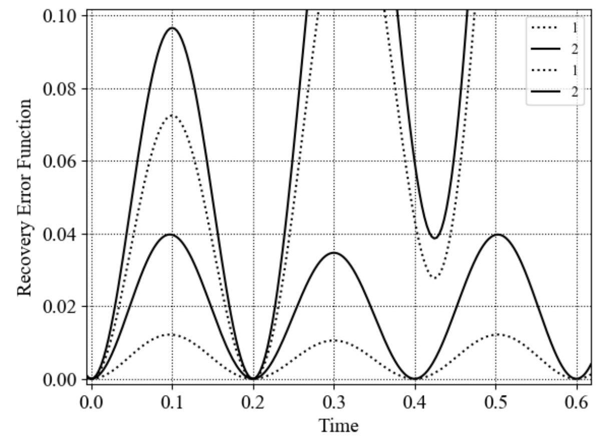

Fig.3. Curve 1- errors of the third process; Curve 2- errors of the second process; $\Delta\omega = 1; \alpha = 2/\pi; \Delta T = 0.2; T_0 = 0.5; N = 4, 2$

The comparison of the curves in Fig. 3 allows us to conclude: the recovery errors of the functions of the third process are less than the errors of the second process. Moreover, Fig. 3 and Fig. 1 show that the errors of the first process are greater when comparing errors in the third and second processes. The reason for this effect is associated with the various properties of the covariance functions (5), (7) and (9): the first process is non-differentiable, its realizations are chaotic. The second and third processes are idealistic; their statistical characteristics (7) - (10) are not realizable; the forms of their realizations are smooth compared to the realizations of the first process.

Case 2. $\Delta T = 0.5T_0$.

In fact, this variant is not acceptable, because the error reconstruction functions are reached the maximum values of errors, i.e.

$$\tilde{\sigma}^2(t = 0.5T_0) = 1. \qquad (11)$$

This result is valid for all types of processes under consideration. The reason of this effect is connected with a specific property of all covariance functions (5), (7) and (9): they have the term $Cos\omega_0 t$. In the case $t = 0.5T_0$, the sum in (4) is zero.

Case 3. $\Delta T > 0.5T_0$

The recovery error graph is shown in Fig. 4. The properties of this curve are typical: the error is zero at the sample points;

the error function among the sampling points (both in extrapolation mode and in interpolation modes) has the character of oscillations corresponding to the covariance function; error values are large. Thus, it is not possible to use sampling intervals greater than $\Delta T > 0.5T_0$.

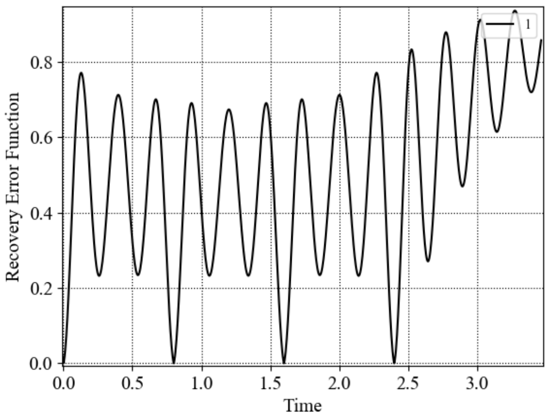

Fig.4. Curve 1- Errors of the first process;
$\Delta\omega = 1; \alpha = 2/\pi; \Delta T = 0.8; T_0 = 0.5; N = 4$

B. *Example 2.*

The spectrum of the first process is limited by frequencies $\omega_2, \omega_1 \left[\omega_2 > \omega_1, \omega_0 = 0.5(\omega_2 - \omega_1)\right]$. A normalized variance operation must be performed to ensure adequate comparison of reconstruction errors. The general formula for the normalized member *M* can be obtained as follows:

$$K(\tau = 0) = \frac{M}{\pi}\int_0^{\omega_b} S(\omega)d\omega = 1, M = \pi\left(\int_0^{\omega_b} S(\omega)\,d\omega\right)^{-1} \quad (12)$$

$$K(\tau) = \left(\int_0^{\omega_b} S(\omega)d\omega\right)^{-1}\int_0^{\omega_b} S(\omega)\cos\omega\tau\, d\omega$$

Once again, we discuss three cases: a) $\Delta T < 0.5T_0$; b) $\Delta T \geq 0.5T_0$.

Case 1. $\Delta T < 0.5T_0$

Graphs of error functions are shown in Fig. 5. Here one can see a rather interesting effect: unlike the graphs in Fig. 1, the recovery errors of the first process are smaller than the error curves of the third process. The explanation of this phenomenon is as follows: when we limit the spectrum of the first process, its statistical properties radically change. Now this process is differentiable, its realizations are smooth, and its covariance function is longer compared to the covariance function of the third process (see graphs in Fig. 6). Despite the equal bandwidth of both processes, the sample quality of the first process with limited spectrum is better than the third process.

Case 2. $\Delta T \geq 0.5T_0$.

The calculations of these variants demonstrate no acceptable values of recovery errors. The types of graphs are similar with the graph in Fig. 4

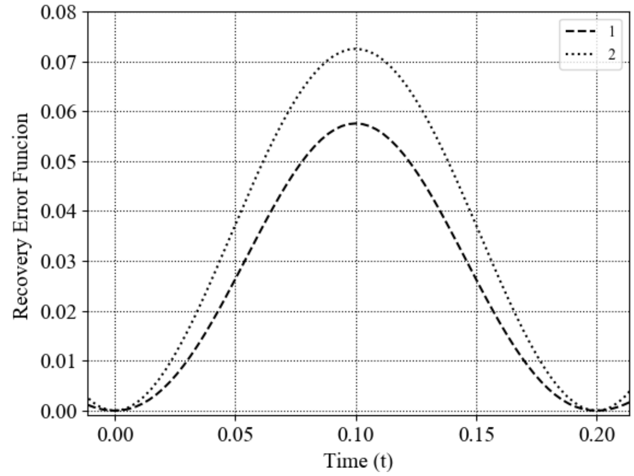

Fig.5. Curve 1- Errors of the first process; Curve 2 –errors of the third process $\Delta\omega = 1, \alpha = 2/\pi, \Delta T = 0.2, T_0 = 0.5, \omega_0 = 2; N = 2$

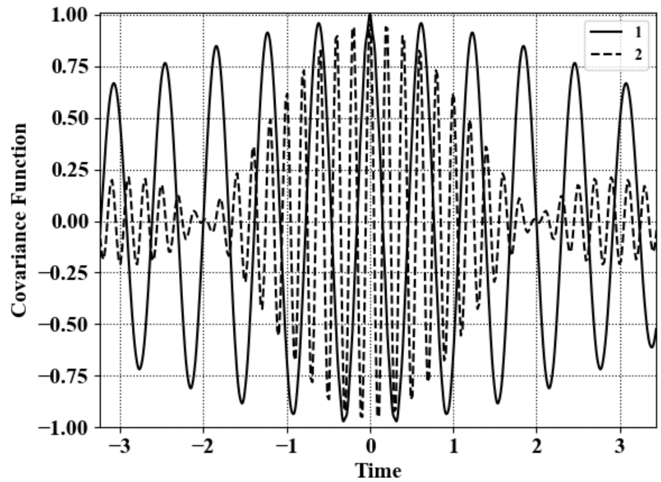

Fig.6. Covariance Functions; Curve 1 - the first process; Curve 2- the third process $\Delta\omega = 2, \alpha = 4/\pi, T_0 = 1, \omega_0 = 2$

V. CONCLUSION

In this paper, we have shown the advantages of using the conditional mean rule (CMR) method for sampling of narrow-band processes in comparison with the well-known method related to the generalization of the Balakrishnan theorem. Namely, using the CMR method: 1) the influence of the form of the covariance function and the shape of the power spectrum on the main statistical characteristics of the sampling – reconstruction algorithms (SRA) of the studied processes was demonstrated; 2) the number and location of samples are arbitrary; 3) it is shown that the most interesting cases of a limited number of samples participating in SRA are analyzed in detail: basic functions (which are not SINC functions) are found, and minimum recovery errors are determined for different sampling intervals $\Delta T < 0.5T_0$ and $\Delta T \geq 0.5T_0$; 4) it was found that the limitation of the spectrum in narrow-band processes radically changes the statistical characteristics of the

SRA, which is important when choosing models of the processes under study.

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