One approach to conducting the hierarchical structuring of various systems

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Abstract: - This paper is intended to present a new approach to the hierarchical structuring of various systems.An approach is thoroughly discussed and its theoretical justification is given, in particular, a theorem is proved that gives formalisms for constructing hierarchical structuring.The paper includes a detailed practical exampleof application of the introduced method. Further ways of development of the presented work are also outlined.

Key-Words: -Binary relation, Hierarchical structuring, Digraph, Reachability matrix, Truncated matrix, Reduction

1 Introduction

The need for hierarchical structuring of various systems (processes, compound objects etc.) often arises in many applied problems. A great many systems can be simulated by pair (P, R), here P is the set of the system structural units (elements, components), while R denotes the set of binaryrelations between these structural units. Binary relations, which earlier were under investigation only from purely mathematical viewpoint, turned out to be very simple, convenient and effective tool for various applications. The language of these relations is very helpful for description and solving different problems.

When modeling various systems, information about these relations every so often is redundant and not sufficiently structured, which creates certain difficulties in developing a formal model of the system By generally accepted opinion, the hierarchical structuring of the created system is an effective tool for the development its formal model.While working on PhD, we proposed a method of hierarchical structuring of design processes when creating CAD systems[6]. The method was based on the concept of D.W. Malone's paper "An Introduction to the Application of Interpretive Structural Modeling" [2]. In his paper Malone introduced the concepts ofthe reachability and the precedence sets and ranged elements by hierarchy levels based on the intersection of these sets. However, he does not provide proofs of the correctness of his assumptions. In our approach, ranking by hierarchy levels is carried out only using the reachability set, the correctness of the corresponding constructions is proved, redundant connections are removed using thereduction of binary relation.

J. Shreider[2] was the first to introduce the concept of reduction of binary relations and proved that any relation of strict order (irreflexive and transitive binary relation) could be one-valued restored by its reduction. A generalization of the concept of the reduction ofbinary relationswas proposed in [4]. The offered approach allows one-valued restoration of any transitive and antisymmetric relation by its reduction."Reduction of binary relations at some extent is a reverse operation towards the operation of implementation of transitive closure. Meanwhile the operation of transitive closure creates relationships between elements of binary relations at each successive level, while the reduction breaks all transitive relations between them in aninstant. Figurate saying, the operation of transitive closure coats binary relations into "transitive clothes", meanwhile the reduction strips them outoff all the "transitive parts", as if extracting the transitive root out of the binary relations" [5].That is why we note the generalized reduction of binary relation R by $R_{\surd}$.

That is why the attempt of creation special mathematical apparatus for restoration full relation by its known quantum followed by hierarchical structuring to our mind has practical significance and is of certain theoretical interest.

The article consists of 5 sections. Section 2 includes all the necessary information to understand the paper. Section 3 of the paper presents a theoretical basis of the offered method. Section 4 demonstrates an example of the practical application of the introduced approach. Section 5summarizes our work and discusses some perspectives to our results.

2 Preliminaries

In this paper we deal with binary relations only and henceforth the word “binary” will be omitted.

Let *X*be a universe. We say that *R* is *relation* in *X**if* $R \subseteq X \times X = X^2$. The characteristic function of relation is determined as follows

$$xRy = \begin{cases} 1 \ \ if \ \ (x,y) \in R, \\ 0 \ \ otherwise \end{cases} \quad \forall (x,y) \in X^2 .$$

Empty relation $\varnothing$: $x \varnothing y = 0 \quad \forall (x,y) \in X^2$

Unitary relation

$E: xEy = 1 \ if \ \ and \ only \ if \ \ x = y, \quad (x,y) \in X^2$.

Equality of relations

$R = S \Leftrightarrow \left\{ (x,y) \in X^2 : \ xRy = xSy \right\}$.

Inclusion of relations

$R \subseteq S \Leftrightarrow \left\{ (x,y) \in X^2 : \ xRy \le xSy \right\}$.

Union of relations

$R \bigcup S = \left\{ \max\left\{ xRy, xSy \right\}, (x,y) \in X^2 \right\}$.

Intersection of relations

$R \bigcap S = \left\{ \min\left\{ xRy, xSy \right\} \ (x,y) \in X^2 \right\}$.

Difference of relations

$R \setminus S = \left\{ (x,y) \in X^2 : \ \left(xRy = 1 \right) \wedge \left(xSy = 0 \right) \right\}$.

Composition ofrelations

$R \circ S = \left\{ (x,y) \in X^2 : \exists z \in X , \ \ xRz = zSy = 1 \right\}$.

Transitivity of R: $R \circ R \subseteq R$.

Reflexive relation R: $E \subseteq R$.

Irreflexive relation R: $R \bigcap E = \varnothing$.

Antisymmetric relation R: $R \bigcap R^{-1} \subseteq E$.

Definition 1. Transitive closure of *R*is the smallest transitive relation R that includes *R*.For any relation *R* given on finite universe

$$R = \bigcup_{k=1}^{n} R^k , \ \ R^n = R^{n-1} \circ R, \ n = 2,3,... \quad (1)$$

Note that for any reflexive relation*R* thisexpression is simplified:

$$R = \bigcup_{k=1}^{n-1} R^k , \ \ R^{n-1} = R^{n-2} \circ R, \ n = 3,4,... \quad (2)$$

It is obvious that $xRy \Rightarrow xRy$.One can easily show that

$$R \subseteq S \Rightarrow R \subseteq S . \quad (3)$$

We say that *R* is relation of a*strict order* if it is irreflexive and transitive.

We say that *R* is relation of a *nonstrict order* if it is reflexive, transitive and antisymmetric.

This definition is equipotent to the following one.

We say that *R* is relation of *nonstrict order* if it is union of strict order and unitary relation:

$$R = R_1 \bigcup E \ , \quad (4)$$

where $R_1$is a relation of strict order and *E* is unitary relation.

Definition 2. *R* contains *cycle* if there exists the subset $\left\{ x_0, x_1, x_{2,} ..., x_n \right\} \in X^n$ such that $x_0 = x_n, \ x_0, x_1, ..., x_{n-1}$ all are different from each other and $x_i R x_{i+1} = 1, \quad i = \overline{0, n-1}, \quad n = 1,2,...$.

Theorem 1[2]. *If R is transitive relation then* $R = R$.

Definition 3[2]. Reduction of relation *R* is relation R_r such that

$$R_r = R \setminus R^2, \quad R^2 = R \circ R . \quad (5)$$

It means that

$xR_r y = 1 \Leftrightarrow xRy = 1 \text{and} \nexists z : xRz = zRy = 1$.

Theorem 2[2]**.***If R is a relation of strict order on finite set, then transitive closure of reduction coincides with the initial order*:

$$R_r = R . \quad (6)$$

Unfortunately this theorem cannot be generalized on infinite sets. For example if *R* is strict order “<” (less then) on the set of real numbers then $R_r = \varnothing \Rightarrow R_r = \varnothing$ and $R_r \ne R$.

Thus, one can restore any initial relation of strict order given on finite set by its reduction.

Now we present the results obtained in [4]. Here and further on *xRy*means *xRy*=1.

Let us give the generalized definition of reduction.

Definition 4. Reduction $R_{\surd}$ of the given relation R is determined as follows

$$R_{\surd} = R \setminus \left(R \setminus E\right)^2, \quad \left(R \setminus E\right)^2 = \left(R \setminus E\right) \circ \left(R \setminus E\right) \quad (7)$$

It means that

$xR_{\surd}y \Leftrightarrow (xRy$ and $\nexists z : x(R \setminus E)z = z(R \setminus E)y)$.

Note that for any irreflexive relation this definition coincides with (5). It is clear that

$$R_{\surd} \subseteq R \; . \quad (8)$$

One can be easily convinced that if R is reflexive then

$$E \subseteq R_{\surd} \, . \quad (9)$$

It is also easy to show that for any relation R

$$xRy \text{ and } x \neq y \Leftrightarrow x(R \setminus E)y \, . \quad (10)$$

Proposition 1[4]**.** *For any relation R*

$$(R)_{\surd} \subseteq R \, . \quad (11)$$

Theorem 3[4]**.** *If R is a relation of nonstrict order on finite set, then transitive closure of reduction coincides with the initial order*:

$$R_{\surd} = R \, . \quad (12)$$

Thus, one can restore any initial relation of nonstrict order given on finite set by its reduction. Moreover $R_{\surd}$ is minimal relation permitting to restore initial relation R. The following theorem clarifies more exact sense of this assertion.

Theorem 4[4]**.** *If relations R and S are such that* $S = R$, *then* $R_{\surd} \subseteq S$.

Let us establish some properties of the reductions.

Definition 4[4]. The relation S is antitransitive if

$$S \bigcap (S \setminus E)^n = \varnothing, \; n \geq 2 \, . \quad (13)$$

By another words if

$x(S \setminus E)x_1, x_1(S \setminus E)x_2, ..., x_n(S \setminus E)y$ then xSy is impossible. It means that in the corresponding graph of the relation $B \setminus E$ immediate connection between vertices x and y cannot be roundabout.

Theorem 5[4]**.** *Reduction of any relation is antitransitive*.

It is not difficult to show that for any relation R

$$R \setminus E = R \setminus E \, . \quad (14)$$

Lemma 1[4]**.** *If S is antitransitive relation, then*

$$(S)_{\surd} = S \, . \quad (15)$$

Remark 1. It is naturally to compare (15) with (12).

Lemma 2[4]. *Transitive closure of any relation S is not antisymmetric if and only if* $S \setminus E$ *contains cycles* .

Now we are able to obtain theorem, which is conversed to the Theorem 3.3.

Theorem 6[4]**.** *If S is antitransitive, then S represents a reduction of some transitive and antisymmetric relation.*

Corollary. *If B is antitransitive, then B represents a reduction of some relation of nonstrict order*

3 Method of Hierarchical Structuring

An arbitrary system (process) which is described by a pair $\{P, R\}$ can be represented as a directed graph (digraph) D (see e.g. [1]), whose vertices are the elements of the set P, and the edges model a certain binary relation R between these elements. In the equivalent binary-matrix representation, the pair $\{P, R\}$ can be expressed as $n \times n$ (0,1) matrix $R = \{r_{ij}\}$:

$$r_{ij} = \begin{cases} 1 \;\; if \;\; p_i R p_j, \; i \neq j \\ 0 \;\; otherwise \end{cases} \qquad (p_i, p_j) \in P^2, \quad (16)$$

here n is the power (number of elements) of the set P. In the literature, such a matrix is called an adjacency matrix.

Definition 3. Transitive closure D of digraph D is the digraph corresponding to transitive closure R of the given relation R.

According to this definition, edge $(p_i, p_j, i \neq j)$ is part of digraph D if and only if there is a path from vertex p_i to vertex p_j in digraph D. In the

DOI: 10.46300/9108.2020.14.3 Volume 14, 2020

same way, pair (p_i, p_i) belongs to digraph D if and only if there is a cycle containing vertex p_i in the structure of digraph *D*.

Definition 5. Vertex $p_j \in D$ is called reachable for vertex $p_i \in D$ if in digraph *D* there exists a path from p_i to p_j.

Agreement 1. $\forall p_i \in D$ is reachable for itself (by a path of zero length).

Definition 6. Reachability matrix $M = (m_{ij})$ of digraph *D* is determined as follows

$$m_{ij} = \begin{cases} 1 \ if \ \ p_i \left(R \cup E\right) p_j \\ 0 \ \ otherwise \end{cases} \quad i, j = \overline{1, n}. \qquad (17)$$

Obviously, taking into account Agreement 1, the reachability matrix of digraph *D* presents the union of the adjacency matrix of *D* and the unitary matrix.

Using the reachability matrix *M*, one can rank the set *P* on hierarchical components, that is, arrange the elements according to the hierarchy levels while preserving informational-oriented relationships between them.

If in a digraph *D* there exists a closed route passing through all the vertices, then any vertex is reachable for any vertex and the hierarchically structured graph will have one hierarchy level. Otherwise, there is a nonempty set $\{p_s\} \in P$ of vertices that are unreachable for all other vertices of the set $P \setminus \{p_s\}, \ s \in \{1, 2, ..., n-1\}$.

From Definition 5 it follows that for all p_j, which are reachable for p_i, the respective columns in the reachability matrix have a value of 1 on the *i*-th row i.e. the condition $m_{ij} = 1, \ m_{ij} \in M$ is satisfied.

Lemma3. Let *M* be the reachability matrix of digraph *D*. Then for vertices unreachable for all other vertices except themselves, the following condition must be fulfilled:

$$\sum_{i=1}^{n} m_{ij} = 1, \ \ m_{ij} \in M, \ \ j \in \{1, ..., n\}.$$

Proof. Assume that $\sum_{i=1}^{n} m_{ij} > 1$ If, for example, this sum equals 2 then there exists the column number $j \in \{1, ..., n\}$ containing two units in matrix *M*. From here it follows that element p_j is reached by another element, but this contradicts to the conditions of the lemma. □

Now we proceed to the hierarchical structuring of systems (processes) that can be described by a couple $\{P, R\}$. First of all, we formulate adequate requirements for hierarchical structuring.

Requirements #1

- At the first (lowest) level of the hierarchy, we arrange all those elements of the set *P* that are not reachable for all other elements. Denote the set of elements of the first level by $P_1 \subseteq P$;
- At each following level of the hierarchy, beginning from the second, we place elements that are different from elements of the previous levels and unreachable for all other elements.

As a result, the set *P* will be divided into hierarchical components

$P = \{P_1, P_2, ..., P_l\}, \ l \in \{1, 2, ..., n\}$ - the number of hierarchy levels.

Definition 7. Unions of type $\bigcup_{k}^{t<k}$ are assumed to be $\varnothing$. $\|P\|$ designates the power of set *P*, *M*(*P*) stands for reachability matrix of set $P, n_k = \left\| \bigcup_{t=1}^{k-1} P_t \right\|$ designates the total number of elements at the first *k*-1 hierarchical levels.

Now we are ready to prove the following theorem, which represents the main result of this paper.

Theorem 7. *In the hierarchical structuring of a system (process), described by couple* (P, R), *carried out according to* Requirements #1, *the elements of the k-th hierarchical level are defined as follows*

$$P_k = \left\{ p_j \in P \setminus \bigcup_{t=1}^{k-1} P_t \mid \sum_{i=1}^{n-n_k} m_{ij} = 1 \right\}, \ m_{ij} \in M\left(P_k\right),$$
$$M\left(P_k\right) = \left(m_{ij}\right) \mid m_{ij} \in M\left(P_{k-1}\right) \wedge \sum_{i=1}^{n-n_k} m_{ij} > 1,$$
$$j \in \{1, ..., n - n_k\}, \ k = \overline{1, l}, \ l \in \{1, ..., n\}. \qquad (18)$$

Proof. We proceed by induction.
(i) k = 1. By Requirements #1, Definition 7 and Lemma 3 we have

$$P_1 = \left\{ p_j \in P \mid \sum_{i=1}^{n} m_{ij} = 1 \right\},\ m_{ij} \in M\left(P_1\right),\quad j \in \{1,...,n\}.$$

Hence for k = 1 (18) is true.
(ii) Let for k> 1 (18) is true.
(iii) Consider P_{k+1}. According to Requirements #1and (ii) elements of level k + 1 should belong to the set

$$P \setminus \left(\left(\bigcup_{t=1}^{k-1} P_t \right) \cup P_k \right) = P \setminus \bigcup_{t=1}^{k} P_t .$$

AnalogouslybyDefinition7the power of the set of elements of the first k levels will be equal to

$$n_{k+1} = n_k + \left\| P_k \right\| = \left\| \bigcup_{t=1}^{k-1} P_t \right\| + \left\| P_k \right\| = \left\| \bigcup_{t=1}^{k} P_t \right\| .$$

Therefore, the reachability matrix $M\left(P_{k+1}\right)$ of the elements of level k+1 will have dimension $\left(n - n_{k+1}\right) \times \left(n - n_{k+1}\right)$. Taking into consideration Requirements # 1, (ii) and Lemma 3, this matrix can be obtained from the matrix $M\left(P_k\right)$ by deleting those columns (and respective rows) where the sum of the elements is equal to 1, i.e.

$$M\left(P_{k+1}\right) = \left(m_{ij}\right) \mid m_{ij} \in M\left(P_k\right) \wedge \sum_{i=1}^{n-n_{k+1}} m_{ij} > 1 .$$

Now by Lemma 3 we can state that at the k + 1 level there are only those elements that satisfy the condition

$$\sum_{i=1}^{n-n_{k+1}} m_{ij} = 1,\quad m_{ij} \in M\left(P_{k+1}\right).$$

Thus we have

$$P_{k+1} = \left\{ p_j \in P \setminus \bigcup_{t=1}^{k} P_t \mid \sum_{i=1}^{n-n_{k+1}} m_{ij} = 1 \right\},\ m_{ij} \in M\left(P_{k+1}\right),$$

$$M\left(P_{k+1}\right) = \left(m_{ij}\right) \mid m_{ij} \in M\left(P_k\right) \wedge \sum_{i=1}^{n-n_{k+1}} m_{ij} > 1,$$

$$j \in \left\{1,...,n - n_{k+1}\right\},\ k = \overline{1, l-1}.$$

and the proof is over. □

The digraph obtained as a result of hierarchical structuring is transformed into the hierarchical graph D_H, which displays all the connections generated by P.

According to the principle of constructing a hierarchical structure at one level of the hierarchy, there are such and only such types of relationships between elements: either unrelated elements, or cycles, or both types together. Without loss of generality, we replace each cycle (if it exists) with some element of it, which we call a representative of the cycle. Let there is a cycle $p_{j_1}, p_{j_2}, ..., p_{j_m}, p_{j_1}$ and let j_1 be the smallest out of row numbers of the matrix M corresponding to the elements of the cycle. Replacing the cycle with its representative is equivalent to removing rows and columns with numbers j_2, j_3,…, j_m from the reachability matrix. We call the resulting matrix the truncated matrix and denote it by M_0. Note that this matrix preserves all information about the reachability set of elements.

Proposition 2. The relation R_0, that corresponds to the truncated reachability matrix $M_0$is a non-strict order.
Proof. The transitivity of the relation R_0 directly follows from the definition of the reachability matrix and Definition 5. Further, since R_0 includes the unit relation, it is reflexive. The antisymmetry is authorized by Lemma 2; indeed, if there are no cycles in R_0, then there will be no more in $R_0 \setminus E$. □

As it was noted above, the digraph obtained as a result of hierarchical structuring contains redundant edges. An application of the reduction of binary relation allows us to remove these redundant edges. By Theorem 4 reduction $R_{\surd}$ is minimal relation permitting to restore initial relation R. By Definition 1 we obtain

$$\left(R_0\right)_{\surd} = R_0 \setminus \left(R_0 \setminus E\right)^2,\ \left(R_0 \setminus E\right)^2 = \left(R_0 \setminus E\right) \circ \left(R_0 \setminus E\right). \quad (19)$$

The hierarchical digraphD_0 corresponding to the relation $\left(R_0\right)_{\surd}$ contains the minimum number of edges, while retaining all the information about the reachability matrix M (if there was a cycles in the matrix M, one can restore it at the place of the representative of the cycle in digraph D_0).

4 Example

Let a certain system be given by a pair (P,R), $P=\{p_1,...,p_5\}$, $R=(r_{ij})$, $i,j=\overline{1.5}$. The digraph corresponding to this pair is shown in Fig. 1

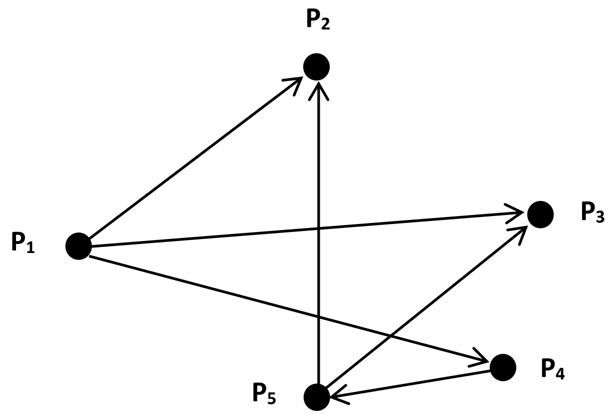

Fig.1

First we find the matrices of adjacency and transitive closure of the relation R:

$$\text{Adjacency matrix } R = \begin{array}{c|ccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \hline p_1 & 0 & 1 & 1 & 1 & 0 \\ p_2 & 0 & 0 & 0 & 0 & 1 \\ p_3 & 0 & 0 & 0 & 0 & 0 \\ p_4 & 0 & 0 & 0 & 0 & 1 \\ p_5 & 0 & 0 & 1 & 0 & 0 \end{array};$$

$$R^2 = R \circ R = \begin{array}{c|ccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \hline p_1 & 0 & 0 & 0 & 0 & 1 \\ p_2 & 0 & 0 & 1 & 0 & 0 \\ p_3 & 0 & 0 & 0 & 0 & 0 \\ p_4 & 0 & 0 & 1 & 0 & 0 \\ p_5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$R^3 = R^2 \circ R = \begin{array}{c|ccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \hline p_1 & 0 & 0 & 1 & 0 & 0 \\ p_2 & 0 & 0 & 0 & 0 & 0 \\ p_3 & 0 & 0 & 0 & 0 & 0 \\ p_4 & 0 & 0 & 0 & 0 & 0 \\ p_5 & 0 & 0 & 0 & 0 & 0 \end{array}.$$

As we see $R^3 \subset R$ and the transitive closure of the relation R will be presented by the following matrix

$$R = R \bigcup R^2 = \begin{array}{c|ccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \hline p_1 & 0 & 1 & 1 & 1 & 1 \\ p_2 & 0 & 0 & 1 & 0 & 1 \\ p_3 & 0 & 0 & 0 & 0 & 0 \\ p_4 & 0 & 0 & 1 & 0 & 1 \\ p_5 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Define the reachability matrix of the set P: $M(P) = R \bigcup E$.

$$M(P) = \begin{array}{c|ccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \hline p_1 & 1 & 1 & 1 & 1 & 1 \\ p_2 & 0 & 1 & 1 & 0 & 1 \\ p_3 & 0 & 0 & 1 & 0 & 0 \\ p_4 & 0 & 0 & 1 & 1 & 1 \\ p_5 & 0 & 0 & 1 & 0 & 1 \end{array}$$

Counting the sums $\sum_{i=1}^{5} m_{ij}$ in the columns of the matrix, we find that such a sum is equal to 1 only in the column corresponding to p_1. From here it follows that at the first level of the hierarchy there is only one element therefore $P_1=\{p_1\}$. Now obtain reachability matrix of set P_1:

$$M(P_1) = \begin{array}{c|ccccc} & \cancel{p_1} & p_2 & p_3 & p_4 & p_5 \\ \hline \cancel{p_1} & \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} \\ p_2 & \cancel{0} & 1 & 1 & 0 & 1 \\ p_3 & \cancel{0} & 0 & 1 & 0 & 0 \\ p_4 & \cancel{0} & 0 & 1 & 1 & 1 \\ p_5 & \cancel{0} & 0 & 1 & 0 & 1 \end{array} = \begin{array}{c|cccc} & p_2 & p_3 & p_4 & p_5 \\ \hline p_2 & 1 & 1 & 0 & 1 \\ p_3 & 0 & 1 & 0 & 0 \\ p_4 & 0 & 1 & 1 & 1 \\ p_5 & 0 & 1 & 0 & 1 \end{array}$$

Counting the sums $\sum_{i=1}^{4} m_{ij}$ in the columns of the matrix, we find that such a sum is equal to 1 in the columns corresponding to p_2 and p_4. It means that at the second level of the hierarchy there are two elements- $P_2 = \{p_2, p_4\}$. Now obtain reachability matrix of set P_2:

$$M(P_2) = \begin{array}{c|cccc} & \cancel{p_2} & p_3 & \cancel{p_4} & p_5 \\ \hline \cancel{p_2} & \cancel{1} & \cancel{1} & \cancel{0} & \cancel{1} \\ p_3 & \cancel{0} & 1 & \cancel{0} & 0 \\ \cancel{p_4} & \cancel{0} & \cancel{1} & \cancel{1} & \cancel{1} \\ p_5 & \cancel{0} & 1 & \cancel{0} & 1 \end{array} = \begin{array}{c|cc} & p_3 & p_5 \\ \hline p_3 & 1 & 0 \\ p_5 & 1 & 1 \end{array}.$$

Counting the sums $\sum_{i=1}^{2} m_{ij}$ in the columns of the matrix, we find that such a sum is equal to 1 in the column corresponding to the element p_5. From here it follows that at the third level of the hierarchy there is one element - $P_3 = \{p_5\}$. Now obtain reachability matrix of set P_3:

$$M(P_3) = \begin{array}{c|cc} & p_3 & \cancel{p_5} \\ \hline p_3 & 1 & \cancel{0} \\ \cancel{p_5} & \cancel{1} & \cancel{1} \end{array} = \begin{array}{c|c} & p_3 \\ \hline p_3 & 1 \end{array}.$$

The matrix $M(P_3)$ consists of one column with a single component which is equal to 1. It means that at the fourth last level of the hierarchy there is one element - $P_4 = \{p_3\}$.

So we get

$$P = \{P_1, P_2, P_3, P_4\} = \{\{p_1\}, \{p_2, p_4\}, \{p_5\}, \{p_3\}\}.$$

The hierarchical digraph corresponding to the reachability matrix is shown in Fig.2.

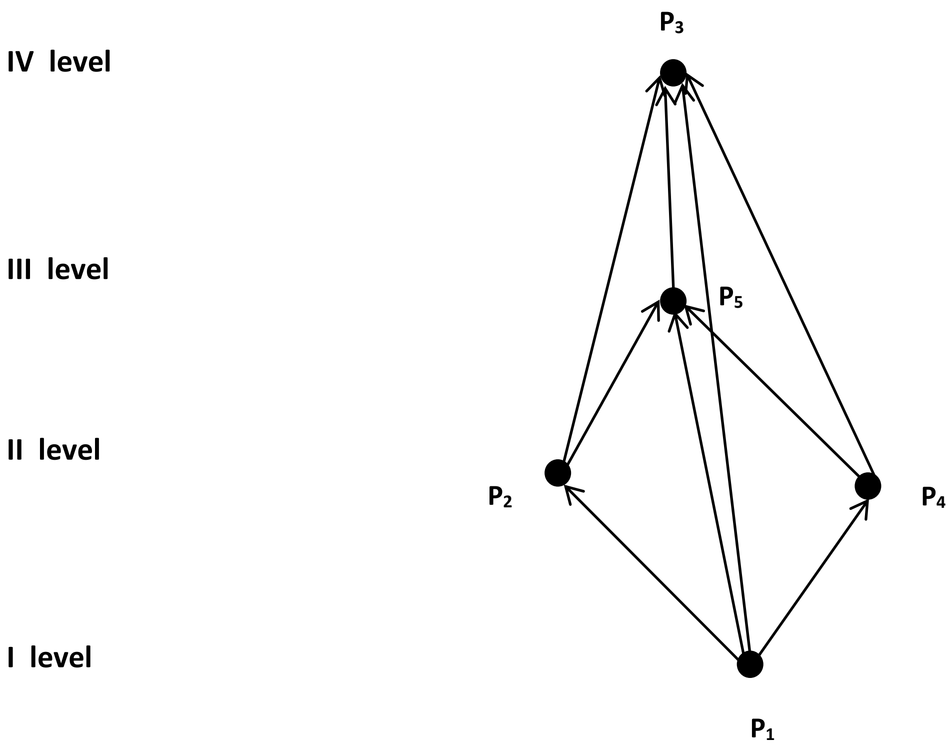

Fig.2

Now remove the redundant edges. To do this, we use the concept of reduction introduced in section 2. Since there are no cycles in the matrix $M(P)$, it is truncated and by Proposition 2 this matrix corresponds to a relation of nonstrict order. It is known that one can restore any initial relation of nonstrict order given on finite set by its reduction, moreover $R_{\surd}$ is minimal relation permitting to restore initial relation R. Now calculate the reduction of the relation corresponding to the matrix $M(P)$by Definition 4:

$$R_{\surd} = R \setminus \left(R \setminus E\right)^2 :$$

$$R = M\left(P\right) = \begin{array}{cccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ p_1 & 1 & 1 & 1 & 1 & 1 \\ p_2 & 0 & 1 & 1 & 0 & 1 \\ p_3 & 0 & 0 & 1 & 0 & 0 \\ p_4 & 0 & 0 & 1 & 1 & 1 \\ p_5 & 0 & 0 & 1 & 0 & 1 \end{array};$$

$$\left(R \setminus E\right)^2 = \begin{array}{cccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ p_1 & 0 & 0 & 1 & 0 & 1 \\ p_2 & 0 & 0 & 1 & 0 & 0 \\ p_3 & 0 & 0 & 0 & 0 & 0 \\ p_4 & 0 & 0 & 1 & 0 & 1 \\ p_5 & 0 & 0 & 1 & 0 & 0 \end{array};$$

$$R_{\surd} = \begin{array}{cccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ p_1 & 1 & 1 & 0 & 1 & 0 \\ p_2 & 0 & 0 & 0 & 0 & 1 \\ p_3 & 0 & 0 & 0 & 0 & 0 \\ p_4 & 0 & 0 & 0 & 0 & 1 \\ p_5 & 0 & 0 & 1 & 0 & 0 \end{array}.$$

The resulting digraph of hierarchical structuring is shown in Fig. 3.

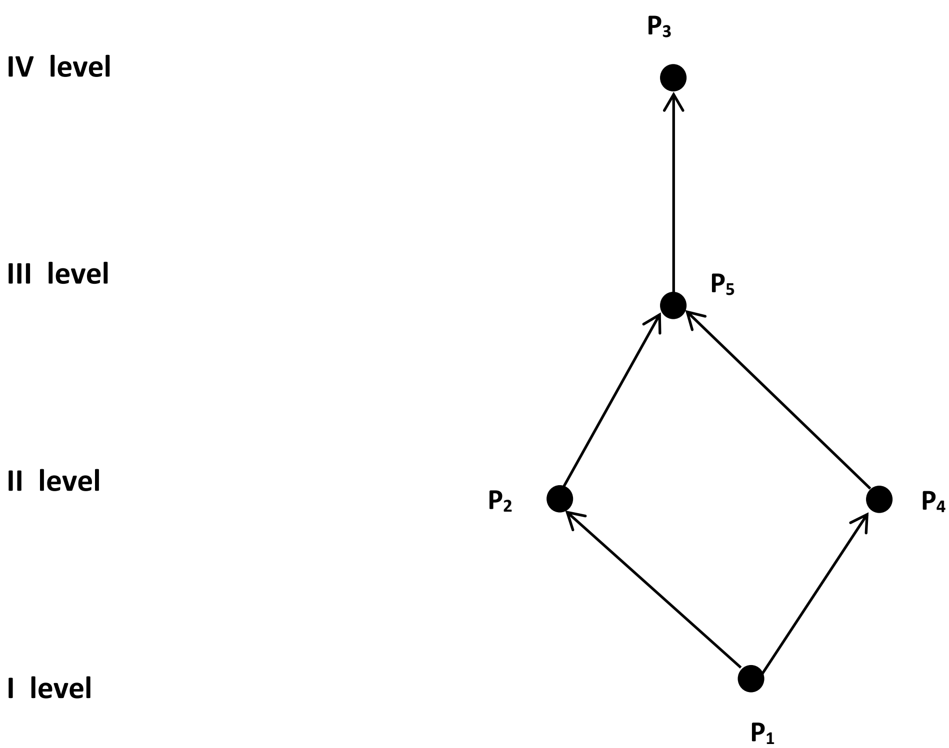

Fig. 3

DOI: 10.46300/9108.2020.14.3 Volume 14, 2020

5 Conclusion

The paper proposes a new approach to building a hierarchical structuring of various systems based on the concepts of transitive closure and reduction of binary relations. Based on already published theorems, as well as on new provisions proved in this work, a theoretically substantiated apparatus is constructed that shows the correctness of the proposed approach.A practical example is presented that clearly demonstrates the steps of the proposed approach.

Let us say a few words about perspective to the obtained results. Nowadays in science and practice, a direction called fuzzy systems is rapidly developing. Fuzzy systems provide opportunities for modeling of conditions which are inherently imprecisely defined. Therefore, a generalization of our approach for the hierarchical structuring of fuzzy systems seems to be a promising and effective direction.

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