

Observability and Geometric Approach of 2D Hybrid Systems

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Abstract—A connection is emphasized between two branches of the Systems Theory, namely the Geometric Approach and 2D Systems, with a special regard to the concept of observability. An algorithm is provided which determines the maximal subspace which is invariant with respect to two commutative matrices and which is included in a given subspace. Observability criteria are obtained for a class of 2D systems by using a suitable 2D observability Gramian and some such criteria are derived for LTI 2D systems, as well as the geometric characterization of the subspace of unobservable states. The presented algorithm is applied to determine this subspace.

Index Terms—2D hybrid linear systems, observability, invariant subspace, geometric approach.

1. INTRODUCTION

Observability is a fundamental concept in Systems Theory which refers to the ability to estimate the state variable from the observation of the output determined by any control. The theory of controllability and observability has been developed in response to problems generated by technological sciences, especially in areas related to control, communication and computers. For "classical" 1D systems the observability theory was developed by Kalman [9,10], Gilbert [6], Weiss [18] etc.

In recent years the theory of two-dimensional (2D) systems became a distinct and important branch of Systems and Control Theory, due to the richness in its potential applications in various areas as digital image processing, seismic signal processing, gravity and magnetic field mapping, as well as to the abundance of non-trivial theoretical approaches. Givone and Roesser [7] [17] or Fornasini and Marchesini [4] introduced the notions of controllability and observability for their 2D models, as natural generalizations of the 1-D case. But their definitions referred to local notions which were not closely related to minimality or to canonical decomposition. Alternative definitions of modal controllability and observability based on coprimeness notions were presented in [11]; these definitions are connected to minimality.

The Geometric Approach is a trend in Systems and Control Theory developed to realize a better and neater investigation of the structural properties of the linear dynamical systems and to provide elegant solutions of problems of controller synthesis such as decoupling and pole-assignment problems

for linear time-invariant multivariable systems. The Geometric Approach leads to a very clear geometric conditions for controllability, reachability, observability, constructibility and minimality of linear systems. The cornerstone of this approach is the concept of invariance of a subspace with respect to a linear transformation.

The history of the Geometric Approach starts in 1969 when Basile and Marro [2] introduced and studied the basic geometric tools called controlled and conditioned invariant subspaces which were applied to disturbance rejection or unknown-input observability and later on to the robust controlled invariant and the emphasis of the duality [3], [13]. Wonham's book [19] imposed the name of "(A,B)-invariant" instead of "(A,B)-controlled invariant". The LQ problem was also studied in a geometric framework by Silverman, Hautus, Willems. Further contributions are due to numerous researchers among which Anderson, Akashi, Bhattacharyya, Kucera, Malabre, Molinari, Pearson, Francis and Schumacher.

The aim of this paper is to emphasize the possible interactions between the two directions of the Systems Theory, concerning the observability of a class of 2D hybrid linear systems, which is the continuous-discrete counterpart of At-tasi's discrete 2D model [1] .

Such systems appear in various problems like the iterative learning control synthesis [12] or repetitive processes [5]. For Roesser or Fornasini - Marchesini models, corresponding hybrid systems were studied by Kaczorek [8].

The advantage of the considered class of 2D systems, studied by Prepelitã et al. in [14], [15] and [16], is the possibility to keep the global character of the natural generalizations of the concepts of observability and controllability, which remain related to minimality and state decomposition.

This paper provides in Section 2 an algorithm which determines the maximal subspace which is invariant with respect to two commutative matrices and included in a given subspace.

The state space representation of the considered class of systems is given in Section 3 and the state and output formulæ for these systems are deduced.

Section 4 is devoted to the study of the observability of 2D time varying systems. This property is characterized by means of a suitable 2D observability Gramian. For completely

observable systems a formula is derived which provides the initial state by knowing the control and corresponding output.

Section 5 studies the time-invariant 2D systems and a list of observability criteria is established. A geometric characterization of the subspace of unobservable states is given in terms of invariant subspaces included in the kernel of the output matrix.

In Section 6 the algorithm presented in Section 2 is used to determine the subspace of unobservable states. A MATLAB Program and an example illustrate the proposed algorithm.

2. MAXIMAL INVARIANT SUBSPACES WITH RESPECT TO TWO COMMUTATIVE MATRICES

Let \mathbf{K} be a field and $A_1, A_2 \in \mathbf{K}^{n \times n}$ commutative matrices.

Definition 2.1: A subspace \mathcal{V} of \mathbf{K}^n is said to be (A_1, A_2) -invariant if

$$\forall v \in \mathcal{V}, A_1 v \in \mathcal{V} \text{ and } A_2 v \in \mathcal{V}. \quad (2.1)$$

Let \mathcal{C} be a proper subspace of \mathbf{K}^n . The sum of the (A_1, A_2) -invariant subspaces included in \mathcal{C} is called the *maximal (A_1, A_2) -invariant subspace included in \mathcal{C}* . We denote it $\max I(A_1, A_2; \mathcal{C})$.

For a subspace \mathcal{V} of \mathbf{K}^n , we consider the subspaces $A_1^{-k} A_2^{-l} \mathcal{V} = \{v \in \mathbf{K}^n | A_1^k A_2^l v \in \mathcal{V}\}$, $k, l \in \mathbf{N}$, $A_1^{-0} A_2^{-0} \mathcal{V} = \mathcal{V}$. If $v \in A_1^{-j} \mathcal{V}$, then $A_i v \in A_i^{-(j-1)} \mathcal{V}$, $i = 1, 2, \forall j \geq 0$.

Proposition 2.1: The maximal (A_1, A_2) -invariant subspace included in \mathcal{C} is

$$\max I(A_1, A_2; \mathcal{C}) = \bigcap_{i=0}^{\infty} \bigcap_{j=0}^{\infty} A_1^{-i} A_2^{-j} \mathcal{C}. \quad (2.2)$$

Proof: Let us denote by \mathcal{U} the subspace $\bigcap_{i=0}^{\infty} \bigcap_{j=0}^{\infty} A_1^{-i} A_2^{-j} \mathcal{C}$. If $v \in \mathcal{U}$ then $v \in A_1^{-i} A_2^{-j} \mathcal{C}$, $\forall i, j \in \mathbf{N}$ and also $v \in A_1^{-(i+1)} A_2^{-j} \mathcal{C}$ and $v \in A_1^{-i} A_2^{-(j+1)} \mathcal{C}$. It results that $A_1 v \in A_1^{-i} A_2^{-j} \mathcal{C}$ and $A_2 v \in A_1^{-i} A_2^{-j} \mathcal{C}$, $\forall i, j \in \mathbf{N}$, hence $A_1 v \in \mathcal{U}$ and $A_2 v \in \mathcal{U}$ i.e. is (A_1, A_2) -invariant. We can write by (2.2) $\mathcal{U} = \mathcal{C} \cap (\bigcap_{i=0}^{\infty} \bigcap_{j=0}^{\infty} A_1^{-i} A_2^{-j} \mathcal{C})$ where $(i, j) \neq (0, 0)$, hence \mathcal{U} is included in \mathcal{C} .

Now, let \mathcal{V} be an (A_1, A_2) -invariant subspace included in \mathcal{C} . Then, for any $v \in \mathcal{V}$, $A_1^i A_2^j v \in \mathcal{V} \subset \mathcal{C}$, $\forall i, j \in \mathbf{N}$, hence $v \in A_1^{-i} A_2^{-j} \mathcal{C}$, $\forall i, j \in \mathbf{N}$, which implies $v \in \mathcal{U}$. Therefore $\mathcal{V} \subset \mathcal{U}$, i.e. $\mathcal{U} = \bigcap_{i=0}^{\infty} \bigcap_{j=0}^{\infty} A_1^{-i} A_2^{-j} \mathcal{C}$ is the maximal such subspace.

Proposition 2.2: The maximal (A_1, A_2) -invariant subspace included in \mathcal{C} is

$$\max I(A_1, A_2; \mathcal{C}) = \bigcap_{i=0}^{n-1} \bigcap_{j=0}^{n-1} A_1^{-i} A_2^{-j} \mathcal{C}. \quad (2.3)$$

Proof: Let us denote by \mathcal{V} the subspace in the right-hand member of (2.3). Obviously, by Proposition 2.1, $\mathcal{U} \subset \mathcal{V}$, where $\mathcal{U} = \bigcap_{i=0}^{\infty} \bigcap_{j=0}^{\infty} A_1^{-i} A_2^{-j} \mathcal{C} = \max I(A_1, A_2; \mathcal{C})$.

Now, for any $v \in \mathcal{V}$, $A_1 A_2^j v \in \mathcal{C}$, $\forall i, j \in \mathbf{N}$, $i, j \leq n-1$. Let $p_k(s) = \det(sI - A_k) = s^n + a_{n-1,k} s^{n-1} + \dots + a_{1,k} s + a_{0,k}$, $k = 1, 2$ be the characteristic polynomial of the matrix A_k , $k = 1, 2$. By Hamilton-Cayley Theorem, each matrix verifies its characteristic equation, hence

$$A_k^n = -a_{n-1,k} A_k^{n-1} - \dots - a_{1,k} A_k - a_{0,k} I, \quad k = 1, 2. \quad (2.4)$$

Then, for any vector $v \in \mathcal{V}$, $A_1^n v = -\sum_{i=0}^{n-1} a_{i,k} A_1^i v$. Since A_1 and A_2 are commutative matrices, we can premultiply this equality by A_2^j and we obtain $A_1^n A_2^j v = -\sum_{i=0}^{n-1} a_{i,k} A_1^i A_2^j v$ for all j , $0 \leq j \leq n-1$. Since \mathcal{C} is a subspace, one obtains $A_1^n A_2^j v \in \mathcal{C}$. Similarly, by applying recurrently the Hamilton-Cayley Theorem for both A_1 and A_2 , we get $A_1^i A_2^j v \in \mathcal{C}$, $\forall i, j \geq 0$, hence $v \in \mathcal{U}$. One obtains $\mathcal{V} \subset \mathcal{U}$, hence $\mathcal{V} = \mathcal{U} = \max I(A_1, A_2; \mathcal{C})$.

The following algorithm determines recurrently the subspace $\max I(A_1, A_2; \mathcal{C})$.

Algorithm 2.1

Stage 1. Construct the sequence of subspaces $(S_{0,j})_{0 \leq j \leq n}$ of the space \mathbf{K}^n :

$$S_{0,0} = \mathcal{C}; \quad (2.5)$$

$$S_{0,j} = \mathcal{C} \cap A_2^{-1} S_{0,j-1}, \quad j = 1, \dots, n; \quad (2.6)$$

Stage 2. Determine j_0 , the first index in $\{0, 1, \dots, n-1\}$ which verifies

$$S_{0,j_0+1} = S_{0,j_0}. \quad (2.7)$$

If $j_0 = n-1$, then $\max I(A_1, A_2; \mathcal{C}) = \{0\}$. STOP

If $j_0 < n-1$, GO TO Stage 3.

Stage 3. Construct the sequence of subspaces $(S_{i,j_0})_{0 \leq i \leq n}$:

$$S_{i,j_0} = S_{i-1,j_0} \cap A_1^{-1} S_{i-1,j_0}. \quad (2.8)$$

Stage 4. Determine i_0 , the first index in $0, 1, \dots, n-1$ which verifies

$$S_{i_0+1,j_0} = S_{i_0,j_0}. \quad (2.9)$$

Then $\max I(A_1, A_2; \mathcal{C}) = S_{i_0,j_0}$. STOP

Proof: Let us consider the doubly-indexed sequence of subspaces of the space \mathbf{K}^n

$$\tilde{S}_{i,j} = \left(\bigcap_{k=0}^i A_1^{-k} \right) \left(\bigcap_{l=0}^j A_2^{-l} \right) \mathcal{C}, \quad i, j \in \{0, 1, \dots, n\}. \quad (2.10)$$

By Proposition 2.1, $\max I(A_1, A_2; \mathcal{C}) \subseteq \tilde{S}_{i,j} \forall i, j$ and by Proposition 2.2, $\tilde{S}_{n-1,n-1} = \max I(A_1, A_2; \mathcal{C})$. Obviously

$$\tilde{S}_{i,j} \supseteq \tilde{S}_{k,l}, \quad \forall k \geq i, l \geq j \quad (2.11)$$

and $\tilde{S}_{0,0} = A_1^{-0} A_2^{-0} \mathcal{C} = \mathcal{C} = S_{0,0}$. For any $j \in \{0, 1, \dots, n\}$, $\tilde{S}_{0,j} = \bigcap_{l=0}^j A_2^{-l} \mathcal{C} = \mathcal{C} \cap A_2^{-1} \left(\bigcap_{l=0}^{j-1} A_2^{-l} \mathcal{C} \right) = \mathcal{C} \cap A_2^{-1} \tilde{S}_{0,j-1}$.

Then, if we assume that $\tilde{S}_{0,j-1} = S_{0,j-1}$, using (2.6) we get by induction the following equality:

$$\tilde{S}_{0,j} = S_{0,j}, \quad \forall j \in \{0, 1, \dots, n\}, \quad (2.12)$$

hence by (2.11)

$$S_{0,j} \supseteq S_{0,j+1}, \quad \forall j \in \{0, 1, \dots, n-1\}, \quad (2.13)$$

Using again Hamilton-Cayley Theorem (see (2.4)) and (2.13) one obtains $S_{0,n} = \tilde{S}_{0,n} = \tilde{S}_{0,n-1} = S_{0,n-1}$, hence $j_0 \leq n-1$.

Let us consider the chain of subspaces

$$\{0\} \subseteq S_{0,n} = S_{0,n-1} \subseteq S_{0,n-2} \subseteq \dots \subseteq S_{0,1} \subseteq S_{0,0} \subseteq \mathbf{K}^n.$$

Since \mathcal{C} is a proper subspace of $X = \mathbf{K}^n$ it results that $\dim S_{0,0} = \dim \mathcal{C} \leq n - 1$. If $j_0 = n - 1$ is the first index which verifies (2.7), we have

$$0 \leq \dim S_{0,n-1} < \dim S_{0,n-2} < \dots < \dim S_{0,0} \leq n - 1$$

hence $\dim S_{0,n-1} = 0$, i.e. $S_{0,n-1} = \{0\}$. Therefore one obtains by Proposition 2.1 $\{0\} = S_{0,n-1} = \tilde{S}_{0,n-1} \supseteq \max I(A_1, A_2; \mathcal{C})$, hence $\max I(A_1, A_2; \mathcal{C}) = \{0\}$.

If $j_0 < n - 1$ one obtains by $S_{0,j_0+1} = S_{0,j_0}$ and by (2.6) $S_{0,j_0+2} = \mathcal{C} \cap A_2^{-1} S_{0,j_0+1} = \mathcal{C} \cap A_2^{-1} S_{0,j_0} = S_{0,j_0+1} = S_{0,j_0}$.

Let us assume that $S_{0,j} = S_{0,j_0}$ for some $j \in \{j_0 + 1, j_0 + 2, \dots, n - 1\}$. Then, again by (2.6), $S_{0,j+1} = \mathcal{C} \cap A_2^{-1} S_{0,j} = \mathcal{C} \cap A_2^{-1} S_{0,j_0} = S_{0,j_0+1} = S_{0,j_0}$ hence

$$S_{0,j} = S_{0,j_0}, \forall j \in \{j_0 + 1, j_0 + 2, \dots, n\}. \quad (2.14)$$

Now, let us assume that $\tilde{S}_{i-1,j_0} = S_{i-1,j_0}$, for some $i \in \{1, 2, \dots, n - 1\}$. We use the obvious equality $\mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3 = (\mathcal{V}_1 \cap \mathcal{V}_2) \cap (\mathcal{V}_2 \cap \mathcal{V}_3)$, where $\mathcal{V}_k, k = 1, 2, 3$ are any subspaces of \mathbf{K}^n . By (2.10) and (2.8) one obtains:

$$\begin{aligned} \tilde{S}_{i,j_0} &= \bigcap_{k=0}^i \bigcap_{l=0}^{j_0} A_1^{-k} A_2^{-l} \mathcal{C} = (\bigcap_{k=0}^{i-1} \bigcap_{l=0}^{j_0} A_1^{-k} A_2^{-l} \mathcal{C}) \cap \\ &(\bigcap_{k=i}^i \bigcap_{l=0}^{j_0} A_1^{-k} A_2^{-l} \mathcal{C}) = (\bigcap_{k=0}^{i-1} \bigcap_{l=0}^{j_0} A_1^{-k} A_2^{-l} \mathcal{C}) \cap \\ &A_1^{-1} (\bigcap_{k=0}^{i-1} \bigcap_{l=0}^{j_0} A_1^{-k} A_2^{-l} \mathcal{C}) = \tilde{S}_{i-1,j_0} \cap A_1^{-1} \tilde{S}_{i-1,j_0} = \\ &S_{i-1,j_0} \cap A_1^{-1} S_{i-1,j_0} = S_{i,j_0}, \text{ hence } \tilde{S}_{i,j_0} = S_{i,j_0}, \forall i \in \\ &\{0, 1, \dots, n\}. \text{ It follows by (2.11) that } S_{i,j_0} \supseteq S_{i+1,j_0} \text{ and} \\ &\text{by Hamilton-Cayley Theorem that } S_{n-1,j_0} = S_{n,j_0}. \end{aligned}$$

Now, let us consider the chain of subspaces

$$\{0\} \subseteq S_{n,j_0} = S_{n-1,j_0} \subseteq S_{n-2,j_0} \subseteq \dots \subseteq S_{0,j_0} \subseteq \mathbf{K}^n.$$

Since $\dim S_{0,j_0} \leq \dim S_{0,0} \leq n - 1$, if $i_0 = n - 1$ we obtain as above that $\dim S_{n-1,j_0} = 0$, hence $S_{n-1,j_0} = \max I(A_1, A_2; \mathcal{C}) = \{0\}$.

If $i_0 < n - 1$, we have by $\tilde{S}_{i_0+1,j_0} = S_{i_0+1,j_0} = S_{i_0,j_0} = \tilde{S}_{i_0,j_0}$ and by (2.9) the following equalities: $\tilde{S}_{i_0+2,j_0} = \tilde{S}_{i_0+1,j_0} \cap A_1^{-1} \tilde{S}_{i_0+1,j_0} = \tilde{S}_{i_0,j_0} \cap A_1^{-1} \tilde{S}_{i_0,j_0} = \tilde{S}_{i_0+1,j_0} = \tilde{S}_{i_0,j_0}$.

If we assume that $\tilde{S}_{i,j_0} = \tilde{S}_{i_0,j_0}$ for some $i \in \{i_0 + 1, i_0 + 2, \dots, n - 1\}$, we get again by (2.9) $\tilde{S}_{i+1,j_0} = \tilde{S}_{i,j_0} \cap A_1^{-1} \tilde{S}_{i,j_0} = \tilde{S}_{i_0,j_0} \cap A_1^{-1} \tilde{S}_{i_0,j_0} = \tilde{S}_{i_0+1,j_0} = \tilde{S}_{i_0,j_0}$, hence

$$\tilde{S}_{i,j_0} = S_{i_0,j_0}, \forall i \in \{i_0 + 1, i_0 + 2, \dots, n - 1\}. \quad (2.15)$$

Using (2.9), (2.14) and (2.15) we obtain $\bigcap_{l=0}^j A_2^{-l} \mathcal{C} = \bigcap_{l=0}^{j_0} A_2^{-l} \mathcal{C}, \forall j \in \{j_0 + 1, j_0 + 2, \dots, n\}$ and $\bigcap_{k=0}^i \bigcap_{l=0}^j A_1^{-k} A_2^{-l} \mathcal{C} = \bigcap_{k=0}^i \bigcap_{l=0}^{j_0} A_1^{-k} A_2^{-l} \mathcal{C} \forall i \in \{i_0 + 1, i_0 + 2, \dots, n\}$.

Then, $\forall i \in \{i_0 + 1, i_0 + 2, \dots, n\}$ and $\forall j \in \{j_0 + 1, j_0 + 2, \dots, n\}$, one obtains $\tilde{S}_{i,j} = \bigcap_{k=0}^i \bigcap_{l=0}^j A_1^{-k} A_2^{-l} \mathcal{C} = \bigcap_{k=0}^i A_1^{-k} (\bigcap_{l=0}^j A_2^{-l} \mathcal{C}) = \bigcap_{k=0}^i A_1^{-k} (\bigcap_{l=0}^{j_0} A_2^{-l} \mathcal{C}) = \bigcap_{k=0}^{i_0} \bigcap_{l=0}^{j_0} A_1^{-k} A_2^{-l} \mathcal{C} = S_{i_0,j_0}$, hence $\tilde{S}_{i,j} = S_{i_0,j_0}, \forall i \in \{i_0 + 1, i_0 + 2, \dots, n\}, \forall j \in \{j_0 + 1, j_0 + 2, \dots, n\}$.

By Proposition 2.2, we obtain

$$\max I(A_1, A_2; \mathcal{C}) = \tilde{S}_{n-1,n-1} = S_{i_0,j_0}, \quad (2.16)$$

which completes the proof of the algorithm.

Given the $n \times n$ matrices A_1, A_2 and the $n \times k$ matrix Cal , the next *Matlab* program calculates an orthonormal basis in the maximal subspace $\max I(A_1, A_2; \text{Cal})$ and its dimension:

```
S = ima(Cal, 0); [n, dimInv] = size(S);
for i= 2:n S= ints(S, invt(A2, S));
[n, ml] = size(S); if ml==dimInv break
else dimInv = ml; end
end
for j= 2:n S= ints(S, invt(A1, S));
[n, ml] = size(S); if dimInv==ml break
else dimInv = ml; end
end
disp('maxI(A1, A2;Cal) has the dim =')
disp( num2str(dimInv))
disp(['and an orthonormal basis for'...
' maxI(A1, A2;Cal) is:'], disp(S))
```

This program will be further adapted for the algorithm in the last section and a concrete example will be given there.

From (2.10), (2.16) and $\tilde{S}_{i_0,j_0} = S_{i_0,j_0}$ one obtains:

Proposition 2.3: *The maximal (A_1, A_2) -invariant subspace included in the subspace \mathcal{C} is*

$$\max I(A_1, A_2; \mathcal{C}) = \tilde{S}_{i_0,j_0} = \bigcap_{i=0}^{i_0} \bigcap_{j=0}^{j_0} A_1^{-i} A_2^{-j} \mathcal{C}. \quad (2.17)$$

3. THE STATE SPACE REPRESENTATION OF THE 2D HYBRID SYSTEMS

The linear spaces $X = \mathbf{R}^n, U = \mathbf{R}^m$ and $Y = \mathbf{R}^p$, are called respectively the *state, input and output spaces* and $T = \mathbf{R} \times \mathbf{Z}$ is the *time set*. By $(s, l) \leq (t, k)$ for $(s, l), (t, k) \in T$ we mean $s \leq t, l \leq k$ and $(s, l) < (t, k)$ means $(s, l) \leq (t, k)$ and $(s, l) \neq (t, k)$.

Definition 3.1: A two-dimensional hybrid (continuous-discrete) linear system (2Dcd) is a quintuplet

$\Sigma = (A_1(t, k), A_2(t, k), B(t, k), C(t, k), D(t, k)) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times m} \times \mathbf{R}^{p \times n} \times \mathbf{R}^{p \times m}$ with $A_1(t, k)A_2(t, k) = A_2(t, k)A_1(t, k) \forall (t, k) \in T$, where all matrices are continuous with respect to $t \in \mathbf{R}$ for any $k \in \mathbf{Z}$; the state space representation of Σ is given by the following state and output equations (where $\dot{x}(t, k) = \frac{\partial x}{\partial t}(t, k)$).

$$\begin{aligned} \dot{x}(t, k+1) &= A_1(t, k+1)x(t, k+1) + A_2(t, k)\dot{x}(t, k) \\ &\quad - A_1(t, k)A_2(t, k)x(t, k) + B(t, k)u(t, k) \end{aligned} \quad (3.1)$$

$$y(t, k) = C(t, k)x(t, k) + D(t, k)u(t, k). \quad (3.2)$$

The vectors $x(t, k) \in X, u(t, k) \in U$ and $y(t, k) \in Y$ are respectively the *state, the input and the output* of the system Σ at the moment $(t, k) \in T$.

We denote by $\Phi(t, t_0; k)$ or $\Phi_{A_1}(t, t_0; k)$ the *continuous fundamental matrix* of $A_1(t, k)$ with respect to $t \in \mathbf{R}$, for any fixed $k \in \mathbf{Z}$. $\Phi(t, t_0; k)$ has the following properties, for

any $t, t_0, t_1 \in \mathbf{R}$:

- i) $\frac{d}{dt}\Phi(t, t_0; k) = A_1(t, k)\Phi(t, t_0; k),$
- ii) $\Phi(t_0, t_0; k) = I_n,$
- iii) $\Phi(t, t_1; k)\Phi(t_1, t_0; k) = \Phi(t, t_0; k),$
- iv) $\Phi(t, t_0; k)^{-1} = \Phi(t_0, t; k),$

$$v) \Phi(t, t_0; k) = I + \sum_{l=1}^{\infty} \int_{t_0}^t A_1(s_1, k) \int_{t_0}^{s_1} A_1(s_2, k) \cdots \int_{t_0}^{s_{l-1}} A_1(s_l, k) ds_l ds_{l-1} \cdots ds_2 ds_1.$$

If A_1 is a constant matrix, then $\Phi(t, t_0; k) = \sum_{l=0}^{\infty} \frac{A_1^l(t-t_0)^l}{l!} = e^{A_1(t-t_0)}.$

We denote by $F(t; k, k_0)$ the discrete fundamental matrix of the matrix $A_2(t, k)$, defined by

$$F(t; k, k_0) = \begin{cases} A_2(t, k-1)A_2(t, k-2) \cdots A_2(t, k_0) & \text{for } k > k_0 \\ I_n & \text{for } k = k_0 \end{cases}$$

for any fixed $t \in \mathbf{R}$.

If A_2 is a constant matrix, then $F(t; k, k_0) = A_2^{k-k_0}.$

The matrices $\Phi(t, t_0; k)$ and $F(s; l, l_0)$ commute for any $t, t_0, s \in \mathbf{R}$ and $k, l, l_0 \in \mathbf{Z}$ since $A_1(t, k)$ and $A_2(t, k)$ are commutative matrices.

Definition 3.2: A vector $x_0 \in X$ is called the initial state of Σ at the moment $(t_0, k_0) \in T$ if, for any $(t, k) \in T$ with $(t, k) \geq (t_0, k_0)$ the following conditions hold:

$$x(t, k_0) = \Phi(t, t_0; k_0)x_0, \quad x(t_0, k) = F(t_0; k, k_0)x_0. \quad (3.3)$$

For $(t_0, k_0) \leq (t, k)$ we denote by $[t_0, t; k_0, k]$ the set $[t_0, t; k_0, k] = [t_0, t] \times \{k_0, k_0 + 1, \dots, k\}.$

From [15, Proposition 2.3] we obtain:

Proposition 3.1: The state of the system Σ at the moment $(t, k) \in T$ determined by the control $u(\cdot, \cdot) : [t_0, t; k_0, k] \rightarrow U$ and by the initial state $x_0 \in X$ is

$$x(t, k) = \Phi(t, t_0; k)F(t_0; k, k_0)x_0 + \int_{t_0}^t \sum_{l=k_0}^{k-1} \Phi(t, s; k)F(s; k, l+1)B(s, l)u(s, l)ds. \quad (3.4)$$

By replacing the state $x(t, k)$ given by (3.4) in the output equation (3.2) we get

Proposition 3.2: The input-output map of the system Σ is given by the formula

$$y(t, k) = C(t, k)\Phi(t, t_0; k)F(t_0; k, k_0)x_0 + \int_{t_0}^t \sum_{l=k_0}^{k-1} C(t, k)\Phi(t, s; k)F(s; k, l+1)B(s, l)u(s, l)ds + D(t, k)u(t, k). \quad (3.5)$$

4. OBSERVABILITY OF TIME-VARYING 2D SYSTEMS

In this section we present the study of the observability of the time-varying system $\Sigma = (A_1(t, k), A_2(t, k), B(t, k), C(t, k), D(t, k))$, along the line of [16].

A triplet $(t, k, \tilde{x}) \in T \times X$ is called a phase of Σ if $\tilde{x} = x(t, k)$ (i.e. \tilde{x} is the state of Σ at the moment (t, k)). One obtains by (3.4) that (t, k, \tilde{x}) is a phase of Σ iff there exist a control $u(\cdot, \cdot)$ and an initial state $x_0 \in X$ such that

$$\tilde{x} = \Phi(t, t_0; k)F(t_0; k, k_0)x_0 + \int_{t_0}^t \sum_{l=k_0}^{k-1} \Phi(t, s; k)F(s; k, l+1)B(s, l)u(s, l)ds. \quad (4.1)$$

We denote by I the set $I = [t_0, t] \times [k_0, k] \subset T$, where $(t_0, k_0), (t, k) \in T$ and $(t_0, k_0) < (t, k)$.

Definition 4.1: A phase (t_0, k_0, x) is said to be unobservable (unobservable on I) if for any control $u(\cdot, \cdot)$ it provides the same output $y(s, l)$ for $(s, l) \geq (t_0, k_0)$ (for $(s, l) \in I$) as the phase $(t_0, k_0, 0)$.

A state $x \in X$ is said to be unobservable (unobservable on I) if there exists $(t_0, k_0) \in T$ such that the phase (t_0, k_0, x) is unobservable (unobservable on I).

Proposition 4.1: The phase (t_0, k_0, x) is unobservable if and only if

$$C(t, k)\Phi(t, t_0; k)F(t_0; k, k_0)x = 0 \quad (4.2)$$

for any $(t, k) \in T, (t, k) \geq (t_0, k_0)$.

The phase (t_0, k_0, x) is unobservable on I if and only if

$$C(s, l)\Phi(s, t_0; l)F(t_0; l, k_0)x = 0 \quad (4.3)$$

for any $(s, l) \in I$.

Proof: The output produced by the initial state x given by (3.5) is denoted $y_x(t, k)$. If we replace x_0 by 0 in (3.5) we get the zero-input response of Σ

$$y_0(t, k) = \int_{t_0}^t \sum_{l=k_0}^{k-1} C(t, k)\Phi(t, s; k)F(s; k, l+1)B(s, l) \cdot u(s, l)ds + D(t, k)u(t, k).$$

By equalizing $y_x(t, k)$ and $y_0(t, k)$ we get (4.2) ((4.3) if it is restricted to the interval I).

It results from (4.2) that the property of observability does not involve the matrices $B(t, k)$ and $D(t, k)$: therefore in the sequel we shall often consider the system Σ reduced to the triplet $\Sigma = (A_1(t, k), A_2(t, k), C(t, k))$

Definition 4.2: The system Σ is said to be completely observable (completely observable on I) if there is no unobservable (unobservable on I) state $x \neq 0$

In order to check whether a system Σ is completely observable we introduce the 2D observability Gramian of Σ denoted by $\mathcal{O}_\Sigma(t, t_0; k, k_0)$

$$\mathcal{O}_\Sigma(t, t_0; k, k_0) = \int_{t_0}^t \sum_{l=k_0}^k F(t_0; l, k_0)^T \Phi(s, t_0; l)^T \cdot C(s, l)^T C(s, l) \Phi(s, t_0; l) F(t_0; l, k_0) ds. \quad (4.4)$$

It results that $\mathcal{O}_\Sigma(t, t_0; k, k_0)$ is a symmetrical, positive semidefinite $n \times n$ matrix.

Proposition 4.2: *The phase (t_0, k_0, x) (the state x) is unobservable on I if and only if*

$$\mathcal{O}_\Sigma(t, t_0; k, k_0)x = 0. \quad (4.5)$$

Proof: Necessity. If the state x is unobservable on I then (4.3) holds for any $(s, l) \in I$, hence $\mathcal{O}_\Sigma(t, t_0; k, k_0)x = \int_{t_0}^t \sum_{l=k_0}^k F(t_0; l, k_0)^T \Phi(s, t_0; l)^T C(s, l)^T C(s, l) \Phi(s, t_0; l) \cdot F(t_0; l, k_0)x ds = 0$.

Sufficiency. If (4.5) holds, then $x^T \mathcal{O}_\Sigma(t, t_0; k, k_0)x = 0$, equality which can be written as $\int_{t_0}^t \sum_{l=k_0}^k \|C(s, l)\Phi(s, t_0; l)F(t_0; l, k_0)x\|^2 ds = 0$. Since the integrand is non-negative, one obtains (4.3), hence by Proposition 4.1 the state x is unobservable on I .

Corollary 4.1: *The set of states which are unobservable on I is the subspace $\text{Ker}\mathcal{O}_\Sigma(t, t_0; k, k_0)$.*

Corollary 4.2: *The phase (t_0, k_0, x) is unobservable if and only if, for any $(t, k) \in T, (t, k) \geq (t_0, k_0)$, $\mathcal{O}_\Sigma(t, t_0; k, k_0)x = 0$.*

Theorem 4.1: *The system $\Sigma = (A_1(t, k), A_2(t, k), C(t, k))$ is completely observable on I if and only if*

$$\text{rank}\mathcal{O}_\Sigma(t, t_0; k, k_0) = n. \quad (4.6)$$

Proof: By Corollary 4.1, Σ is completely observable on I if and only if $\text{Ker}\mathcal{O}_\Sigma(t, t_0; k, k_0) = \{0\}$, condition which is equivalent to (4.6).

Now it is possible to determine the initial state by knowing the input and the output of Σ .

Theorem 4.2: *Let Σ be completely observable on I . If the control $u(s, l)$ produces the output $y(s, l)$, $(s, l) \in I$, then the initial state x_0 is given by*

$$x_0 = \mathcal{O}_\Sigma(t, t_0; k, k_0)^{-1} \int_{t_0}^t \sum_{l=k_0}^k F(t_0; l, k_0)^T \cdot \Phi(s, t_0; l)^T C(s, l)^T \tilde{y}(s, l) ds \quad (4.7)$$

where

$$\tilde{y}(s, l) = y(s, l) - \int_{t_0}^s \sum_{\lambda=k_0}^{l-1} C(s, l)\Phi(s, \tau; l) \cdot F(\tau; l, \lambda+1)B(\tau, \lambda)u(\tau, \lambda)d\tau - D(s, l)u(s, l). \quad (4.8)$$

Proof: From (3.5) and (4.8) we obtain $C(s, l)\Phi(s, t_0; l)F(t_0; l, k_0)x_0 = \tilde{y}(s, l)$. If we premultiply this equality by $F(t_0; l, k_0)^T \Phi(s, t_0; l)^T C(s, l)^T$, then we integrate on $[t_0, t]$ and we sum up for $l = \overline{k_0, k}$, we get

$$\mathcal{O}_\Sigma(t, t_0; k, k_0)x_0 = \int_{t_0}^t \sum_{l=k_0}^k F(t_0; l, k_0)^T \cdot \Phi(s, t_0; l)^T C(s, l)^T \tilde{y}(s, l) ds \quad (4.9)$$

Since Σ is completely observable on I , by Theorem 4.1 the matrix $\mathcal{O}_\Sigma(t, t_0; k, k_0)$ is nonsingular and (4.7) results by premultiplying (4.9) by its inverse.

Remark 4.1: By replacing x_0 given by (4.7) into (3.4) one can conclude that completely observable systems allow the reconstitution of the whole trajectory of the system Σ $x(t, k), (t, k) \in T, (t, k) \geq (t_0, k_0)$.

5. OBSERVABILITY OF TIME-INVARIANT HYBRID SYSTEMS

The system Σ given by Definition 3.1 is said to be *time-invariant* (or *stationary*) if A_1, A_2, B, C, D are constant matrices. In this case we can consider the initial moment $(t_0, k_0) = (0, 0)$.

Since for constant matrices A_1 and A_2 the two fundamental matrices become $\Phi(t, s; 0) = e^{A_1(t-s)}$ and $F(0; k, l) = A_2^{k-l}, (t, k), (s, l) \in T^+ := \mathbf{R}^+ \times \mathbf{Z}^+, (t, k) \geq (s, l)$, the input-output map (3.5) can be written in the form

$$y(t, k) = Ce^{A_1 t} A_2^k x_0 + \int_0^t \sum_{l=0}^{k-1} Ce^{A_1(t-s)} A_2^{k-l-1} \cdot Bu(s, l) ds + Du(t, k). \quad (5.1)$$

Definition 5.1: Two time-invariant systems $\Sigma = (A_1, A_2, B, C, D)$ and $\tilde{\Sigma} = (\tilde{A}_1, \tilde{A}_2, \tilde{B}, \tilde{C}, \tilde{D})$ are said to be *isomorphic* if there exists a nonsingular matrix $T \in \mathbf{R}^{n \times n}$ such that

$$\tilde{A}_i = T^{-1} A_i T, i = 1, 2; \tilde{B} = T^{-1} B; \tilde{C} = CT; \tilde{D} = D.$$

These relations are the consequence of the change of basis $x = T\tilde{x}$ in the state space $X = \mathbf{R}^n$. Since $e^{\tilde{A}_1 t} = T^{-1} e^{A_1 t} T$ and $\tilde{A}_2^k = T^{-1} A_2^k T$, if we write the input-output maps of both systems Σ and $\tilde{\Sigma}$ we get

Proposition 5.1: *Two isomorphic systems have the same input-output map.*

In the case of time-invariant systems, the observability Gramian (denoted by $\mathcal{O}(t, k)$ instead of $\mathcal{O}_\Sigma(t, t_0; k, k_0)$) becomes

$$\mathcal{O}(t, k) = \int_0^t \sum_{l=0}^k (A_2^T)^l e^{A_1^T s} C^T C e^{A_1 s} A_2^l ds$$

and Proposition 4.1 and Theorem 4.1 can be restated as

Proposition 5.2: *The state $x \in X$ is unobservable if and only if*

$$Ce^{A_1 t} A_2^k x = 0 \quad (5.2)$$

for any $(t, k) \in T^+$.

Theorem 5.1: *The system $\Sigma = (A_1, A_2, C)$ is completely observable if and only if*

$$\text{rank} \mathcal{O}(t, k) = n$$

for any $(t, k) \in T^+$.

Definition 5.2: The matrix

$$O_\Sigma = [C^T \ A_1^T C^T \ \dots \ (A_1^T)^{n-1} C^T \ A_2^T C^T \ A_1^T A_2^T C^T \ \dots \\ \dots \ (A_1^T)^{n-1} A_2^T C^T \ \dots \\ (A_2^T)^{n-1} C^T \ A_1^T (A_2^T)^{n-1} C^T \ \dots \ (A_1^T)^{n-1} (A_2^T)^{n-1} C^T]^T.$$

is called the *observability matrix* of the system Σ .

Theorem 5.2: *The system $\Sigma = (A_1, A_2, C)$ is completely observable if and only if*

$$\text{rank} O_\Sigma = n. \quad (5.3)$$

Proof: Necessity. Let us assume that (5.3) fails, i.e. $\text{rank} O_\Sigma < n$. Then there exists $x \in X \setminus \{0\}$, such that $O_\Sigma x = 0$. Then $Cx = 0, CA_1 x = 0, \dots, CA_1^{n-1} A_2^{n-1} x = 0$. By Hamilton-Cayley Theorem applied to A_1 and A_2 respectively (see [2.4]), it results that $CA_1^l A_2^k x = 0$ for any $l, k \in \mathbf{N}$. Then $Ce^{A_1 t} A_2^k x = C(\sum_{l=0}^{\infty} \frac{A_1^l t^l}{l!}) A_2^k x = \sum_{l=0}^{\infty} \frac{t^l}{l!} CA_1^l A_2^k x = 0$. By Proposition 5.2 x is unobservable, hence Σ is not completely observable.

Sufficiency. Let us suppose that Σ is not completely observable. By Proposition 5.2 there exists $x \in X \setminus \{0\}$ such that $Ce^{A_1 t} A_2^k x = 0$ for any $(t, k) \in T^+$. By deriving successively this equality with respect to t and by taking $t = 0$ we get $CA_2^k x = 0, CA_1 A_2^k x = 0, \dots, CA_1^l A_2^k x = 0, \dots, l, k \in \mathbf{N}$; this implies $O_\Sigma x = 0$, hence $\text{rank} O_\Sigma < n$.

The proof of Theorem 5.2 suggests the following result:

Theorem 5.3: *The system $\Sigma = (A_1, A_2, C)$ is completely observable if and only if $Ce^{A_1 s} A_2^l x = 0, \forall (s, l) \in [0, t] \times [0, k] \subset T^+$ for some $(t, k) \in T^+$ implies $x = 0$.*

The following statement results from Theorem 5.2 and [15, Theorem 4.2], by noticing that the 2D observability matrix O_Σ of the system $\Sigma = (A_1, A_2, B, C, D)$ coincides with the 2D controllability matrix C_{Σ^*} of $\Sigma^* = (A_1^T, A_2^T, C^T, B^T, D^T)$. We say that the system Σ^* is the *dual* of Σ .

Theorem 5.4: *The system $\Sigma = (A_1, A_2, B, C, D)$ is completely observable if and only if its dual $\Sigma^* = (A_1^T, A_2^T, C^T, B^T, D^T)$ is completely reachable.*

Now we can give a characterization of the set of unobservable states, denoted by X_{uo} .

Proposition 5.3: *The set X_{uo} of all unobservable states of Σ is the maximal (A_1, A_2) -invariant subspace of X which is contained in $\text{Ker} C$.*

Proof: From the proof of Theorem 5.2 one can derive that X_{uo} is the subspace $X_{uo} = \{x \in X | CA_1^l A_2^k x = 0, \forall l, k \in \mathbf{N}\}$. Obviously, if $x \in X_{uo}$ then $l = k = 0$ implies $x \in \text{Ker} C$; moreover $A_1 x \in X_{uo}$ since $CA_1^l A_2^k (A_1 x) = CA_1^{l+1} A_2^k x = 0, \forall l, k \in \mathbf{N}$. Similarly $A_2 x \in X_{uo}$ hence X_{uo} is (A_1, A_2) -invariant and it is contained in $\text{Ker} C$.

Now, let \mathcal{V} be an (A_1, A_2) -invariant subspace of X contained in $\text{Ker} C$. Let x be an element of \mathcal{V} . Since $\mathcal{V} \subset \text{Ker} C, Cx = 0$. \mathcal{V} being (A_1, A_2) -invariant we get $A_1 x \in \mathcal{V}$ and $A_2 x \in \mathcal{V}$ and by recurrence $A_1^l A_2^k x \in \mathcal{V} \subset \text{Ker} C$, hence $CA_1^l A_2^k x = 0, \forall l, k \in \mathbf{N}$, that is $x \in X_{uo}$. Therefore $\mathcal{V} \subset X_{uo}$ and X_{uo} is the maximal such subspace of X .

By using Hamilton-Cayley Theorem for A_1 and A_2 (see [2.4]), we obtain $X_{uo} = \{x \in X | CA_1^l A_2^k x = 0, \forall l, k = 0, n-1\} = \text{Ker} O_\Sigma$. We proved:

Corollary 5.1: *The set of all unobservable states of Σ is $X_{uo} = \text{Ker} O_\Sigma$.*

We obtain from Proposition 5.3:

Theorem 5.5: *The system $\Sigma = (A_1, A_2, C)$ is completely observable if and only if $\{0\}$ is the greatest subspace of X which is (A_1, A_2) -invariant and is contained in $\text{Ker} C$.*

The next result gives the canonical form of unobservable systems, by using the geometric characterization of X_{uo} .

Theorem 5.6: *The system $\Sigma = (A_1, A_2, C)$ is not completely observable if and only if it is isomorphic to a system $\tilde{\Sigma} = (\tilde{A}_1, \tilde{A}_2, \tilde{C})$ with*

$$\tilde{A}_i = \begin{bmatrix} A_{11i} & 0 \\ A_{21i} & A_{22i} \end{bmatrix}, i = 1, 2, \quad \tilde{C} = [C_1 \ 0], \quad (5.4)$$

where $A_{11i} \in \mathbf{R}^{\tilde{n} \times \tilde{n}}, C_1 \in \mathbf{R}^{p \times \tilde{n}}$ and $\tilde{n} < n$. The triplet $\Sigma_1 = (A_{111}, A_{112}, C_1)$ is completely observable.

Proof: We obtain the direct sum decomposition $X = X_1 \oplus X_2$ where $X_2 = X_{uo} \neq \{0\}$ by considering a basis of X_2 and by completing it to a basis \mathcal{B} of X . We denote by \tilde{n} the dimension of the subspace X_1 . Since Σ is not completely observable $n - \tilde{n} = \dim X_2 > 0$, hence $\tilde{n} < n$.

In the basis \mathcal{B} we have $X_2 = \left\{ x = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \mid x_2 \in \mathbf{R}^{n-\tilde{n}} \right\}$ and we denote the corresponding matrices of Σ in this basis by \tilde{A}_1, \tilde{A}_2 and \tilde{C} . The transition matrix is denoted by T . Let us partition the matrix \tilde{C} as $\tilde{C} = [C_1 \ C_2]$ where C_1 has \tilde{n} columns. By Proposition 5.3 X_2 is included in $\text{Ker} \tilde{C}$; then for any $x_2 \in \mathbf{R}^{n-\tilde{n}}$ we have $[C_1 \ C_2] \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = 0$, i.e. $C_2 x_2 = 0$, hence C_2 is a null matrix and $\tilde{C} = [C_1 \ 0]$.

Since by Proposition 5.3 X_2 is \tilde{A}_i -invariant, $i = 1, 2$, we have $\tilde{A}_i \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \in X_2$ for any $x_2 \in \mathbf{R}^{n-\tilde{n}}$. Then if we partition \tilde{A}_i as $\tilde{A}_i = \begin{bmatrix} A_{11i} & A_{12i} \\ A_{21i} & A_{22i} \end{bmatrix}, i = 1, 2$ we

obtain $\begin{bmatrix} A_{12i}x_2 \\ A_{22i}x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ A_{22i}x_2 \end{bmatrix}$ hence $A_{12i}x_2 = 0$ for any $x_2 \in \mathbf{R}^{n-\tilde{n}}$, i.e. $A_{12i} = 0, i = 1, 2$. Therefore in the basis \mathcal{B} the system has the form (5.4).

By corollary 4.1 $\tilde{n} = \dim X_1 = n - \dim X_2 = n - \dim \text{Ker} O_\Sigma = \text{rank} O_\Sigma = \text{rank} O_{\tilde{\Sigma}}$ since $O_{\tilde{\Sigma}} = O_\Sigma T$ and T is nonsingular. From (5.4) we get $\tilde{C}A_i^k = [C_1A_{11i}^k \ 0], \forall k \in \mathbf{Z}^+$. Then $O_{\tilde{\Sigma}}$ has the structure $O_{\tilde{\Sigma}} = \begin{bmatrix} C_1A_{11i}^k & 0 \end{bmatrix}_{0 \leq k \leq n-1}$. By Hamilton-Cayley Theorem, for any $j \in \mathbf{N}, A_{11i}^{\tilde{n}-1+j}$ is a linear combination of $A_{11i}^l, l = 0, \tilde{n} - 1$, hence

$$\begin{aligned} \tilde{n} &= \text{rank} O_{\tilde{\Sigma}} = \text{rank} \begin{bmatrix} C_1A_{11i}^k & 0 \end{bmatrix}_{0 \leq k \leq \tilde{n}-1} = \\ &= \text{rank} [C_1A_{11i}^k]_{0 \leq k \leq \tilde{n}-1}; \end{aligned}$$

But $[C_1A_{11i}^k]_{0 \leq k \leq \tilde{n}-1}$ is the observability matrix O_{Σ_1} of the system $\Sigma_1 = (A_{111}, A_{112}, C_1)$. By Theorem 5.2, $\text{rank} O_{\Sigma_1} = \tilde{n}$ implies the complete observability of Σ_1 .

This Theorem can be restated as

Theorem 5.7: *The system $\Sigma = (A_1, A_2, C)$ is completely observable if and only if it is not isomorphic to a system of the form (5.4).*

Theorem 5.8: *The system $\Sigma = (A_1, A_2, C)$ is completely observable if and only if there is no common eigenvector of the matrices A_1 and A_2 belonging to $\text{Ker} C$.*

Proof: Necessity. Let us assume that $\exists x \in \mathbf{R}^n \setminus \{0\}$ such that $A_1x = \lambda_1x, A_2x = \lambda_2x$ for some $\lambda_1, \lambda_2 \in \mathbf{C}$ and $Cx = 0$. Then $CA_1^i A_2^j x = \lambda_1^i \lambda_2^j Cx = 0$ for any $i, j \geq 0$; we get $O_\Sigma x = 0$, hence $\text{rank} O_\Sigma < n$ and by Theorem 5.2 Σ is not completely observable.

Sufficiency. Let us assume that Σ is not completely observable. We denote by S_1 the subspace of $\mathbf{R}^n, S_1 = \text{Ker} O_\Sigma$. Since $\text{rank} O_\Sigma < n, \exists x \in \mathbf{R}^n \setminus \{0\}$ such that $O_\Sigma x = 0$, hence S_1 contains nonzero vectors. For any $x \in S_1$ we get $CA_1^i A_2^j x = 0, i, j = \overline{0, n-1}$ and by Hamilton-Cayley Theorem, applied to matrices A_1 and A_2 , this equality is true $\forall i, j \geq 0$, hence $S_1 = \{v \in \mathbf{R}^n | CA_1^i A_2^j x = 0, \forall i, j \geq 0\}$.

We can prove that S_1 is (A_1, A_2) -invariant. Indeed, if $x \in S_1$, then for any $i, j \in \mathbf{N}$ we have $CA_1^i A_2^j (A_1x) = CA_1^{i+1} A_2^j x = 0$, hence $A_1x \in S_1$ and analogously $A_2x \in S_1$. Since S_1 is a proper A_1 -invariant subspace of \mathbf{R}^n it contains an eigenvector x_1 of A_1 ; let λ be the corresponding eigenvalue of A_1 . Let S_2 be the subspace $S_2 = \{v \in \mathbf{R}^n | A_1v = \lambda v\}$; obviously $0 \neq x_1 \in S_2$. S_2 is A_2 -invariant since $x \in S_2$ implies $A_1(A_2x) = A_2(A_1x) = \lambda(A_2x)$, hence the subspace $S_3 := S_1 \cap S_2$ is A_2 -invariant and contains the nonzero vector x_1 . It results that S_3 contains an eigenvector x_2 of A_2 which is also an eigenvector of A_1 since $S_3 \subset S_2$. This eigenvector belongs to $\text{Ker} O_\Sigma$ since $x_2 \in S_3 \subset S_1$ implies $O_\Sigma x = 0$ and for $i = j = 0$ we get $Cx_2 = 0$.

A consequence of this Theorem is a Popov-Hautus-Belevitch type criterion of observability:

Theorem 5.9: *The system $\Sigma = (A_1, A_2, C)$ is completely observable if and only if for any $s_1, s_2 \in \mathbf{C}$*

$$\text{rank} \begin{bmatrix} C \\ s_1I - A_1 \\ s_2I - A_2 \end{bmatrix} = n. \quad (5.5)$$

Proof: By Theorem 5.8 the statement that Σ is not completely observable is equivalent to the existence of $x \in \mathbf{R}^n \setminus \{0\}$ with $Cx = 0, A_i x = \lambda_i x, i = 1, 2$ for some $\lambda_1, \lambda_2 \in \mathbf{C}$, that is such that $Cx = 0, (\lambda_1 I - A_1)x = 0, (\lambda_2 I - A_2)x = 0, x \neq 0$; this is equivalent to the fact that the matrix in (4.7) has the rank less than n .

Since $\text{rank}(s_i I - A_i) = n \ \forall s_i \notin \sigma(A_i)$, Theorem 5.9 can be restated as

Corollary 5.2: *The system $\Sigma = (A_1, A_2, C)$ is completely observable if and only if (5.5) holds for any $s_1 \in \sigma(A_1), s_2 \in \sigma(A_2)$.*

6. THE DETERMINATION OF THE SUBSPACE OF UNOBSERVABLE STATES

Let us consider an LTI system $\Sigma = (A_1, A_2, C) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times n} \times \mathbf{R}^{p \times n}$ with A_1, A_2 commutative matrices. We will adapt Algorithm 1 to determine the subspace X_{uo} of unobservable states of the system Σ .

Algorithm 6.1

Stage 1. Determine the observability matrix O_Σ .

Stage 2. Compute $\text{rank} O_\Sigma$.

If $\text{rank} O_\Sigma = n$, then $X_{uo} = \{0\}$. STOP

If $\text{rank} O_\Sigma < n$ GO TO Stage 3.

Stage 3. Construct the sequence of subspaces $(S_{0,j})_{0 \leq j \leq n-2}$ of the space $X = \mathbf{R}^n$:

$$S_{0,0} = \text{Ker} C; \quad (6.1)$$

$$S_{0,j} = \text{Ker} C \cap A_2^{-1} S_{0,j-1}, j = 1, \dots, n-2. \quad (6.2)$$

Stage 4. Determine j_0 , the first index in $\{0, 1, \dots, n-2\}$ which verifies

$$S_{0,j_0+1} = S_{0,j_0}. \quad (6.3)$$

Stage 5. Construct the sequence of subspaces $(S_{i,j_0})_{0 \leq i \leq n-2}$ of the space $X = \mathbf{R}^n$:

$$S_{i,j_0} = S_{i-1,j_0} \cap A_1^{-1} S_{i-1,j_0}. \quad (6.4)$$

Stage 6. Determine i_0 , the first index in $\{0, 1, \dots, n-2\}$ which verifies

$$S_{i_0+1,j_0} = S_{i_0,j_0}. \quad (6.5)$$

Then $X_{uo} = S_{i_0,j_0}$. STOP

Proof: By Proposition 5.3, $X_{uo} = \max I(A_1, A_2; C)$ where $C = \text{Ker} C$.

If $\text{rank} O_\Sigma = n$, then Σ is completely observable (by Theorem 5.1), hence $X_{uo} = \{0\}$ (and in this case i_0 or j_0

would be equal to $n - 1$). Otherwise, $i_0 < n - 1$, $j_0 < n - 1$ and $\max I(A_1, A_2; C) = S_{i_0, j_0}$, hence $X_{uo} = S_{i_0, j_0}$.

The *Matlab* program presented below and based upon the algorithm above calculates an orthonormal basis of the unobservable states subspace and its dimension for the bi-dimensional case.

The instructions make use of the m-functions *ima*, *ints* and *invt* included in the Geometric Approach toolbox published by G. Marro and G. Basile at <http://www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm>; this GA toolbox works with Matlab 5, Matlab 6 and Matlab 7 and the Control System Toolbox.

More precisely, given the matrices A_1 , A_2 that commute and the matrix C , the next commands will compute and display the dimension and an orthonormal basis in the space $S = \max I(A_1, A_2; B)$.

```
n= length(A1); Osigma=[];
for k = 0:n-1; for l = 0:n-1
Osigma = [Osigma ; C*A2^l*A1^k]; end
end
r = rank(Osigma);
if (r < n); S = ker(C);
[n, dimInv] = size(S);
for i= 2:n; S = ints(S, invt(A2, S));
[n, m1] = size(S);
if m1 == dimInv break
else dimInv = m1; end
end
for j= 2:n; S = ints(S, invt(A1, S));
[n, m1] = size(S);
if dimInv == m1 break
else dimInv = m1; end
end
disp(['The dimension of the '...
'unobservable space is '])
disp([ num2str(dimInv)])
disp('and an orthonormal basis is:')
disp(S)
else
disp('Completely observable system.')
end
```

For example, given the matrices

$$A_1 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -2 & 5 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 0 & 0 & 5 \\ -3 & 0 & 0 & -2 \end{bmatrix},$$

the above Matlab program gives the answers:

The dimension of the unobservable space is 1 and an orthonormal basis is:

$$B = \{[0 \ 1 \ 0 \ 0]^T\}.$$

7. CONCLUSION

The minimal subspace which is invariant with respect to some commutative matrices and which is included in a given subspace is determined by a suitable algorithm. This algorithm is applied to determine the subspace of the unobservable states of a hybrid 2D system. The state space representation of these systems is studied and observability criteria are obtained. Necessary and sufficient conditions of observability are derived for LTI 2D systems as well as the characterization of the unobservable states subspace.

These results and the proposed algorithms can be extended to (q, r) -D systems with $(q, r) > (1, 1)$, i.e. to hybrid systems with q continuous-time and r discrete-time variables.

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