Sufficient Condition for Min-Plus Deconvolution to Be Closed in the Service-Curve Set in Computer Networks

Ming Li [1] and Wei Zhao [2]

***Abstract*—** This paper studies the inverse of min-plus convolution, i.e., min-plus deconvolution, in the set of non-negative, wide-sense increasing and causal functions. A sufficient condition for min-plus deconvolution to be closed in this set of functions is presented. Possible application of min-plus deconvolution to the service curve design is discussed.

***Keywords*—** Min-plus convolution, inverse problem, computer networks, real-time communications.

I. Introduction

NETWORK calculus gains applications to real-time communications, intrusion detection, and so forth, see e.g. [1-6], where min-plus convolution plays a role. Define the operation ⊗ by

$$f(t)\otimes g(t) = \inf_{0\le u\le t} \{f(u) + g(t-u)\}. \quad (1)$$

Then, the symbol ⊗ represents the operation of min-plus convolution, which corresponds to ordinary convolution in ordinary linear systems. Purely in mathematics, functions involved in $f(t) \otimes g(t)$ can be any real functions. Nevertheless, taking into account the practical computer networks, we only consider functions that are non-negative, wide sense increasing and causal. By wide sense increasing, we mean $f(s) \le f(t)$ for $s \le t$. By causal, we mean $f(t) = 0$ for $t < 0$ [7, p. 13]. Examples of such functions are arrival curves of traffic, service curves, and departure curves of servers [4]. For simplicity and without confusion causing, we call the set of such functions "service-curve set" and denote it by $\mathcal{G}$.

Example 1: Let $f(t) = t^2$ for $t > 0$ and 0 elsewhere. Then, $f(t) \otimes f(t) = t^2/2$. □

The practical significance of min-plus convolution in computer networks is to linearize servers in series so that servers in series with each other are linearly connected in the sense of min-plus convolution. Let $S_i(t)$ be the service curve of the ith server in series ($i = 1, 2, \ldots, I$). Then, two servers in series, say the ith server and the (i + 1)th one, see Fig. 1, construct a server that has the service curve $S_{i,\,i+1}(t)$ given by

$$S_{i,\,i+1}(t) = S_i(t) \otimes S_{i+1}(t). \quad (2)$$

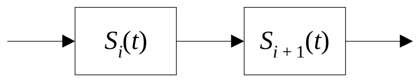

Fig. 1. Two servers in series.

We now consider the issue of the inverse of ⊗. Given $S_{i,\,i+1}(t)$ and $S_i(t)$, find $S_{i+1}(t)$ in the sense of min-plus convolution. The solution to this issue is desired for the network control. Mathematically, it relates to the inverse operation of ⊗. It is termed min-plus deconvolution.

Denote the inverse of ⊗ by ∅. Then [1,3,8], the operation of ∅ is traditionally expressed by

$$f(t) \varnothing g(t) = \sup_{u\ge 0}\{ f(t+u) - g(u)\}. \quad (3)$$

Note that ⊗ is closed in $\mathcal{G}$ for $f, g \in \mathcal{G}$ but in general, unfortunately, ∅ in Eq. (3) may not be closed as can be seen from the following example.

Example 2: Denote the affine function by

$$f(t) = kt + b \text{ for } t > 0 \text{ and } 0 \text{ elsewhere}, \quad (4)$$

where $k, b > 0$ are constants. Denote the rate-latency function by

$$g(t) = K(t - \mathrm{T})u(t - \mathrm{T}), \quad (5)$$

where K, $\mathrm{T} > 0$ are constants and $u(t)$ the Heaviside unit step function. Two functions are obviously elements in $\mathcal{G}$. However,

$$f(t) \varnothing g(t) = b + f(t + \mathrm{T}) \notin \mathcal{G}, \quad (6)$$

since it is no longer causal. □

The example 2 implies a difficulty to consider min-plus deconvolution in computer networks. For that reason, we greatly desire to find the conditions for ∅ to be closed in $\mathcal{G}$. To our best knowledge, reports in this regard are rarely seen. This paper may be the first attempt in introducing a sufficient condition of ∅ to be closed in $\mathcal{G}$ in the next section. In Section 3, we shall explain the algorithms for the min-plus convolution and its inverse. Discussions are given in Section 4. Finally, Section 5 concludes the paper.

Manuscript received March 2, 2007; Revised version received Sept. 29, 2007. This work was supported in part by the National Natural Science Foundation of China under the project grant number 60573125. Wei Zhao's work was also partially supported by the NSF (USA) under Contracts 0808419, 0324988, 0721571, and 0329181. Any opinions, findings, conclusions, and/or recommendations in this paper, either expressed or implied, are those of the authors and do not necessarily reflect the views of the agencies listed above.

[1] Ming Li (corresponding author) is with the School of Information Science & Technology, East China Normal University, No. 500, Dong-Chuan Road, Shanghai 200241, PR. China. (Tel.: +86-21-5434 5193; fax: +86-21-5434 5119; e-mails: ming_lihk@yahoo.com, mli@ee.ecnu.edu.cn).

[2] Wei Zhao is with Rensselaer Polytechnic Institute, 110 Eighth Street, 1C05 Science, Troy, NY 12180-3590, USA. (e-mail: zhaow3@rpi.edu).

II. Sufficient Condition

We first discuss the existence of min-plus deconvolution in $\mathcal{G}$. Then, a sufficient condition for $\varnothing$ to be closed in $\mathcal{G}$ is derived.

Define $\wedge$ such that

$$f \wedge g = \inf[f, g] \text{ for } f, g \in \mathcal{G}. \tag{6}$$

To facilitate discussions, we list some properties of $\otimes$ in the algebra system $(\mathcal{G}, \wedge, \otimes)$ ([8, p. 135]).

Lemma 1 (Closure of $\otimes$): Let $f, g \in \mathcal{G}$. Then, $f \otimes g \in \mathcal{G}$. □

Lemma 2: $\otimes$ with respect to $\wedge$ is distributive. That is, for $f, g, h \in \mathcal{G}$,

$$(f \wedge g) \otimes h = (f \otimes h) \wedge (g \otimes h). \tag{7}$$

□

Lemma 3: The operation $\otimes$ is commutative. That is,

$$f \otimes g = g \otimes f \text{ for } f, g \in \mathcal{G}. \tag{8}$$

□

Lemma 4: For $K \in \mathbb{R}$,

$$(f + K) \otimes g = f \otimes g + K. \tag{9}$$

□

Definition [9]: A function of rapid decay is a smooth function $\phi: \mathbb{R} \to \mathbb{C}$ such that $t^n \phi^{(r)}(t) \to 0$ as $t \to \pm\infty$ for all $n, r \geq 0$, where $\mathbb{C}$ is the space of complex numbers. The set of all functions of rapid decay is denoted by $\mathcal{S}$. □

Lemma 5 [9]: Every function belonging to $\mathcal{S}$ is absolutely integrable. □

Now, define the norm and inner product of $f \in \mathcal{G}$ by

$$\| f \|^2 = \langle f, f \rangle = \int_0^{+\infty} f^2(u) w(u) du, \tag{10}$$

where $w \in \mathcal{S}$. Combining any $f \in \mathcal{G}$ with its limit, $\mathcal{G}$ is a Hilbert space.

Let $g \in \mathcal{G}$ be a system function such that it transforms its input $f \in \mathcal{G}$ to the output by

$$y = (f \otimes g) \in \mathcal{G}. \tag{11}$$

Denote the system by the operator L. Then, we purposely force the functionality of L such that it maps an element $f \in \mathcal{G}$ to another element $(f \otimes g) \in \mathcal{G}$.

Note that L is a linear operator. In fact, according Lemma 2, we have

$$L(f \wedge g) = L(f) \wedge L(g). \tag{12}$$

In addition, from Lemma 4, one has

$$L(f + K) = L(f) + K. \tag{13}$$

Therefore, L is a linear mapping from $\mathcal{G}$ to $\mathcal{G}$.

Denote the space consisting of all such operators by

$$\mathcal{L}(\mathcal{G}, \mathcal{G}) = \mathcal{L}(\mathcal{G}).$$

From Lemmas 2 and 4, one can easily see that $\mathcal{L}(\mathcal{G})$ is a linear space.

Lemma 6 (Archimedes criterion): For any real number $a > 0$ and $b \in \mathbb{R}$, there exists positive integer $n \in \mathbb{N}$ such that $na > b$ [10, Chap. 15]. □

Lemma 7 (Archimedes): If $b \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $b < n$ [11]. □

Lemma 8: An operator $T: X \mapsto Y$ is invertible if and only if there exists constant $m > 0$ such that for all $x \in X$, $\| Tx \| \geq m \| x \|$, where X and Y are linear normed spaces [12]. □

From the above discussions, we obtain the following theorem.

Theorem 1 (Existence of min-plus deconvolution): For $f, g \in \mathcal{G}$ and $f(0) \neq 0$ and $g(0) \neq 0$, if $L(f) = f \otimes g$ or $L_1(g) = g \otimes f$, then both L and L_1 are invertible.

Proof: Consider

$$\| Lf \| = \sqrt{\| f \otimes g \|} = \sqrt{\int_0^\infty [\inf\{f(u) + g(t-u)\}]^2 w(u) du}.$$

Since

$$\inf\{f(u) + g(t-u)\} \geq \inf\{f(u)\} = f(0)$$

and $f(u) \in \mathcal{G}$, one has

$$0 < f(0) \leq f(u).$$

According to Lemmas 6 and 7, there exists $m > 0$ such that

$$f(0) \geq m^2 f(u).$$

Therefore,

$$\| Lf \| \geq \sqrt{\int_0^\infty [\inf\{f(u)\}]^2 w(u) du} = \sqrt{\int_0^\infty f(0)^2 w(u) du}$$
$$\geq m \sqrt{\int_0^\infty f(u)^2 w(u) du} = m \| f \|.$$

Similarly, if $L_1 \in \mathcal{L}(\mathcal{G})$ is such that $L_1(g) = g \otimes f$, we have $\|L_1 g\| \geq m_1 \|g\|$ since $g(0) \neq 0$, where $m_1 > 0$ is a constant. Thus, according to Lemma 8, Theorem 1 holds. □

Theorem 1 exhibits the existence of L^{-1} that corresponds to $\varnothing$. The following theorem gives the sufficient condition that $\varnothing$ is closed in $\mathcal{G}$.

Theorem 2: A sufficient condition for $\varnothing$ to be closed in $\mathcal{G}$ is $f(0) \neq 0$ and $g(0) \neq 0$.

Proof: Note that $\| Lf \|$ and $\| f \|$ are finite. From Theorem 1, we have

$$\| Lf \| \geq m \| f \|$$

Thus,

$$\frac{\| Lf \|}{\| f \|} = M > 0.$$

This means L is bounded.

Define

$$\| L \| = \sup_{f \neq 0} \frac{\| Lf \|}{\| f \|}.$$

As $\mathcal{G}$ is a Hilbert space and also Banach space, $\mathcal{L}(\mathcal{G})$ is a Banach space. According to the inverse theorem by Banach, therefore, L^{-1} is bounded. Recall that

$$L(f) = (f \otimes g) \in \mathcal{G}.$$

Thus,

$$L^{-1} L(f) = y \varnothing g = f \in \mathcal{G}.$$

Similarly, suppose $L_1 \in \mathcal{L}(\mathcal{G})$ is such that

$$L_1(g) = g \otimes f.$$

Then, $\|L_1 g\| \geq m_1 \|g\|$ according to Theorem 1. Thus, we have

$$\|L_1 g\| = M_1 \|g\|, \text{ where } M_1 > 0.$$

Consequently, L_1 is bounded. Hence,

$$L_1^{-1} L_1(g) = g \in \mathcal{G}.$$

□

III. Algorithms

Note that it may not be easy, in general, to carry out analytic solutions from either min-plus convolution or its inverse for f, $g \in \mathcal{G}$. Therefore, we give their numeric solutions by Algorithm 1 and Algorithm 2, respectively.

```
Algorithm 1 (Min-plus convolution):
Input Arguments:
f(t), g(t)
end:    index of the end of f(t) and g(t)
Output Arguments:
X(t):   Result of f(t) ⊗ g(t)

Variables:
t, u:   integers
temp[]:   one-dimensional matrix

BEGIN
  X(t):=0, t < 0
  For t:=0 to end do
    For u:=0 to t do
      temp[u]:=f(u)+g(t-u)
    END FOR
    X(t):=min(temp[:])
  END FOR
END
```

```
Algorithm 2 (Demin-plus convolution):
Input Arguments:
f(t), g(t)
left end:  index of the left end of f(t) and g(t)
right end: index of the right end of f(t) and g(t)

Output Arguments:
Y(t):   Result of f(t) ∅ g(t)
Variables:
t, u:   integers
temp[]:      one-dimensional matrix

BEGIN
  For t:=leftend to rightend do
    IF (t<=0)
      For u:=0 to rightend
 do

  temp[u]:=f(t+u)+g(u)
      END FOR
    ELSE
      For u:=0 to rightend-t do

  temp[u]:=f(t+u)+g(u)
      END FOR
    END IF
    Y(t):=max(temp[:])
  END FOR
END
```

IV. Discussiosn

Having proved a sufficient condition of the min-plus deconvolution to be closed in $\mathcal{G}$, we attempt to explain the practical significance of the present result. In doing so, we first brief some preliminaries of the relationship between the input and output of a server.

Considering an application sends a series of packets from the source to the destination through a network, the network is decomposed into a sequence of servers. The servers, in this sense, are classified into two categories, namely, constant servers and variable servers. By constant server, we mean that it imposes a constant delay to each packet and does not modify the traffic flow characteristics of a connection. Examples of constant servers are physical links, input ports, and most common switching fabrics. Variable servers, on the other side, add a non-constant delay to each packet, and so modify the traffic characteristics of connections. An example of the variable servers is output port, which acts a multiplexor. It may simultaneously receive packets belonging to different connections competing for transmission on the link associated with the port. Therefore, packet blocking may occur and packets may be forwarded in an order that is determined by the scheduling policy adopted by the switch.

From a view of communication, constant servers do not affect the traffic flows and they need not be further considered to be involved in the analysis of servers. In this paper, therefore, a network is considered to consist of a sequence of variable servers, and servers for short unless otherwise stated.

Note that a server serves arrival traffic on an interval-by-interval basis. Let $a_j^i(t)$ be instantaneous arrival traffic, implying the bytes of a packet at time t from connection j at the input port of the server i with the service curve $S_i(t)$. Then, the accumulated function regarding $a_j^i(t)$ in the time interval $[0, t]$ is given by

$$A_j^i(t) = \int_0^t a_j^i(t)dt. \tag{10}$$

Thus, $A_j^i(t)$ is always wide-sense increasing and it is assumed to be causal with the starting time $t = 0$.

Denote $D_j^i(t)$ the accumulated function characterizing the departing the server i (Fig. 2). Then, min-plus convolution provides a tool to establish the relationship between $A_j^i(t)$, $S_i(t)$, and $D_j^i(t)$ by

$$D_j^i(t) \ge A_j^i(t) \otimes S_i(t) = \inf_{0\le u\le t} \{S_i(u) + A_j^i(t-u)\}. \tag{11}$$

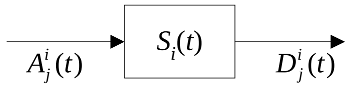

Fig. 2. Single server with arrival and departure traffic.

Suppose a traffic series passes through I servers from the first server with the service curve $S_1(t)$ to the Ith server with the service curve $S_I(t)$ to reach the destination (Fig. 3). Denote the departure traffic of the Ith server by $D_j^I(t)$. Then,

$$D_j^I(t) \geq A_j^1(t) \otimes S_I^1(t) = \inf_{0 \leq u \leq t} \{ S_I^1(u) + A_j^1(t-u) \}, \quad (12)$$

where

$$S_I^1(t) = S_1(t) \otimes S_2(t) \otimes \ldots \otimes S_i(t) \ldots \otimes S_I(t). \quad (13)$$

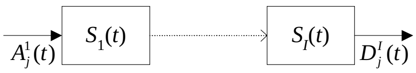

Fig. 3. A series of servers with arrival and departure traffic.

Taking into account the inverse of $\otimes$, the practical significance of the present result can be in the aspect of service curve design. Given $S_I^1(t)$ and $S_I(t)$, there exists a function in $\mathcal{G}$ such that

$$S_{I-1}^1(t) = S_I^1(t) \; \varnothing \; S_I(t), \quad (14)$$

where

$$S_{I-1}^1(t) = S_1(t) \otimes S_2(t) \otimes \ldots \otimes S_i(t) \ldots \otimes S_{I-1}(t). \quad (15)$$

Repeating the above procedure, one can determine each service curve $S_i(t)$.

Our future work will focus on the computational formula of the min-plus deconvolution instead of Eq. (3).

V. Conclusion

We have given a sufficient condition for min-plus deconvolution to be closed in $\mathcal{G}$. The potential application of the min-plus deconvolution in service curve design has been discussed.

References

[1] C. Li and E. Knightly, "Schedulability criterion and performance analysis of coordinated multihop schedulers," *IEEE/ACM Trans. Networking*, Vol. 13, No. 2, 2005, pp. 276-287.
[2] H. Zhang, "Service disciplines for guaranteed performance service in packet-switching networks," *Proc. the IEEE*, Vol. 83, No. 10, 1995, pp. 1374-1396.
[3] R. Agrawal, F. Baccelli, and R. Rajan, *An Algebra for Queueing Networks with Time Varying Service and Its Application to the Analysis of Integrated Service Networks*, RR-3435, INRIA, 1998.
[4] R. L. Cruz, "A calculus for network delay, part i: network elements in isolation, part ii: network analysis," *IEEE Trans. Information Theory*, Vol. 37, No. 1, 1991, pp. 114-141.
[5] M. Li and W. Zhao, "A statistical model for detecting abnormality in static-priority scheduling networks with differentiated services," *CIS 2005*, Springer LNAI 3802, Dec. 2005, 267-272.
[6] S. Wang, D. Xuan, R. Bettati, and W. Zhao, "Providing absolute differentiated services for real-time applications in static-priority scheduling networks," *IEEE/ACM Trans. Networking*, Vol. 12, No. 2, 2004, pp. 326-339.
[7] A. Papoulis, *The Fourier Integral and Its Applications*, McGraw-Hill, 1962.
[8] J.-Y. Boudec and P. Thiran, *Network Calculus, A Theory of Deterministic Queuing Systems for the Internet*, Springer LNCS 2050, 2003.
[9] D. H. Griffel, *Applied Functional Analysis*, John Wiley & Sons, 1981.
[10] A. D. Aleksandro, et al., *Mathematics, Its Essence, Methods and Role*, Vol. 3, Publisher USSR Academy of Sciences, 1952.
[11] R. G. Bartle and D. R. Sherbert, *Introduction to Real Analysis*, 3rd Edition, John Wiley & Sons, 2000.
[12] VI Istratescu, *Introduction to Linear Operator Theory*, NY, USA, Marcel Dekker, 1981.

Ming Li was born in 1955 in Wuxi, China. He completed his undergraduate program in electronics engineering at Tsinghua University. He received the M.S. degree in ship structural mechanics from China Ship Scientific Research Center (CSSRC) and Ph.D. degree in computer science from City University of Hong Kong, respectively. From 1990 to 1995, he was a researcher in CSSRC. From 1995-1999, he was with the Automation Department, Wuxi University of Light Industry. From 2002 to 2004, he was with the School of Computing, National University of Singapore.

In 2004, he joined East China Normal University (ECNU) as a professor in both electronics engineering and computer science. He is currently a Division Head for Communications & Information Systems at ECNU. His research areas relate to applied statistics, computer science, measurement & control. He has published over 70 papers in international journals and international conferences in those areas. Li is a member of IEEE, member of the Chinese Institute of Electronics, member of the Chinese Society of Theoretical and Applied Mechanics, and senior member of the China Electrotechnical Society.

Wei Zhao is a professor of computer science and the dean for the School of Science at Rensselaer Polytechnic Institute. His research interests include distributed computing, real-time systems, computer networks, and cyberspace security. Zhao received a PhD in computer and information sciences from the University of Massachusetts, Amherst. He is a Fellow of the IEEE. He has published over 280 papers in international journals, international conferences, and book chapters.