Economic-mathematical Models of Progressive Income Taxation in Russia

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Abstract— Transition methods from flat to progressive linear and nonlinear tax scales of income taxation, on the basis of the offered mathematical tools, are developed in this article. The generalized mathematical model of transition to the progressive taxation is offered. This model allows estimating all possible options of alleged reform of a flat scale transformation. The probabilistic model of the income taxation considering possibility of tax payment evasion is also developed.

Keywords— Progressive tax scale, income taxation, social taxation, personal income tax.

I. Introduction

CURRENTLY in Russia personal income tax (PIT) is one of the major taxes and ranks third after Value added tax (VAT) and corporate income tax. Along with the corporate income tax, it is central to regional and local budgets. In 2013, consolidated budget revenue of federal subjects of Russia accounted for about 5.9 trillion rubles, more than 1/3 is taken by income tax on individuals [9].

The international experience shows that the similar situation also exists in the OECD countries, where the first rank in budget revenues is taken by VAT, and not by PIT [7].

However, despite the third largest share of fiscal performance, the share of personal income tax in consolidated revenue is about 13%. According to experts, this is due to low tax potential of most of the population and the use of a flat income tax scale.

At the moment, the most common form of social orientation of taxation system in international practice is using progressive taxation. In 2001 progressive income taxation scale was declined in the Russian Federation, a flat tax rate of 13% was introduced instead. The main argument justifying the introduction of a flat tax scale was the idea that large revenues will be withdrawn from the shadow economy.

The identified features of income taxation in developed countries and in the Russian Federation, as well as cross-country analysis allow us to outline the following features of the income taxation system in Russia:

– Linear scale of taxation is used, whereas developed countries apply progressive scale;

– The lowest maximum rate of income tax is used;

– The share of tax revenues from personal income tax in the consolidated budget of the country and in the GDP is significantly lower compared to developed countries;

– There is no concept of non-taxable income, it is replaced by a nonequivalent standard tax deduction; in developed countries non-taxable income is generally equal to the minimum consumer basket;

– The gap between the wealthiest and poorest sectors of the population is significantly greater than in developed countries;

– Specific features of taxation lead to a greater stratification of society;

– Antisocial nature of taxation manifests itself when providing social and property-related tax deductions.

According to these points we can formulate the conclusion that flat income tax is socially unfair in RF. Russian government increased PIT rate by 1%, which had a negative effect on the population with low and middle income, whereas for rich and super-rich population groups the tax rate became significantly lower. Moreover, after the reform, taxes on income from equity investments in companies (dividends) were imposed at the rate of 6%, i.e. 24% lower than before. It should be noted that in European countries this problem also exists, when the population with lower income pays more, than the population with higher income [4].

Also it is necessary to notice that the problem of payment burden differentiation is actual not only in the social sphere, but also in the sphere of business. An interesting analogy can be drawn, comparing low earning groups of the population and small business. Particularly for decreasing the payment burden for small business it is possible to consider a question of decreasing in costs of the information technologies used for business support: there is an approach which allows to use the same information technologies, but with application of cheaper software [6].

All this suggests the necessity of introducing a progressive income taxation system in RF, which will replenish national budgets and achieve social equity by income redistribution.

In this regard it is necessary to develop "social" algorithm and methods of tax scales creation upon transition from the existing flat scale to the progressive. Thus it is necessary to develop and prove the following methodological provisions:

• what law of progressive rates growth is optimum or socially accepted;

• how first group of the population can be created in case of the taxation exception of a needy segment of the population;

• to keep or to change taxable base before tax scale

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reforming;

• to keep or to increase the income of the budget at the expense of increase in a total personal income tax;

• how to consider probability of tax avoidance by part of taxpayers as a result of an ascending scale introduction.

The experimental and statistical base for the study is a unified system of indicators of the Federal State Statistics Service.

II. Modeling of the progressive income taxation model

A. Development of the tax scale formation and optimization algorithm

In the economic literature we can find various terms to refer to tax scales, depending on the tax rate and the income subject to taxation. In this paper, we use the following terms:

Taxation scale is called *flat* if the value of the tax rate is the same for all taxpayers regardless of their individual income. In case when the tax rate rises with the increase of taxpayer's income the scale is called *progressive*. With proportional increase of tax rates and income, the scale is defined as *linearly progressive*, in other cases - as *nonlinear*. If the value of the tax rate decreases with increasing income of a taxpayer - the scale is *regressive*.

For building progressive scales of income taxation it is necessary to break all taxpayers into groups by the size of their income. We will assume that the number of such groups is m (the number of levels). We will arrange these groups with the increase of average income of taxpayers in the group and give them their respective number $i = 1; 2; \ldots m$.

Currently, according to the Federal State Statistics Service of the Russian Federation, all Russian taxpayers are divided into 5 levels by 20 per cent (quintile) groups of distribution of total monetary income ($m = 5$), and 8 levels by average income per capita ($m = 8$) [8]; the first group is with the lowest income, the fifth and eighth with the highest (the Federal State Statistics Service also forms 10 percent (decile) groups, $m = 10$).

Each group has its own taxable base S_i:

$\sum_{i=1}^{m} S_i = S_0$ in which S_0 is existing taxable base.

With a flat scale of taxation (flat tax rate n_0) the total tax on personal income will be equal to:

$$C_0 = S_0 \cdot n_0 = \sum_{i=1}^{m} S_i \cdot n_0$$

For a progressive tax scale we introduce tax rates for each group, indicating them n_i accordingly.

Then the total tax base will be $S = \sum_{i=1}^{m} S_i$, and the general total tax:

$$C = \sum_{i=1}^{m} C_i = \sum_{i=1}^{m} S_i \cdot n_i \qquad (1)$$

B. Development of the linear progressive tax scale: constancy of tax levies

We will consider the case of transformation of a flat scale into progressive, in which the reform is not aimed at increasing the total income tax, i.e. with equal value of the tax base and the total income tax before and after the introduction of a progressive tax scale:

$$S = \sum_{i=1}^{m} S_i = S_0 \text{ and } C = \sum_{i=1}^{m} n_i \cdot S_i = C_0 \qquad (2)$$

We assume that the tax rate increases linearly with the taxpayer's income; such tax scales are used in a number of developed countries, such as USA, Canada, UK, France, Germany, etc.

The condition for linear increase of the tax rate is a constant value of the tax rate increment Δ from group to group

$$\Delta = n_{i+1} - n_i = c \quad o$$

and the rate in the group will be:

$$n_i = n_1 + \Delta \cdot (i-1), \qquad (3)$$

where n_1 is the tax rate on personal income for the first lowest earning group of taxpayers.

From (2) follows:

$$S_1 n_1 + S_2 n_2 + \ldots S_m n_m - S_0 n_0 = 0 \qquad (4)$$

Inserting (3) into (4), we obtain:

$$S_1 n_1 + S_2 (n_1 + \Delta) + S_3 (n_1 + 2\Delta) + S_m [n_1 + (m-1)\Delta] - S_0 n_0 = 0$$

or

$$n_1 (S_1 + S_2 + \ldots S_m) + \Delta [S_2 + 2S_3 + \ldots (m-1) S_m] - S_0 n_0 = 0$$

Given the condition of preservation of the tax base (2), we have:

$$(n_1 - n_0) S_0 + \Delta [S_2 + 2S_3 + \ldots (m-1) S_m] = 0$$

hence the increase in the tax rate will be determined by the formula:

$$\Delta = \frac{(n_0 - n_1) S_0}{S_2 + 2S_3 + \ldots (m-1) S_m} \qquad (5)$$

Inserting (5) into (3), we find the values of all the rates for groups for a linear progressive tax scale:

$$n_i = n_1 + \frac{S_0 (n_0 - n_1)(i-1)}{\sum_{j=2}^{m} (j-1) S_j} = n_1 + \frac{(n_0 - n_1)(i-1)}{\sum_{j=2}^{m} (j-1) \eta_j}, \qquad (6)$$

in which $\eta_j = \frac{S_j}{S_0}$.

The results of calculation of tax rates by formula (6) are shown in Table 1. The values of income distribution η_i are taken from the Federal State Statistics Service of the RF for 2014. The range of tax rates for low income groups η_i fluctuated from 10% to 0%. Note that many countries practice a complete exemption from tax for the poor, so considering option $n_1 = 0$ is relevant.

Table 1. Linear progressive tax scale: the results of calculations, 2014

i	**1**	**2**	**3**	**4**	**5**	
η_i **n_1 %**	0.052	0.099	0.149	0.225	0.475	
13	13	13	13	13	13	**n_i, %**
10	10	11	12	13	14	
5	5	7.7	10.4	13	15.8	
0	0	4.4	8.7	13	17.5	
13	1.000	1.000	1.000	1.000	1.000	**n_i /n_0**
10	0.769	0.847	0.925	1.000	1.080	
5	0.385	0.592	0.799	1.000	1.213	
0	0.000	0.336	0.673	1.000	1.346	

Source: author's calculations

The table shows redistribution of the tax burden from low earning groups to the wealthy. We should point out that the tax burden for the forth group of taxpayers has not changed, whereas for the fifth group it has increased by 35%.

We can show that if in transforming the flat scale into linear progressive we take as a parameter the tax rate for "the wealthy" n_m, and not the rate for the poor n_1, then the formula for calculating all the other rates will be:

$$n_i = n_m - \frac{(n_m - n_0)(m-i)}{\sum_{j=1}^{m-1}(m-j)\eta_j} \qquad (6a)$$

The received formula (6a) on the basis of the offered approach, expressed in (1)–(6), actually is "one-parametrical" dependence: transformation of an initial flat scale of the taxation into the progressive is carried out in socially significant only parameter – tax rate size of n_1 for the low-income group. Elaborated provisions of transition from a flat to the progressive linear scale form a certain method which we will call method of linear transformation.

Advantage of this suggested method is that it allows to analyze, to reveal features and to offer concrete sizes of tax rates of the formed ascending linear scale of the income taxation when using the only initial parameter – rates of the low-income group.

On the basis of the made calculations we receive that (see Tab. 1):

1. At the size of a tax rate for the first group n_1 = 13% the method of linear transformation automatically gives a flat scale of the taxation;
2. Decreasing in a tax rate to 10% for the low-income group leads to increasing in a tax rate for group with the greatest income for 1%;
3. Decreasing in a tax rate to 5% for the first group leads to increasing in a tax rate for the latter group for 2.8%.

The third option of the received results is possible to consider socially accepted for the linear ascending scale of income taxation creation.

After considering change options of a tax rate n_1=10% and 5% it is topical to study a question of liberation from the taxation for the low-income group of the population. The carried-out analysis of foreign experience in income taxation in the developed countries showed that there are high tax-free income minima (for example, in France, the free income makes about 4200 euros, in Germany about 8000 euros). According to the Russian tax law there is no concept of the free income nowadays; this issue is partially resolved by granting standard tax deductions in the form of the fixed sum for some categories of citizens.

Speaking about calculations of tax rates for progressive linear taxation at liberation from a tax of the low-income group, it is visible from Table 1 that the rate of a tax on group with the greatest income will make 17.5%.

C. Development of the linear progressive tax scale: increasing in tax levies

Let us consider the case when the transformation of a flat tax should be accompanied by an increase in total income tax.

We introduce the coefficient of the planned increase in tax revenue by increasing the total tax on the income of individuals in τ times: $C = \tau \cdot C_0$

With a flat rate tax, additional tax burden is equally imposed on all people; with a linear progressive scale, tax rates will be calculated for the tax base $\tau \cdot S_0$ by the formula:

$$n_i = n_1 + \frac{(\tau \cdot n_0 - n_1)(i-1)}{\sum_{j=1}^{m}(j-1)\eta_i} \qquad (7)$$

The results of calculations of tax rates by (7) required for increasing tax revenues by 20% are presented in Table 2. Gradations of tax rates n_1 and the distribution of monetary income in groups η_i are similar to those contained in Table 1.

Table 2. Linear progressive tax scale, increasing in tax collections: the results of calculations, 2014

***τ* = 1,2**						
i	**1**	**2**	**3**	**4**	**5**	
η_i **n_1 %**	0.052	0.099	0.149	0.225	0.475	
13	13	13.9	14.7	15.6	16.5	**n_i %**
10	10	11.9	13.8	15.7	17.5	
5	5	8.6	12.1	15.7	19.3	
0	0	5.2	10.5	15.7	21	
13	1.000	1.067	1.135	1.202	1.269	**n_i /n_0**
10	0.769	0.914	1.059	1.204	1.349	
5	0.385	0.659	0.933	1.208	1.482	
0	0.000	0.404	0.808	1.211	1.615	

Source: author's calculations

The obtained results shown in Tables 1 and 2, allow to visually compare the tax rates with an increase in the total tax revenue from income tax by 20%.

D. Development of the nonlinear progressive tax scale

Let us consider a transformation mechanism of a nonlinear progressive tax scale. Due to the ambiguity of the use of terms in the economic literature, we will focus on the terms used in this paper for describing different taxation scales.

We will consider the case when the tax scale is a continuous function of the income *W*, $n = f(W)$. Figure 1 shows examples of three types of progressive taxation $n = f(W)$:

1 **–** flat scale (the tax rate does not depend on the size of income);

2 – linear progressive tax scale, whose specific features are described above;

3 – nonlinear progressive scale with increasing dynamics of tax rate growth;

4 – nonlinear progressive scale with decreasing dynamics of tax rate growth.

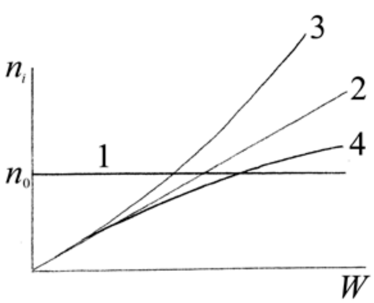

Fig. 1. Various types of tax scales

These terms will be also applied in case of tax distribution by groups to a step change of tax rate n_i from group to group. Consideration of such taxation scales is of interest both to the study of shifting the tax burden on the highest income group, and in addressing the problem of tax evasion.

Exponential distribution and parameters distribution as a geometric progression are among the most common in describing various relations in the economy, and the corresponding mathematical models match physical nature of real processes and lead to constructive solutions in analytical and in algorithmic form [1]. In contrast to traditional methods, we will consider a nonlinear scale, changing according to the law of "double arithmetic progression", as a more convenient when describing a step tax scale.

For building a linear progressive scale the following ratios were used (3):

$n_i = n_1 + \Delta \cdot (i-1)$; $\Delta = n_i - n_{i-1} = c \quad o$.

For a nonlinear scale, the value of tax rate increment Δ does not remain constant; we will assume that Δ itself changes in an arithmetic progression:

$$\Delta_i = \Delta_1 + \delta \cdot (i-1);\ \delta = \Delta_i - \Delta_{i-1} = c \quad o \qquad (8)$$

Inserting (8) into (3) we obtain:

$$n_i = n_1 + \left[\Delta_1 + \delta(i-1)\right](i-1) \qquad (9)$$

Value δ will be defined as a part of Δ_1 :

$\delta = k \cdot \Delta_1$, $-1 < k < 1$,

where *k* is a coefficient of “nonlinearity”.

Then

$$n_i = n_1 + \Delta_1\left[1 + k(i-1)\right] \cdot (i-1) \qquad (10)$$

The proposed method of forming a non-linear scale is illustrated by Figure 2.

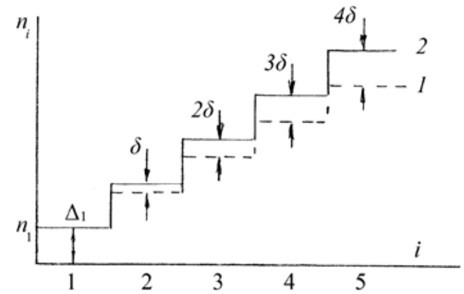

Fig. 2. Nonlinear progressive tax scale, where **1** (dotted) is a linear scale, **2** is a nonlinear scale ($\delta = k \cdot \Delta_1$, $k > 0$)

Inserting (10) into (4) we will obtain:

$$S_1 n_1 + S_2\left[n_1 + \Delta_1(1+k)\right] + S_3\left[n_1 + \Delta_1(1+2k)\right] \cdot 2 + ...$$
$$+ S_m\left[n_1 + \Delta_1(1 + (m-1)k)\right](m-1) - S_0 n_0 = 0$$

or

$$n_1(S_1 + S_2 + ...S_m) + \Delta_1\{S_2(1+k) + 2S_3(1+2k) + ...\}$$
$$\{+(m-1)S_m\left[1 + (m-1)k\right]\} - S_0 n_0 = 0$$

Taking into account that $\sum_{i=1}^{m} S_i = S_0$, we have:

$$\Delta_1 = \frac{(n_0 - n_1)S_0}{\sum_{j=2}^{m} S_j(j-1)\left[1 + (j-1) \cdot k\right]} \qquad (11)$$

Inserting (11) into (10) and shifting to non-dimensional coefficients η_i, we will eventually get:

$$n_i = n_1 + \frac{(n_0 - n_1) \cdot S_0\left[1 + k(i-1)\right](i-1)}{\sum_{j=2}^{m} S_j(j-1)\left[1 + (j-1) \cdot k\right]} =$$
$$= n_1 + \frac{(n_0 - n_1) \cdot \left[1 + k(i-1)\right] \cdot (i-1)}{\sum_{j=2}^{m} \eta_j(j-1)\left[1 + (j-1) \cdot k\right]} \qquad (12)$$

According to the elaborated provisions and the received mathematical dependences of transition from linear to a nonlinear progressive tax scale, it is possible to speak about development of a nonlinear transformation method of income taxation.

Advantages of this method are the following: it allows to reveal features and to receive the concrete sizes of tax rates for formation of a nonlinear scale of the progressive taxation. Thus, as well as in case of the method of linear transformation, socially significant key parameter is the tax rate of the low-income group.

The results received by this method can be used to solve various social and economic problems, in particular, when there is an urgent need of transition to a nonlinear progressive scale of the taxation, resolved by means of fast increase in tax rates of the high-income population. So, for example, catastrophic natural disaster in Australia (flooding of extensive territories in the country) forced the government to transform an income tax scale urgently: the parliament increased PIT on taxpayers, whose income was more than 50 thousand dollars per year [5].

The calculation results of nonlinear transformation method came up with different options of nonlinear progressive scales using varied parametric index n_1 *(n_1=10%, n_1=15%, n_1=0%)* and "nonlinearity" index *k*, *(for k = 0.2 and k = - 0.12)* see Tables 3 and 4.

Initial parameters correspond to Table 1.

Table 3. Nonlinear progressive tax scale, *k*>0: the results of calculations, 2014

***k* = 0.2**						
i	**1**	**2**	**3**	**4**	**5**	
η_i **n_1 %**	0.052	0.099	0.149	0.225	0.475	
13	13	13	13	13	13	**n_i %**
10	10	10.7	11.7	11.9	14.3	
5	5	6.9	9.4	12.6	16.4	
0	0	3.1	7.2	12.4	18.6	
13	1.000	1.000	1.000	1.000	1.000	**n_i/n_0**
10	0.769	0.824	0.898	0.989	1.099	
5	0.385	0.531	0.727	0.971	1.264	
0	0.000	0.238	0.556	0.953	1.430	

Source: author's calculations

Table 4. Nonlinear progressive tax scale, *k*<0: the results of calculations, 2014

***k* = -0.12**						
i	**1**	**2**	**3**	**4**	**5**	
η_i **n_1 %**	0.052	0.099	0.149	0.225	0.475	
13	13	13	13	13	13	**n_i %**
10	10	11.5	12.6	13.3	13.6	
5	5	9.1	12.0	13.9	14.6	
0	0	6.6	11.4	14.4	15.6	
13	1.000	1.000	1.000	1.000	1.000	**n_i/n_0**
10	0.769	0.886	0.972	1.025	1.046	
5	0.385	0.697	0.924	1.066	1.123	
0	0.000	0.508	0.877	1.108	1.200	

Source: author's calculations

Examples of creation of progressive scales (for n_1 = 5% and *k* = 0 and 0.2) by methods of linear and nonlinear transformation are given in Figure 3 and 4.

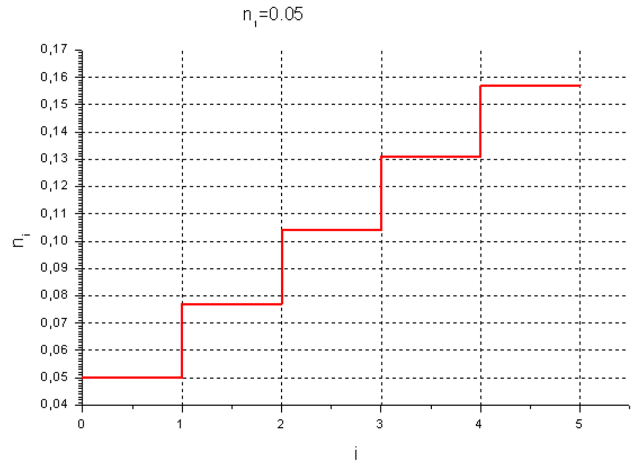

Fig. 3. Method of linear transformation, *k* = 0

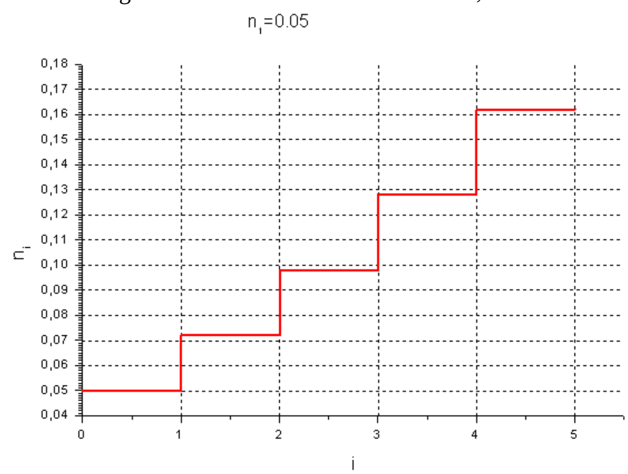

Fig. 4. Method of nonlinear transformation, *k* = 0.2

E. Calculation of social return and development of the generalized model of transition to progressive taxation

Transition from flat to linear and nonlinear progressive taxation scales has a social nature of redistribution of tax burden among population. Social return means the difference between the values of tax rates for the most and least wealthy groups of taxpayers. In other words, for instance, under the conditions of Table 3 social return (*d*) can be calculated according to the formula $d = n_5 - n_1$.

Comparing the obtained values of building a nonlinear progressive scale with the tax rates values of a linear progressive scale, we can conclude that:

– the greatest social return is observed when building a nonlinear scale at $k > 0$;

– the value of social return when using a linear income taxation scale ranks second;

– minimum, yet existing, social return is present when building a nonlinear progressive scale at $k < 0$.

We will note that calculation of social effect in the income taxation is an important component of the state tax policy. Along with the public authorities, participating in socially oriented distribution of financial resources, business structures also carry out an important role in this process, providing the flow of social investments in the national economy [10].

Summarizing the results, we obtain a generalised mathematical model of transition to progressive income taxation, adaptive to changing political, economic and social conditions, which allows to calculate the coefficients of taxation for various tax scales simultaneously:

$$n_i = n_1 + \frac{(\tau \cdot n_0 - n_1) \cdot [1 + k(i-1)] \cdot (i-1)}{\sum_{j=2}^{m} \eta_j (j-1) \cdot [1 + (j-1) \cdot k]}, \quad (13)$$

where:

n_0 – tax rate of a flat taxation scale, n_0 = 13%;

n_1 – tax rate for the first, lowest-income group (selected as a "basic" socially significant parameter);

n_i – tax rate in group i;

$\tau = \frac{C}{C_0}$ - coefficient of a planned tax revenue increase;

m – the number of taxation groups;

k – coefficient of "nonlinearity" of the scale, which takes into account the pace of progressive scale changes;

$\eta_i = \frac{S_i}{S_0}$ - coefficients of monetary income distribution by groups in relation to total income S_0;

j – index used for summing the shares of coefficient n_i .

The generalised model allows us to estimate all possible options for the reform of the proposed transformation of a flat scale into progressive [3]. In this case the above mentioned coefficients vary depending on external conditions, as well as targets and challenges which face public authorities developing tax policy for the short and medium term.

It must be emphasized that this model remains valid for different taxpayers breakdowns, such as their division by average income per capita, when the entire population is divided into 8 levels (m = 8), and the corresponding values of monetary income are given in percentages [8].

An important tax scale optimization problem is to ensure the highest total income. A possible solution is connected with optimizing taxpayers breakdown by the value of coefficients n_i and selecting the number of levels m. It can be shown that since the values of coefficients η_i are included in (13) with their weight coefficients $(j-1)[1+(j-1)k]$, the possible maximum value of $\tau_{\max}$ depends on dividing the values n_i into groups.

The practical value of the proposed mechanism of transition to progressive taxation scales and social return valuation is that taxation coefficients estimation can be used by state authorities when selecting a strategy of progressive income taxation.

III. Modeling of the Probabilistic Progressive Model of the Income Taxation

A. Development of the algorithm of progressive scale modeling considering possibility of tax evasion

The following part of the research is development of mathematical tools to create the probabilistic tax model, which can reflect processes of the taxation and problems of tax avoidance. The problem of income tax avoidance is well considered in papers [2, 11].

Forming a progressive scale of the income taxation, we set the following task: to find such rates of the taxation of n_i that the size of tax levies was maximum provided the conditions that probabilities of real tax payments depend on levels of tax rates.

We designate the coefficient p_i which characterizes what part of a tax will be really paid by taxpayers at the size of the income in i-interval and n_i taxation group. These coefficients can be defined from the statistical data on the taxation counted by Federal State Statistics Service, the Ministry of Finance or the Ministry of Taxes and Tax Collection of the RF. Thus the coefficients p_i are called the probabilities of tax payments by payers.

We will assume that the taxpayer pays tax with probability of p_i, or doesn't pay it at all. Note that the variant of partial tax payment in this elementary probabilistic model isn't considered.

According to this assumption the formula (2) is transformed into:

$$C = \sum_{i=1}^{m} S_i \cdot n_i \cdot p_i \quad (14)$$

This formula allows us to estimate the size of tax levies taking into account the probability of real tax payments.

It is necessary to consider that if the tax rate is higher, the taxpayer will pay it with a smaller probability.

We will assume that p_i and n_i are connected by a ratio:

$$p_i = 1 - n_i \quad (15)$$

For example, if the tax rate isn't so huge: n_i = 0.1, according to (15) the probability of payment is bigger: p_i = 0.9; if n_i = 0.6, the probability decreases to p_i = 0.4.

The existence in (14) probabilities of p_i makes a barrier of "forcible" n_i rates increasing because this will lead to decreasing in probabilities of tax payments.

Also we can consider that the existence in n_i and p_i the multidirectional tendencies of changing allows to hope for possibility of correct problem solution statement of optimization the tax scale parameters.

Taking into account (15) the formula (14) is transformed into:

$$C = \sum_{i=1}^{m} S_i \cdot n_i \cdot (1 - n_i) \quad (16)$$

Thus, the problem of optimization consists of selecting n_i tax rates so that to provide the maximum values of tax collections $C_{\max}$:

$$C_{\max} = \max_{n_i} \sum_{i=1}^{m} S_i \cdot n_i \cdot (1 - n_i) \quad (17)$$

As a result the task was reduced to search the maximum of function C depending on several variables n_i. By rules of searching the extremum of function with several variables we have:

$$\left.\begin{array}{l} \dfrac{\partial C}{\partial n_1} = S_1(1-2n_1) = 0 \\ \dfrac{\partial C}{\partial n_2} = S_2(1-2n_2) = 0 \\ \dots\dots\dots\dots\dots\dots \\ \dfrac{\partial C}{\partial n_m} = S_m(1-2n_m) = 0 \end{array}\right\} \quad (18)$$

Thus, the optimum values provided the maximum to the tax levies $C_{\max}$ will be equal to $n_1^* = n_2^* = ...n_m^* = 0.5$.

Further we consider the exact, more real, probabilistic model of progressive taxation system. Assume that n_i and p_i are connected by a ratio:

$$p_i = 1 - b_i \cdot n_i, \quad (19)$$

where:

b_i – weight coefficients which can be defined on the basis of statistical data, $b_i = \dfrac{1-p_i}{n_i}$.

According to (19) the examples of graphs depended on different values of b_i are presented in Figure 5.

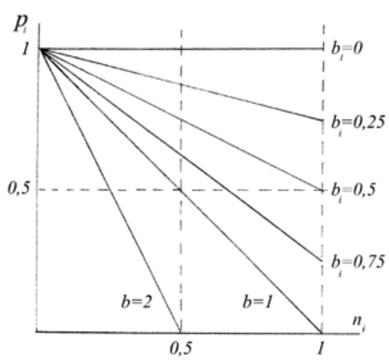

Fig. 5. The dependences $p_i = 1 - b_i \cdot n_i$

Considering (19) the formula (14) can be transformed into:

$$C = \sum_{i=1}^{m} S_i \cdot n_i \cdot (1 - b_i n_i) \quad (20)$$

and the problem of optimization (20) will be reduced to search the maximum of $C_{\max}$ function with several variables n_i:

$$C_{\max} = \max_{n_i} \sum_{i=1}^{m} S_i \cdot n_i \cdot (1 - b_i n_i) \quad (21)$$

Then

$$\left.\begin{array}{l} \dfrac{\partial C}{\partial n_1} = S_1(1-2b_1n_1) = 0 \\ \dfrac{\partial C}{\partial n_2} = S_2(1-2b_2n_2) = 0 \\ \dots\dots\dots\dots\dots\dots \\ \dfrac{\partial C}{\partial n_m} = S_m(1-2b_mn_m) = 0 \end{array}\right\} \quad (22)$$

Thus, optimum values of tax rates n_i^* will be equal to:

$$n_1^* = \frac{1}{2b_1};\ n_2^* = \frac{1}{2b_2};\ \cdots n_m^* = \frac{1}{2b_m} \quad (23)$$

The optimum decision provides the maximum of tax levies and can serve as a practical recommendation about a choice of the progressive taxation scale parameters. Thus the accuracy of the constructed model will depend on reliability of statistical data on the taxation, i.e. p_i.

The developed mathematical tools including probabilistic approach to the problem of the tax payment evasion, allow us to speak about modeling the probabilistic model of the taxation.

Advantages of this model are: it allows transforming the progressive tax scale into the specified scale, taking into account the additional possible tax avoidance connected with a change of tax rates as a result of introduction the progression on income taxation. According to the suggested model it is possible to find the optimum tax rates received from probabilistic approach to tax payments.

The typical type of the corrected progressive scale is presented in Figure 6.

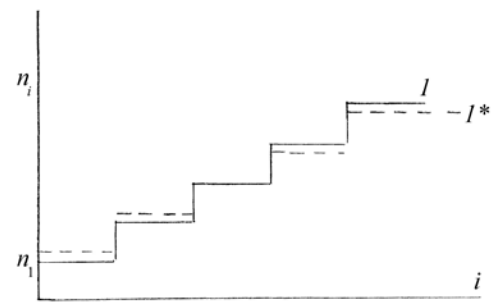

Fig. 6. The typical type of the corrected progressive scale: *1* – initial scale; *1** – corrected scale

Fig. 6 shows the mathematical recommendation allowing to maximize tax levies taking into account probabilities of their payment p_i, that can be formulated as: it is necessary to lower a little some taxation rates for groups with the big income and to raise them for groups with the small income.

From the point of view of a social orientation this recommendation should not be carried out literally, i.e. for groups with low incomes it is necessary to leave rates of not corrected tax scale. At the same time, for bigger tax levies of PIT it is necessary to listen to the received recommendation about groups with high incomes: it is expedient to correct a little (to reduce) the rate of their taxation for indemnification of negative consequences from possible non-payments at high rates. The size of these updates is defined on the basis of the solved problem of optimization under formulas (23).

We will notice that the accuracy of correction of optimum tax rates depends on reliability of statistical data on p_i.

According to the probabilistic model, the calculations show that the correction of the tax scale leads to correction in the coefficient of nonlinearity $k \approx 0.05$ and to correction in the taxation coefficients at a size ≈ 5-10% for extreme groups of the population.

B. Development of the progressive probabilistic model and tax rate levels optimization

We will set the following task: it is required to construct the progressive tax scale guaranteeing the same level of tax levies as at a flat scale, but thus considering probabilities of tax payment evasion.

According to the same approach in (3), the formula (14) is transformed into (20). In this case we will demand that the progressive scale constructed in (20) provides the same sum of the tax levies as the flat scale:

$$C_0 = n_0 \sum_{i=1}^{m} S_i = n_0 \cdot S_0 , \tag{24}$$

and

$$C = C_0 \tag{25}$$

From (20), (24), (25) follows:

$$S_1 n_1 (1 - b_1 n_1) + S_2 n_2 (1 - b_2 n_2) + ... + S_m n_m (1 - b_m n_m) - S_0 n_0 = 0 \tag{26}$$

Inserting (3) into (26) we obtain:

$$\begin{aligned} & n_1 (S_1 + S_2 + S_3 + ... + S_m) - n_0 S_0 + \Delta (S_2 + 2S_3 + ... + (m-1) S_m) - \\ & - n_1^2 (S_1 b_1 + S_2 b_2 + S_3 b_3 + ... + S_m b_m) - \\ & - 2\Delta n_1 (S_2 b_2 + 2S_3 b_3 + ... + (m-1) S_m b_m) - \\ & \Delta^2 (S_2 b_2 + 4S_3 b_3 + ... + (m-1)^2 S_m b_m) = 0 \end{aligned} \tag{27}$$

Finally this equation can be written as:

$$\alpha \Delta^2 + \beta \Delta + \gamma = 0 , \tag{28}$$

where:

$$\left. \begin{aligned} \alpha &= \sum_{j=1}^{m} (j-1)^2 S_j b_j \\ \beta &= \sum_{j=1}^{m} (j-1) S_j - 2n_1 \sum_{j=1}^{m} (j-1) S_j b_j \\ \gamma &= (n_1 - n_0) S_0 - n_1^2 \sum_{j=1}^{m} S_j b_j \end{aligned} \right\} \tag{29}$$

From (28) we find tax rate gain size Δ in group:

$$\Delta_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \tag{30}$$

The negative root in (30) has to be rejected:

$$\Delta_{1,2} = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \tag{31}$$

Substituting the found values Δ in (3), we will receive a final formula for finding n_i:

$$n_i = n_1 + \Delta \cdot (i-1), \qquad i = 1, 2, \ldots, m \tag{32}$$

Thus, the formulas (29), (31) and (32) for development of the progressive tax scale provide the same tax levies, as the flat scale on condition of the probabilities of tax payment evasion are generally received.

Further it is convenient to pass in formulas (29) to dimensionless sizes η_j:

$$\left. \begin{aligned} \alpha &= S_0 \sum_{j=1}^{m} (j-1)^2 \eta_j b_j \\ \beta &= S_0 \left[\sum_{j=1}^{m} (j-1) \eta_j - 2n_1 \sum_{j=1}^{m} (j-1) \eta_j b_j \right] \\ \gamma &= S_0 \left[n_1 - n_0 - n_1^2 \sum_{j=1}^{m} \eta_j b_j \right] \end{aligned} \right\} , \tag{33}$$

where: $\eta_j = \dfrac{S_j}{S_0}$.

We will consider a special case, when in (19) all $b_i = 1$, i.e. $p_i = 1 - n_i$.

Then:

$$\left. \begin{aligned} \alpha &= S_0 \sum_{j=1}^{m} (j-1)^2 \eta_j \\ \beta &= S_0 (1 - 2n_1) \sum_{j=1}^{m} (j-1) \eta_j \\ \gamma &= S_0 \left[n_1 - n_0 - n_1^2 \sum_{j=1}^{m} \eta_j \right] \end{aligned} \right\} \tag{34}$$

On the basis of the real statistical data, provided by Federal State Statistics Service of RF for 2014, we will construct the linear progressive tax scales considering possibility of tax payment evasion in the conditions of saving the budget income.

Substituting these data in (34) we will obtain:

$$\alpha = S_0 \cdot 10.344$$
$$\beta = S_0 (1 - 2n_1) \cdot 2.976$$
$$\gamma = S_0 (n_1 - n_0 - n_1^2)$$

After statement of these clauses in (31) it is possible to receive required value Δ at various n_1 values.

The results of the corresponding calculations at three values $n_1 = 0.1$; $n_1 = 0.05$ and $n_1 = 0$ are classified in Table 5, section I ($n_0 = 0.13$).

Thus, according to a method of linear transformation the progressive linear tax scales considering possibility of evasion from payment PIT have been generated and constructed, at preservation of budget incomes in comparison with a flat scale of the taxation.

Then we will consider the situation of possible budget income increase from tax payments due to introduction of the progressive tax scale.

If it is required to collect tax payments in τ-times more, than at the flat scale of the taxation, we have:

$$C = \tau \cdot n_0 \sum_{j=1}^{m} S_j = n_0^{(\tau)} \sum_{j=1}^{m} S_j \tag{35}$$

The solved task is reduced to earlier considered one in the conditions that n_0 is changed on $n_0^{(\tau)} = \tau \cdot n_0$.

The results of the corrected tax rates calculations taking into account the probabilities of tax payment evasion and planned increasing of tax revenues in the budget, are presented in Table 5, section II.

Table 5. Corrected progressive tax scale, probabilities of tax payment evasion: the results of calculations, 2014

i	**1**	**2**	**3**	**4**	**5**	
η_i n_1 %	0.052	0.099	0.149	0.225	0.475	
13	13	13	13	13	13	Section I
10	10	11.6	13.2	14.8	16.4	
5	5	7.8	10.6	13.4	16.2	
0	0	3.9	7.8	11.7	15.6	
13	13	13.9	14.7	15.6	16.5	Section II
10	10	12.5	15.0	17.5	20.0	
5	5	8.5	12.0	15.5	19.0	
0	0	4.5	9.0	13.5	18	

Source: author's calculations

Comparing the obtained results presented in Tables 1, 2 and 5, we can conclude that:

– it is possible to develop progressive tax scales taking into account probability of tax payment evasion providing the equality of tax revenue levies in comparison with the flat scale (section I), or providing the increase in τ-time (section II);

– the increase in tax revenue levies as a result of using the progressive tax scale considering probability of tax payment evasion is reached at the expense of some increase in tax rates: in particular at n_1 = 10%, tax rates increase for the 2-5-th groups are equal to 0.6% – 2.4%; and at n_1 = 5% respectively 0.1% – 0.4%;

– the accounting of probability of tax payment evasion conducts to some increase in tax rates, however tax levies don't decrease.

The specified conclusions and results are received for a linear progressive scale. It is also similarly possible to receive the corresponding results for a nonlinear tax scale.

IV. Conclusion

This article is focused on two methods of building linear and nonlinear progressive tax scales, on the basis of which we calculate the social return emerging from redistribution of tax burden among the population. On basis of these methods a generalized mathematical model, which allows us to take into account various external factors with the process of transition to progressive taxation, is presented. In conclusion we introduce a probabilistic progressive model of taxation considering possibility of tax payment evasion.

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