On the Existence of Maximum Likelihood Estimates in Modulated Gamma Process

Alicja Jokiel-Rokita and Ryszard Magiera

***Abstract*—** The problem of existence of maximum likelihood estimates (MLE's) of unknown parameters in a modulated gamma process (MGP) is considered. It is shown that the MLE's do not always exist. A theorem is established which gives the conditions under which the MLE's can be determined. A simulation study was conducted for some chosen values of the process model parameters for small numbers of the process events observed. The aim of the simulation was: 1) to provide frequency of appearing the respective conditions formulated under which the MLE's do exist, 2) to demonstrate the frequency of appearing the condition formulated under which the MLE's do not exist and 3) in the case when the MLE's do exist to give their values and accuracy. The MGP considered is a member of the rich class of the trend renewal processes which were considered among others in the fields of reliability, economics and medicine.

***Keywords*—** Modulated gamma process, inhomogeneous gamma process, trend-renewal process, maximum likelihood estimation.

I. Introduction

A modulated gamma process (MGP) is a special case of the inhomogeneous gamma process (IGP) defined in [2]. As the class of trend renewal processes (TRP's) introduced in [8] includes IGP, the MGP is also the special case of TRP. The MGP, so others TRP models, are a compromise between the renewal process (RP) and the nonhomogeneous Poisson process (NHPP), since its failure probability at a given time t depends both on the age t of the system and on the distance of t from the last failure time. Thus it seems to be quite realistic model in many practical situations.

In the paper we consider the problem of existence of MLE's of the parameters of the MGP. We show that the MLE's in the model considered do not always exist and we give conditions under which the MLE's can be determined.

Statistical inference for the MGP was considered in [2] and for modulated Poisson process (a special case of modulated gamma process) in [3]. Both papers only seriously addressed questions of hypothesis testing (via the likelihood ratio test), but did not satisfactorily solve the problem of parameter estimation. Inferential and testing procedures for log-linear nonhomogeneous Poisson process (a special case of the modulated Poisson process) can be found in [1], [4], [6], [7], [12].

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The article is organized as follows. In Section II we recall the definition of the IGP and its special case – the MGP. In Section III we establish a theorem which gives the conditions under which the MLE's in the models considered can be determined. In Section IV a simulation study was conducted for some chosen values of the process model parameters to demonstrate the frequency of appearing the condition formulated under which the MLE's do not exist, and to provide frequency of appearing the respective conditions formulated under which the MLE's do exist, and to give the values and accuracies of the MLE's in the cases when the MLE's exist. Section V contains conclusions and some prospects.

II. Definitions and preliminaries

The IGP was defined in [2] in the following manner. Consider a Poisson process with intensity function $\lambda(t)$. Suppose that an event occurs at the origin, and that thereafter only every κ-th event of the Poisson process is observed. Then, if $T_1, \ldots, T_n$ are the times of the first n events observed after the origin, their joint density is the following

$$f_n(t_1, \ldots, t_n) = \left\{ \prod_{i=1}^{n} \lambda(t_i)[\Lambda(t_i) - \Lambda(t_{i-1})]^{\kappa-1} \right\} \cdot \exp[-\Lambda(t_n)]/[\Gamma(\kappa)]^n, \quad (1)$$

where

$$\Lambda(t) = \int_0^t \lambda(u)du \quad (2)$$

and $t_0 = 0$. If κ is any positive number, not necessarily an integer, then (1) is still a joint density function. A point process $\{T_i, i = 1, 2, \ldots\}$ with the joint density (1) for all positive integers n is called the IGP with rate function $\lambda(t)$ and shape parameter κ.

The MGP is the IGP with rate function of the form

$$\lambda(t) = \rho \exp\{\beta z(t)\}, \quad (3)$$

where $\beta = (\beta_1, \ldots, \beta_p)$ and $z(t) = (z_1(t), \ldots, z_p(t))^T$. When $\kappa = 1$, this reduces to Cox's modulated Poisson process (see [3]).

An alternative method of deriving the IGP is the following one. Suppose that the random variables

$$W_i := \Lambda(T_i) - \Lambda(T_{i-1}),\ i = 1, 2, \ldots, \quad (4)$$

for $i = 1, \ldots, n$ are independently and identically distributed according to the gamma distribution $\mathcal{G}(\kappa, 1)$ with unit scale parameter and shape parameter κ. It then follows that (1) is the joint distribution of $t_1, \ldots, t_n$.

It follows from Definition 1 given below that the IGP can be regarded as a special case of a TRP introduced and investigated first in [8] and [11] (see also [9] and [10]).

Definition 1: The TRP is defined as follows. Let $\lambda(t), t \geq 0$, be a nonnegative function, and let $\Lambda(t)$ be given by (2). The point process $\{T_i, i = 1, 2, \ldots\}$ is called a TRP with a renewal distribution function $F(t)$ and a trend function $\lambda(t)$ if the time-transformed process $\{\Lambda(T_i), i = 1, 2, \ldots\}$ is an RP(F) with the renewal distribution function F, i.e. if the random variables W_i, given by (4), are i.i.d. with cumulative distribution function F. The TRP is denoted by TRP$(F, \lambda(\cdot))$.

Parametric inference in the TRP model was studied in [10]. In [5] the problem of estimating unknown trend parameters of a TRP in the case when its renewal distribution function F is completely unknown was considered.

Definition 2: The MGP is the TRP$(F, \varrho \exp\{\beta z(t)\})$, $\beta = (\beta_1, \ldots, \beta_p)$, $z(t) = (z_1(t), \ldots, z_p(t))^T$, where F corresponds to the gamma distribution $\mathcal{G}(\kappa, 1)$.

It follows from the definition of the MGP that $\varrho > 0$ and $\kappa > 0$. In the sequel the MGP with $p = 1$ and $z_1(t) = t$ is considered. We will denote it by MGP(ϱ, β, κ). We suppose that the MGP(ϱ, β, κ) is observed up to the nth event (failure) appears for the first time, and the values $t_1, \ldots, t_n$ of the jump times $T_1, \ldots, T_n$ are recorded. In other words, we consider the so called failure truncation (or inverse sequential) procedure. Denote $\mathbf{t} = (t_1, \ldots, t_n)$. The likelihood function of the MGP(ϱ, β, κ), observed until the nth failure occurs is

$$\begin{aligned}\mathcal{L}_n(\varrho, \beta, \kappa; \mathbf{t}) &= \left[\frac{\varrho^\kappa}{\Gamma(\kappa)}\right]^n \exp\left(\beta \sum_{i=1}^n t_i\right) \\ &\cdot \exp\left[-\varrho \int_0^{t_n} \exp(\beta x)dx\right] \prod_{i=1}^n \left[\int_{t_{i-1}}^{t_i} \exp(\beta x)dx\right]^{\kappa-1} \\ &= \left[\frac{\varrho^\kappa}{\Gamma(\kappa)\beta^{\kappa-1}}\right]^n \exp\left[\beta \sum_{i=1}^n t_i - \frac{\varrho}{\beta}\left(\exp(\beta t_n) - 1\right)\right] \\ &\cdot \prod_{i=1}^n \left[\exp(\beta t_i) - \exp(\beta t_{i-1})\right]^{\kappa-1}.\end{aligned}$$

Remark that we can distinguish the following special cases of the MGP(ϱ, β, κ):

1) For $\beta = 0$, i.e. for MGP$(\varrho, 0, \kappa)$ the intensity is $\lambda(t) = \varrho$ and we have to deal with the RP(F) with $F(x) \sim \mathcal{G}(\kappa, \varrho)$.
2) If $\kappa = 1$, then the MGP$(\varrho, \beta, 1)$ becomes the NHP$(\lambda(t))$, Cox's modulated Poisson process.

We will consider the MGP(ϱ, β, κ) for which $\varrho > 0, \beta > 0, \kappa > 0$.

III. The ML estimation in the MGP model

The log-likelihood function of the MGP(ϱ, β, κ) is of the following form

$$\begin{aligned}\ell(\varrho, \beta, \kappa; \mathbf{t}) &= n \log\left[\frac{\varrho^\kappa}{\Gamma(\kappa)\beta^{\kappa-1}}\right] + \beta S(\mathbf{t}) \\ &- \frac{\varrho}{\beta}\left[\exp(\beta t_n) - 1\right] + (\kappa - 1)V(\beta; \mathbf{t}),\end{aligned}$$

where

$$\begin{aligned}S(\mathbf{t}) &= \sum_{i=1}^n t_i, \\ V(\beta; \mathbf{t}) &= \sum_{i=1}^n \log\left[\exp(\beta t_i) - \exp(\beta t_{i-1})\right].\end{aligned}$$

Therefore, the possible MLE's of MGP(ϱ, β, κ) parameters are solutions to the following system of the log-likelihood equations

$$\frac{\partial \ell(\varrho, \beta, \kappa; \mathbf{t})}{\partial \varrho} = \frac{n\kappa}{\varrho} - \frac{1}{\beta}\left[\exp(\beta t_n) - 1\right] = 0, \tag{5}$$

$$\begin{aligned}\frac{\partial \ell(\varrho, \beta, \kappa; \mathbf{t})}{\partial \beta} &= -\frac{n(\kappa - 1)}{\beta} + S(\mathbf{t}) \\ &+ \frac{\varrho}{\beta^2}\left[(1 - \beta t_n)\exp(\beta t_n) - 1\right] \\ &+ (\kappa - 1)W(\beta; \mathbf{t}) = 0,\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\partial \ell(\varrho, \beta, \kappa; \mathbf{t})}{\partial \kappa} &= n \log \varrho - n\Psi(\kappa) - n \log \beta \\ &+ V(\beta; \mathbf{t}) = 0,\end{aligned} \tag{7}$$

where

$$W(\beta; \mathbf{t}) = \sum_{i=1}^n \frac{t_i \exp(\beta t_i) - t_{i-1}\exp(\beta t_{i-1})}{\exp(\beta t_i) - \exp(\beta t_{i-1})},$$

and $\Psi(\kappa)$ denotes the digamma function.

Proposition 1: The MLE's $\widehat{\varrho}_{ML}$, $\widehat{\beta}_{ML}$ and $\widehat{\kappa}_{ML}$ of the parameters ϱ, β and κ in the failure truncation procedure for the MGP(ϱ, β, κ) exist if and only if, given data $\mathbf{t}$, there exists the solution to the equation

$$\begin{aligned}L_\kappa(\beta; \mathbf{t}) &=: \log[n\kappa(\beta; \mathbf{t})] - \Psi[\kappa(\beta; \mathbf{t})] \\ &- \log[\exp(\beta t_n) - 1] + \frac{1}{n}V(\beta; \mathbf{t}) = 0,\end{aligned} \tag{8}$$

with respect to β, where

$$\kappa(\beta; \mathbf{t}) = \frac{W(\beta; \mathbf{t}) - S(\mathbf{t}) - \dfrac{n}{\beta}}{W(\beta; \mathbf{t}) - nt_n \dfrac{\exp(\beta t_n)}{\exp(\beta t_n) - 1}}. \tag{9}$$

Proof. Observe that the equation $\dfrac{\partial \ell(\varrho, \beta, \kappa; \mathbf{t})}{\partial \varrho} = 0$ holds for

$$\varrho^* = \varrho(\beta, \kappa; \mathbf{t}) = \frac{n\beta\kappa}{\exp(\beta t_n) - 1}. \tag{10}$$

Substituting $\varrho = \varrho^*$, where ϱ^* is defined by (10), into (6) and (7) we obtain the log-likelihood equations for β and κ of the following form

$$\begin{aligned}-\frac{n(\kappa - 1)}{\beta} + S(\mathbf{t}) &+ \frac{n\kappa}{\beta}\left[1 - \frac{\beta t_n \exp(\beta t_n)}{\exp(\beta t_n) - 1}\right] \\ &+ (\kappa - 1)W(\beta; \mathbf{t}) = 0;\end{aligned} \tag{11}$$

$$\begin{aligned}\log(n\kappa) - \Psi(\kappa) &- \log[\exp(\beta t_n) - 1] \\ &+ \frac{1}{n}V(\beta; \mathbf{t}) = 0.\end{aligned} \tag{12}$$

Notice that equation (11) is a linear function with respect to κ and the parameter $\kappa = \kappa(\beta; \mathbf{t})$ can be easily determined to be of the form given by (9). Substituting $\kappa = \kappa(\beta;\mathbf{t})$ into equation (12) leads to equation (8) for β. $\square$

It will be shown in Theorem 1 below that in the MGP(ϱ,β,κ) model the MLE's of ϱ,β and κ not always exist. The conditions will be given under which the MLE's do exist. If the MLE's exist, then they can be evaluated using Theorem 2, which gives the formulas determining the MLE's of ϱ,β and κ of the MGP(ϱ,β,κ) in the case of failure truncation observation scheme.

Denote

$$\begin{aligned}
L^L_\kappa(\beta;\mathbf{t}) &:= \log[\kappa(\beta;\mathbf{t})] - \Psi[\kappa(\beta;\mathbf{t})], &(13)\\
L^R_\kappa(\beta;\mathbf{t}) &:= -\log n + \log[\exp(\beta t_n) - 1]\\
&\quad -\frac{1}{n}V(\beta;\mathbf{t}), &(14)\\
D_1(\mathbf{t}) &:= S(\mathbf{t}) - (n+1)\frac{t_n}{2}, &(15)\\
\kappa_N(\beta;\mathbf{t}) &:= W(\beta;\mathbf{t}) - S(\mathbf{t}) - \frac{n}{\beta} &(16)\\
&\quad \text{(the nominator of } \kappa(\beta;\mathbf{t})), &(17)\\
\kappa_D(\beta;\mathbf{t}) &:= W(\beta;\mathbf{t}) - nt_n\frac{\exp(\beta t_n)}{\exp(\beta t_n)-1} &(18)\\
&\quad \text{(the denominator of } \kappa(\beta;\mathbf{t})),\\
\beta_0(\mathbf{t}) &:= \kappa_D(0^+;\mathbf{t}), &(19)\\
\kappa_0(\mathbf{t}) &:= -\frac{t_n}{2D_1(\mathbf{t})} = \frac{\kappa_N(0^+;\mathbf{t})}{\kappa_D(0^+;\mathbf{t})}\\
&= \frac{\kappa_N(0^+;\mathbf{t})}{\beta_0(\mathbf{t})}, &(20)\\
D_2(\kappa) &:= \log(\kappa) - \Psi(\kappa), &(21)\\
Z_0(\mathbf{t}) &:= -\frac{1}{n}\sum_{i=1}^{n}\log\left(n\frac{t_i - t_{i-1}}{t_n}\right)\\
&= L^R_\kappa(0^+;\mathbf{t}). &(22)
\end{aligned}$$

Theorem 1: The MLE $(\widehat{\varrho}_{ML},\widehat{\beta}_{ML},\widehat{\kappa}_{ML})$ of the vector parameter (ϱ,β,κ) for the MGP(ϱ,β,κ) model exists if one of the following two cases holds for the data $\mathbf{t} = (t_1,\ldots,t_n)$ of the process observed, namely in
CASE 1^0: $D_1(\mathbf{t}) > 0$ for $\beta > \beta_0(\mathbf{t})$
or in
CASE 2^0: $D_1(\mathbf{t}) < 0$ and $D_2(\kappa_0(\mathbf{t})) < Z_0(\mathbf{t})$ for $\beta > 0$.
Otherwise, i.e. in
CASE 3^0: $D_1(\mathbf{t}) < 0$ and $D_2(\kappa_0(\mathbf{t})) \geq Z_0(\mathbf{t})$
the MLE's do not exist.

Proof. According to Proposition 1 the problem of existence of the MLE's comes down essentially to the question of whether there exists a solution to likelihood equation (8) with respect to β. Equivalently, consider the equation

$$L^L_\kappa(\beta;\mathbf{t}) = L^R_\kappa(\beta;\mathbf{t}), \tag{23}$$

where $L^L_\kappa(\beta;\mathbf{t})$ and $L^R_\kappa(\beta;\mathbf{t})$ are defined by (13) and (14), respectively.

Let us take into account the right hand side function $L^R_\kappa(\beta;\mathbf{t})$ of the likelihood equation. The function $L^R_\kappa(\beta;\mathbf{t})$ can be expressed in the following form:

$$L^R_\kappa(\beta;\mathbf{t}) = -\frac{1}{n}\sum_{i=1}^{n}\log[nh_i(\beta;\mathbf{t})],$$

where

$$\begin{aligned}
h_i(\beta;\mathbf{t}) &= \frac{\exp(\beta t_i) - \exp(\beta t_{i-1})}{\exp(\beta t_n) - 1}\\
&= \frac{\exp(\beta(t_i - t_n)) - \exp(\beta(t_{i-1} - t_n))}{1 - \exp(\beta t_n)}.
\end{aligned}$$

One can show that

$$\lim_{\beta\to 0^+} h_i(\beta;\mathbf{t}) = \frac{t_i - t_{i-1}}{t_n}, \quad i = 1,\ldots,n.$$

Moreover,

$$\lim_{\beta\to+\infty} h_i(\beta;\mathbf{t}) = 0, \quad i < n; \quad \lim_{\beta\to+\infty} h_n(\beta;\mathbf{t}) = 1.$$

Thus,

$$\begin{aligned}
\lim_{\beta\to 0^+} L^R_\kappa(\beta;\mathbf{t}) &= -\frac{1}{n}\sum_{i=1}^{n}\log\left[\frac{n(t_i - t_{i-1})}{t_n}\right]\\
&=: Z_0(\mathbf{t}).
\end{aligned}$$

Now we show that $\lim_{\beta\to 0^+} L^R_\kappa(\beta;\mathbf{t}) > 0$. To do this we use the fact that the arithmetic mean for positive numbers is not smaller than their geometric mean. Namely, for $a_i = \dfrac{n(t_i - t_{i-1})}{t_n}$ we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\frac{n(t_i - t_{i-1})}{t_n} &= 1 = \frac{1}{n}\sum_{i=1}^{n}a_i \geq \sqrt[n]{\prod_{i=1}^{n}a_i}\\
&= \sqrt[n]{\prod_{i=1}^{n}\frac{n(t_i - t_{i-1})}{t_n}}.
\end{aligned}$$

Thus,

$$\lim_{\beta\to 0^+} L^R_\kappa(\beta;\mathbf{t}) = -\sqrt[n]{\prod_{i=1}^{n}\frac{n(t_i - t_{i-1})}{t_n}} \geq \log(1) = 0,$$

i.e. $\lim_{\beta\to 0^+} L^R_\kappa(\beta;\mathbf{t}) > 0$ for every realization $\mathbf{t} = (t_1,\ldots,t_n)$. Moreover, $\lim_{\beta\to+\infty} L^R_\kappa(\beta;\mathbf{t}) = +\infty$.

Remark that

$$\frac{\partial L^R_\kappa(\beta;\mathbf{t})}{\partial\beta} = -\frac{1}{n}\kappa_D(\beta;\mathbf{t}), \tag{24}$$

where $\kappa_D(\beta;\mathbf{t})$ is defined by (18) (denominator of the function $\kappa(\beta;\mathbf{t})$). Notice also that

1) if $D_1(\mathbf{t}) \leq 0$, then $\kappa_D(\beta;\mathbf{t}) \leq 0$ for every $\beta > 0$, where $D_1(\mathbf{t})$ is defined by (15);
2) if $D_1(\mathbf{t}) > 0$, then $\kappa_D(\beta;\mathbf{t}) > 0$ for $\beta \in (0,\beta_0(\mathbf{t}))$ and $\kappa_D(\beta;\mathbf{t}) < 0$ for $\beta \in (\beta_0(\mathbf{t}),\infty)$, where $\beta_0(\mathbf{t})$ is defined by (19). If $\kappa_D(\beta;\mathbf{t}) < 0$, then $\kappa < 0$ which does not concern the process model considered.

Thus, we have shown that the function $L^R_\kappa(\beta;\mathbf{t})$ is increasing as a function of β for $\beta \in \{\beta > 0 : \kappa(\beta;\mathbf{t}) > 0\}$.

From formula (24) and the fact

$$\kappa_D(\beta;\mathbf{t}) \in \left(S(\mathbf{t}) - nt_n, S(\mathbf{t}) - (n+1)\frac{t_n}{2}\right)$$

we infer that

$$\frac{\partial L^R_\kappa(\beta;\mathbf{t})}{\partial\beta} \in \left(-S(\mathbf{t}) + \frac{(n+1)t_n}{2n}, -S(\mathbf{t}) + t_n\right).$$

It then follows that the function $L^R_\kappa(\beta;\mathbf{t})$ increases not faster than $t_n - S(\mathbf{t})$. Thus the function $L^R_\kappa(\beta;\mathbf{t})$ increases from $L^R_\kappa(0^+;\mathbf{t}) = Z_0(\mathbf{t}) > 0$ but no faster than $t_n - S(\mathbf{t})$ provided that $D_1(\mathbf{t}) < 0$ or decreases on the interval $(0, \beta_0(\mathbf{t}))$ and increases on $(\beta_0(\mathbf{t}), +\infty)$ but not faster than $t_n - S(\mathbf{t})$.

Consider now the left hand side function $L^L_\kappa(\beta;\mathbf{t})$ of the likelihood equation defined by (23). Observe that

$$\begin{aligned} &L^L_\kappa(\beta;\mathbf{t}) \\ &\quad = \log[\kappa(\beta;\mathbf{t})] - \Psi[\kappa(\beta;\mathbf{t})] \in \left[\frac{1}{2\kappa(\beta;\mathbf{t})}, \frac{1}{\kappa(\beta;\mathbf{t})}\right] \end{aligned}$$

and it is decreasing as a function of $\kappa(\beta;\mathbf{t})$. Since $\kappa(\beta;\mathbf{t})$ is decreasing as a function of β in the interval $(\beta_0(\mathbf{t}), \infty)$ or on $(0, \infty)$, the function $L^L_\kappa(\beta;\mathbf{t})$ is increasing as a function of β. Thus we have the following conclusions:

1) If $D_1(\mathbf{t}) > 0$, then $L^L_\kappa(\beta;\mathbf{t})$ increases from 0 on the interval $(\beta_0(\mathbf{t}), +\infty)$ to $\lim_{\beta\to+\infty} L^L_\kappa(\beta;\mathbf{t}) = +\infty$.
2) If $D_1(\mathbf{t}) < 0$, then $L^L_\kappa(\beta;\mathbf{t})$ increases from $\log[\kappa_0(\mathbf{t})] - \Psi[\kappa_0(\mathbf{t})]$ to $\lim_{\beta\to+\infty} L^L_\kappa(\beta;\mathbf{t}) = +\infty$, where $\kappa_0(\mathbf{t})$ is defined by (20) .

Finally, we gather from the considerations carried above that

1^0 If $D_1(\mathbf{t}) > 0$, then the equation (8) has a solution $\widehat{\beta}_{ML}$ in the interval $(\beta_0(\mathbf{t}), \infty)$.
2^0 If $D_1(\mathbf{t}) < 0$, then the equation (8) has a solution $\widehat{\beta}_{ML} > 0$ provided that $\log[\kappa_0(\mathbf{t})] - \Psi[\kappa_0(\mathbf{t})] < Z_0(\mathbf{t})$.
3^0 In other cases, equation (8) has no solution in the interval $(0, \infty)$.

Thus the proof of the theorem is complete. □

Figures 1 – 6 illustrate the statements of Theorem 1 in the three situations, Case 1^0, Case 2^0 and Case 3^0, i.e. for three possible realizations of the process under the same triple (ϱ, β, κ) of the process model.

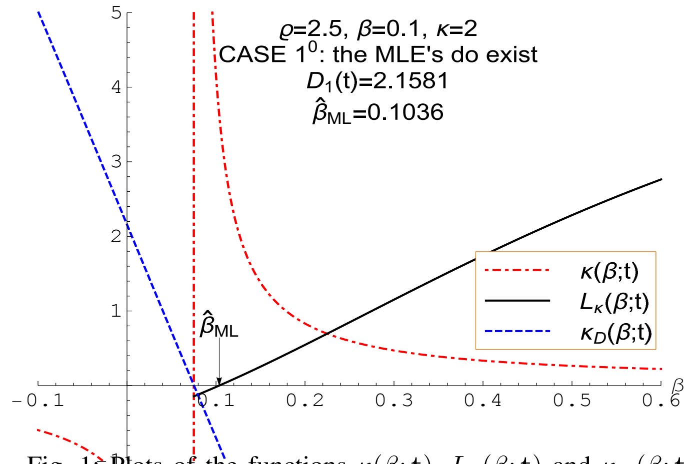

Fig. 1: Plots of the functions $\kappa(\beta;\mathbf{t})$, $L_\kappa(\beta;\mathbf{t})$ and $\kappa_D(\beta;\mathbf{t})$ for the data $\mathbf{t} = (t_1, \ldots, t_n), n = 10$, of a trajectory for which Case 1^0 holds

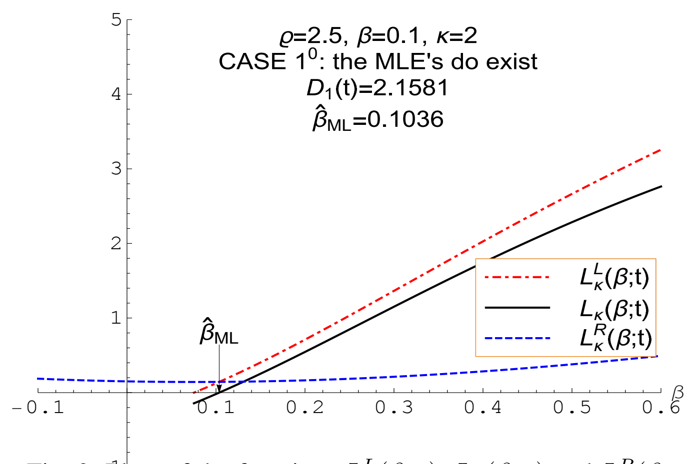

Fig. 2: Plots of the functions $L^L_\kappa(\beta;\mathbf{t})$, $L_\kappa(\beta;\mathbf{t})$ and $L^R_\kappa(\beta;\mathbf{t})$ for the data $\mathbf{t} = (t_1, \ldots, t_n), n = 10$, of a trajectory for which Case 1^0 holds

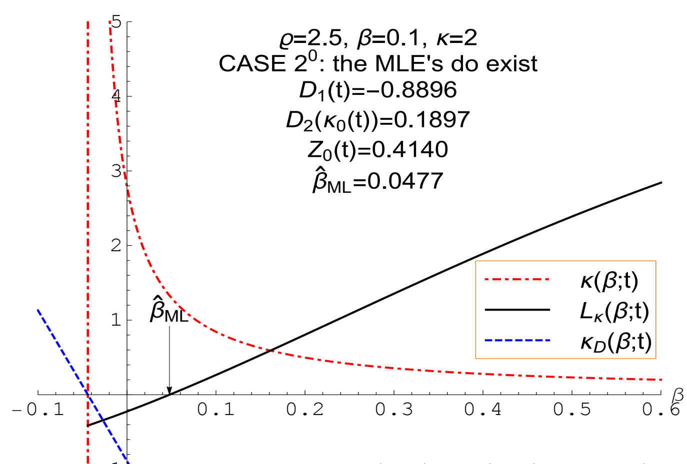

Fig. 3: Plots of the functions $\kappa(\beta;\mathbf{t})$, $L_\kappa(\beta;\mathbf{t})$ and $\kappa_D(\beta;\mathbf{t})$ for the data $\mathbf{t} = (t_1, \ldots, t_n), n = 10$, of a trajectory for which Case 2^0 holds

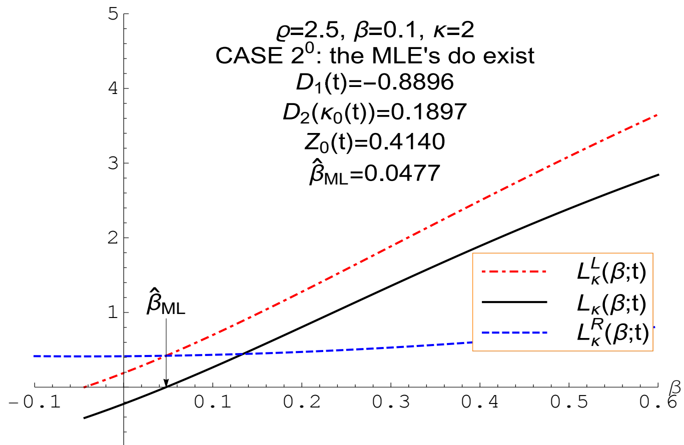

Fig. 4: Plots of the functions $L_\kappa^L(\beta;\mathbf{t})$, $L_\kappa(\beta;\mathbf{t})$ and $L_\kappa^R(\beta;\mathbf{t})$ for the data $\mathbf{t}=(t_1,\ldots,t_n), n=10$, of a trajectory for which Case 2^0 holds

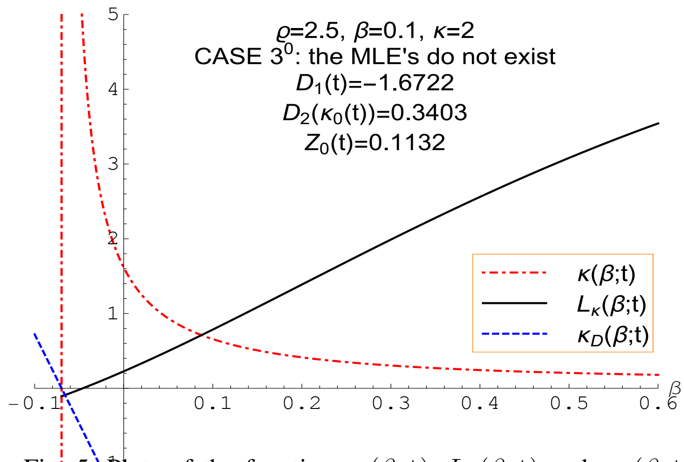

Fig. 5: Plots of the functions $\kappa(\beta;\mathbf{t})$, $L_\kappa(\beta;\mathbf{t})$ and $\kappa_D(\beta;\mathbf{t})$ for the data $\mathbf{t}=(t_1,\ldots,t_n), n=10$, of a trajectory for which the MLE's do not exist

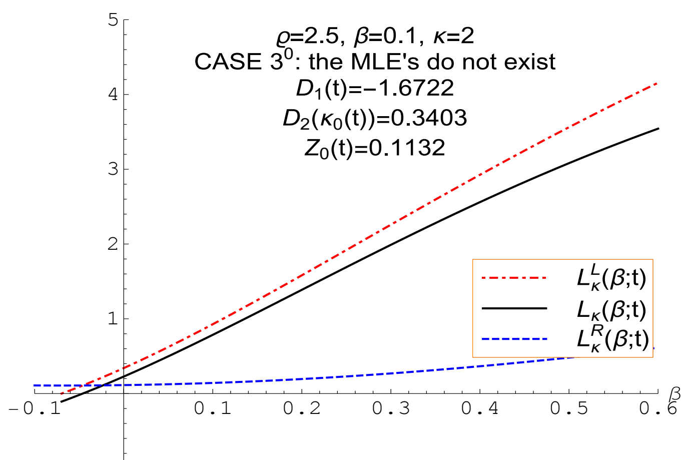

Fig. 6: Plots of the functions $L_\kappa^L(\beta;\mathbf{t})$, $L_\kappa(\beta;\mathbf{t})$ and $L_\kappa^R(\beta;\mathbf{t})$ for the data $\mathbf{t}=(t_1,\ldots,t_n), n=10$, of a trajectory for which the MLE's do not exist

As a consequence of Proposition 1 and Theorem 1 we have the following theorem.

Theorem 2: If Case 1^0 or 2^0 holds, then the MLE's $\widehat{\varrho}_{ML}$, $\widehat{\beta}_{ML}$ and $\widehat{\kappa}_{ML}$ can be determined as follows: $\widehat{\beta}_{ML}$ is the solution to equation (reflogLikehoodBeta), $\widehat{\kappa}_{ML}$ is determined by the formula

$$\widehat{\kappa}_{ML}=\kappa(\widehat{\beta}_{ML};\mathbf{t}), \tag{25}$$

where $\kappa(\beta;\mathbf{t})$ is defined by (9) and

$$\widehat{\varrho}_{ML}=\frac{n\widehat{\beta}_{ML}\widehat{\kappa}_{ML}}{\exp(\widehat{\beta}_{ML}t_n)-1}.$$

IV. Simulation study

The main purpose of the simulation study is to show using numerical program how often on average the Cases 1^0, 2^0 and 3^0 of Theorem 1 may occur observing the process corresponding to the MGP(ϱ,β,κ) model considered.

The MGP(ϱ,β,κ) can be generated according to the formula

$$t_i=\frac{1}{\beta}\log\left[\frac{\beta}{\varrho}G_{\kappa,1}+\exp(\beta t_{i-1})\right],\ i=1,2,\ldots,n,$$

where $G_{\kappa,1}$ is a random number generated according to the gamma $\mathcal{G}(\kappa,1)$ distribution.

Each sample of the MGP(ϱ,β,κ) is generated up to a fixed number n of jumps is reached.

The simulation study was carried out for small values of the number n of jumps (failures). For each chosen triple (ϱ,β,κ), the number k repetitions of the realization of the MGP(ϱ,β,κ) were generated.

The 'real' last failure times T_n are evaluated as the means from the k end-time points of the k repetitions of realizations of the process generated up to nth jump (failure), and all the ML estimates are evaluated as the means from the k estimates, such that each of these estimates was derived on the basis of the individual realization of the process considered.

To evaluate the ML estimates of the parameters ϱ,β and κ, the numerical program was constructed using Mathematica 10.4 package.

The accuracy of any MLE, say $\widehat{\eta}$ of η, is measured by the variability of an estimator $\widehat{\eta}$ which under squared error loss is determined by the root mean squared error RMSE$(\widehat{\eta})=\sqrt{(sd(\widehat{\eta}))^2+(mean(\widehat{\eta})-\eta)^2}$, where sd stands for the standard deviation, and by the absolute error (ABSE) which under absolute error loss is defined by ABSE$(\widehat{\eta})=|\widehat{\eta}-\eta)|$.

In tables the abbreviations se, ae and re are used for the RMSE's, ABSE's and RE's, respectively.

The results of the simulation study are given in Tables I and II in the case of the MGP(ϱ,β,κ) for some combinations of the parameters ϱ, β and κ. The parameters of the model have been chosen to maintain the last failure time $T_n\approx 1$. For comparison, analogous numerical results are presented in Tables III and IV.

Tables I and III contain the values of the ML estimates $\widehat{\varrho}_{ML},\widehat{\beta}_{ML},\widehat{\kappa}_{ML}$ and the percentages of occurrence of the Cases 1^0, 2^0 and 3^0 in the two situations, when $T_n\approx 6$ and $T_n\approx 1$, respectively. In the tables these percentages are denoted by ML1, ML2 and noML, respectively.

As Tables I and III show, it does not rarely happens that the MLE's do not exist. For many triples (ϱ, β, κ) the percentage of the case of non-existence of the MLE's exceeds 20% and for some cases is even close to 30%. The Case 2^0 appears significantly less often than the Case 1^0.

In Tables II and IV there are given the RMSE's of the estimators considered. In the whole range of parameter triples (ϱ, β, κ) of data set for $T_n \approx 6$, corresponding to Table II, all the RMSE's of the estimators $\widehat{\beta}_{ML}$ are considerably smaller than the RMSE's of the other two estimators.

TABLE I: The ML estimates of ϱ, β and κ and the percentages ML1, ML2 and noML of occurrence of the Cases 1^0, 2^0 and 3^0 in the MGP$(\varrho, \beta, \kappa); n = 10, T_n \approx 6$

No.	ϱ	β	κ	T_n	T_{n+1}	$\widehat{\varrho}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\kappa}_{ML}$	ML1	ML2	noML
1	1.2	0.01	0.75	5.9456	6.5131	1.0947	0.2609	1.0099	48.0	23.0	29.0
2	1	0.05	0.75	6.2094	6.8129	0.9739	0.2835	1.1288	64.5	16.0	19.5
3	0.9	0.1	0.75	6.0937	6.5584	0.8146	0.3264	1.0372	71.5	13.0	15.5
4	0.6	0.2	0.75	6.2758	6.6220	0.7142	0.3421	1.0134	77.5	11.5	11.0
5	1.6	0.01	1	6.1347	6.7268	1.4247	0.2249	1.3607	52.0	18.0	30.0
6	1.4	0.05	1	5.9702	6.4884	1.3972	0.2286	1.3598	57.5	16.5	26.0
7	1.2	0.1	1	5.9238	6.4150	1.3083	0.2495	1.3646	67.5	16.5	16.0
8	0.9	0.2	1	5.7609	6.0930	1.0852	0.3541	1.4257	83.0	10.5	6.5
9	2.5	0.01	1.5	5.8328	6.4049	2.9294	0.1375	2.3288	51.5	19.0	29.5
10	2.2	0.05	1.5	5.8665	6.3331	2.5470	0.1755	2.2453	60.0	13.5	26.5
11	1.9	0.1	1.5	5.7150	6.1694	2.2966	0.2222	2.2920	77.0	9.5	13.5
12	1.3	0.2	1.5	5.8799	6.2349	1.8867	0.2832	2.3131	90.5	5.5	4.0
13	3.3	0.01	2	5.7701	6.3219	3.4108	0.1469	2.7643	57.5	11.5	31.0
14	3	0.05	2	5.7311	6.2397	3.6625	0.1633	3.1918	66.5	11.5	22.0
15	2.5	0.1	2	5.7536	6.1689	3.5182	0.1900	3.1997	78.0	6.0	16.0
16	1.8	0.2	2	5.7276	6.0841	2.8356	0.2589	3.2374	89.0	4.0	7.0

TABLE II: The RMSE's of the ML estimates of ϱ, β and κ in the MGP$(\varrho, \beta, \kappa); n = 10, T_n \approx 6$

No.	ϱ	β	κ	se$(\widehat{\varrho}_{ML})$	se$(\widehat{\beta}_{ML})$	se$(\widehat{\kappa}_{ML})$
1	1.2	0.01	0.75	0.9511	0.3381	1.1357
2	1	0.05	0.75	0.9407	0.3348	1.3344
3	0.9	0.1	0.75	0.6986	0.3321	1.0706
4	0.6	0.2	0.75	0.8413	0.2854	0.9901
5	1.6	0.01	1	1.1860	0.2825	1.5121
6	1.4	0.05	1	1.2162	0.2708	1.4894
7	1.2	0.1	1	1.1443	0.2300	1.4677
8	0.9	0.2	1	1.2471	0.2784	1.5933
9	2.5	0.01	1.5	2.2497	0.1674	2.6645
10	2.2	0.05	1.5	2.0492	0.1814	2.5541
11	1.9	0.1	1.5	1.8147	0.1871	2.6319
12	1.3	0.2	1.5	1.7026	0.1787	2.4736
13	3.3	0.01	2	2.2349	0.1801	3.1222
14	3	0.05	2	2.6006	0.1675	3.7865
15	2.5	0.1	2	2.8601	0.1541	3.5604
16	1.8	0.2	2	3.1108	0.1512	3.8563

TABLE III: The ML estimates of ϱ, β and κ and the percentages ML1, ML2 and noML of occurrence of the Cases 1^0, 2^0 and 3^0 in the MGP$(\varrho, \beta, \kappa); n = 10, T_n \approx 1$

No.	ϱ	β	κ	T_n	T_{n+1}	$\widehat{\varrho}_{ML}$	$\widehat{\beta}_{ML}$	$\widehat{\kappa}_{ML}$	ML1	ML2	noML
1	7	0.1	0.75	1.0280	1.1239	6.0504	1.4320	1.0225	59.0	20.0	21.0
2	6	0.5	0.75	0.9499	1.0265	5.4462	1.8513	1.0028	64.0	20.0	16.0
3	4	1	0.75	1.0169	1.0810	4.9747	1.9949	1.1759	77.0	14.0	9.0
4	2	2	0.75	1.0717	1.1119	2.6204	2.9725	1.0799	96.3	3.6	0.1
5	10	0.1	1	0.9658	1.0546	8.6341	1.4219	1.3310	51.0	15.0	34.0
6	8	0.5	1	0.9541	1.0327	8.4871	1.5803	1.4112	73.0	11.0	16.0
7	6	1	1	0.9648	1.0275	6.7206	2.0185	1.4432	82.0	12.0	6.0
8	3	2	1	1.0107	1.0507	4.4054	2.6017	1.4649	95.0	4.0	1.0
9	15	0.1	1.5	0.9571	1.0463	17.2093	0.9155	2.3549	57.0	14.0	29.0
10	12	0.5	1.5	0.9652	1.0387	14.7420	1.0871	2.1843	68.0	15.0	17.0
11	9	1	1.5	0.9600	1.0212	10.8644	1.7009	2.1439	88.0	5.0	7.0
12	5	2	1.5	0.9667	1.0066	7.1646	2.5862	2.2518	99.4	0.5	0.1
13	20	0.1	2	0.9544	1.0474	23.2363	0.8064	2.9694	56.0	16.0	28.0
14	16	0.5	2	0.9747	1.0529	19.2950	1.0140	2.9353	74.0	10.0	16.
15	12	1	2	0.9849	1.0407	16.1466	1.4061	2.9422	88.5	5.0	6.5
16	6	2	2	1.0029	1.0452	9.3126	2.2567	2.9578	99.8	0.1	0.1

TABLE IV: The RMSE's of the ML estimates of ϱ, β and κ in the MGP(ϱ, β, κ); $n = 10, T_n \approx 1$

No.	ϱ	β	κ	se$(\widehat{\varrho}_{ML})$	se$(\widehat{\beta}_{ML})$	se$(\widehat{\kappa}_{ML})$
1	7	0.1	0.75	4.7626	1.7605	1.0336
2	6	0.5	0.75	5.1487	1.9632	0.7297
3	4	1	0.75	5.1027	1.6686	0.8532
4	2	2	0.75	3.2797	1.8863	1.0805
5	10	0.1	1	9.4946	1.8121	1.4686
6	8	0.5	1	8.4933	1.5654	1.2734
7	6	1	1	6.2604	1.7741	0.9360
8	3	2	1	4.7995	1.4662	0.9459
9	15	0.1	1.5	14.2683	1.0502	2.7438
10	12	0.5	1.5	11.5191	1.0301	2.0652
11	9	1	1.5	8.7395	1.2546	1.5103
12	5	2	1.5	8.1666	1.2879	1.2802
13	20	0.1	2	18.3434	0.9877	3.5316
14	16	0.5	2	13.6210	0.8326	2.9712
15	12	1	2	15.6613	0.9532	2.5069
16	6	2	2	10.8388	0.9356	2.3028

V. Concluding remarks

The result presented in Theorem 1 gives the prescription whether on the basis of a concrete realization of the process, subjected to the MGP(ϱ, β, κ) model, the statistician may try to estimate the unknown model parameters using the ML method. As the simulation study shows, it does not rarely happens that the MLE's do not exist and the existence of MLE's strongly depends on data represented by the event times of the process observed. For many triples (ϱ, β, κ) the percentage of the case of non-existence of the MLE's exceeds 20% and for some cases it exceeds even 30%. The Case 2^0

appears significantly less often than the Case 1^0. Theorem 2 presents the formulas for evaluating the MLE's in the Cases 1^0 and 2^0.

In the failure truncation procedure and the model parameters chosen to maintain approximately the same last failure time, we observe that the greater the parameter β (the smaller the parameter ϱ under the same κ) is, the greater the frequency ML1 is and the smaller the frequencies ML2 and noML are.

The estimators $\widehat{\beta}_{ML}$ are considerably less accurate than the estimators $\widehat{\varrho}_{ML}$ and $\widehat{\kappa}_{ML}$.

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