An Optimal Control Problem to Determine Optimal Paths of Some Macroeconomic Indicators. Empirical Study for Romania

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Abstract— In this paper we focus on the empirical application for Romania of an optimal control model having as state variables the the real wealth and expected inflation, as control variable real government expenditures and as objective function, the minimisation of the integral during initial and final time of the squared differences between actual real government expenditures and their corresponding political values. The dynamic system coefficients were calculated based on the equilibrium trajectory of IS-LM-SARS model using statistical data for Romania during 2001Q2-2016Q2 and computed the multipliers. We solved the model according to the Pontryagin's theorem and we computed the paths in MATLAB by Runge-Kutta algorithm. Finally, we computed and forecast the main macroeconomic indicators, replacing the optimal values resulted from optimal control problem, in the equilibrium equations given by IS-LM-SRAS model.

Keywords—IS-LM, Runge-Kutta algorithm, Pontryagin's theorem, optimal paths.

I. Introduction

BLANCHARD [1] improved Hicks and Hansen's IS-LM model by extending its dynamic version and introducing endogenous expectations of forward-looking agents. The model developed by Blanchard is deterministic, since he considers rational expectations and perfect foresight actions of agents.

In recent years various dynamic IS-LM models have been used to analyse the monetary policy. [2] proposed a dynamic IS-LM model with exchange rate adjustments. He studied the effects of interest rate parity and purchasing power parity, under the setting that prices are not fixed.

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[4] studied the limitations existing in the classic aggregate supply and demand in the IS-LM model, such as: the lack of micro foundation, short-term static nature, inconsistent logic, functional defect and research divergence. In order to overcome these limitations, they introduced the dynamic rational expectations in the IS-LM model. Another improvement of the IS-LM model could be obtained by adding the new Phillips curve.

[5] initiated a theoretical approach to a recently developed IS-LM model, called "expectational IS-LM model", since it updates the classic IS-LM model by adding rational expectations. The new IS-LM model has five endogenous variables: the log level of real output/spending, the log price level, the real interest rate, the inflation rate and the nominal interest rate. It comprises the forward-looking IS equation, the Fisher equation and the expectational Phillips curve.

[6] proposed a continuous time version of the Keynesian model and analyzed the effect of monetary policy according to the loss function and then according to the Taylor rule.

[7] used a two-element setting (national level and price setting) to minimize the expected value of a weighted sum of squared deviations of target variables from their equilibrium values.

[8] found a quantification of the economic growth, based on the dynamic IS-LM-SRAS model. IS and LM are taken linear functions of the real GDP, nominal interest rate and real wealth, and SRAS is derived from the linear Phillips curve. Two dynamic equations describe the evolution of the economy: expected inflation rate and wealth. The authors studied the evolution of the economy when the policy of money financial deficit is applied, the real stock of bonds being constant, under the assumption of adaptive inflationary expectations.

[9] continued the previous research from [8] by providing a short-term solution of the dynamic IS-LM-SRAS equilibrium. The model is applied assuming linear functions and all calculated multipliers are used for the static and dynamic analysis of sensitivity to determine the effects of shocks in the economy. Estimations functions for investments, consumption, LM function and SRAS function were obtained. Their results concluded that the estimations reveal a consistency with the observed data and the economic processes that were faced by the Romanian economy.

II. THE MODEL

In our previous works ([8], [19]) we considered a linear static IS-LM-SRAS model augmented by wealth and constant bonds monetary policy for budget deficit financing, completed with an equilibrium dynamic model for real wealth and expected inflation which is based on Sidrauski and Turnovski "complete macroeconomic dynamic model" [10], [11].

The wealth dynamics $\dot{a}(t) = \dot{m}(t) + \dot{b}(t)$ is given by the monetary policy for budget deficit financing $DB(t) = g(t) - \tau y(t) + i(t)b(t) - \pi(t)a(t)$. The real wealth, denoted by $a(t)$, is set up in real stock of money, denoted by $m(t)$ considered as the real monetary aggregate M1 (demand deposits and currency) and the real stock of bonds, denoted by $b(t)$.

The budget deficit, $DB(t)$ is set up in the primary deficit $(g(t) - \tau y(t))$, where $g(t)$ represents the real government expenditures, τ denotes the tax rate and $y(t)$ the real GDP. The debt service (interest payments) on government bonds is represented by $i(t)b(t)$, where $i(t)$ denotes the nominal interest rate and $b(t)$ the real stock of bonds.

In addition, the real actual wealth is eroded by inflation running, so it is reduced by the product $\pi(t)a(t)$, for each t.

The expected inflation dynamics is given by the adaptive mechanism of expectations i.e. the gap between actual inflation, denoted by $\pi(t)$ and expected inflation, denoted by $\pi^e(t)$. The adjustment speed is given by the nonnegative parameter α which reveals the adaptive inflationary expectations.

Thus, we consider the following dynamic model [8] :

$$\begin{cases} \dot{\pi}^e(t) = \alpha(\pi(t) - \pi^e(t)), \quad \alpha > 0 \\ \dot{a}(t) = g(t) - \tau y(t) + i(t)b(t) - \pi(t)a(t) \end{cases} \quad (1)$$

The dynamic model, considered on the equilibrium IS-LM-SRAS path, under the assumption of constant stock of bonds for deficit financing policy ($\tilde{y}(t)$, $\tilde{i}(t)$) is given by the following ordinary differential equation system (See Annex) [9]:

$$\begin{cases} \dot{a}(t) = \alpha_1 \pi^e(t)a(t) - \alpha_2 a^2(t) + \alpha_3 \pi^e(t) + \alpha_4 g(t) + \alpha \\ \dot{\pi}^e(t) = \beta_1 a(t) - \beta_2 \pi^e(t) + \beta_3 g(t) + \beta_4 \end{cases} \quad (2)$$

where $a(0)$ and $\pi^e(0)$ are given data.

The parameters $\alpha_1, \ldots, \alpha_5$ and $\beta_1, ..., \beta_4$ are defined and computed in [10].

We notice that in the above model (2), $a(t)$ and $\pi^e(t)$ are the state variables, while $g(t)$ is the instrumental variable. Considering a known fiscal policy $g(t)$, we have already studied the stability of the path using the state space analysis of the stationary trajectories in [9], [13].

As the paths are situated in an unstable zone, with real wealth increasing and expecting inflation decreasing down to negative values and given that the model already includes a macroeconomic monetary policy of constant stock of bonds for budget deficit financing, we introduced an alternative macroeconomic policy that could drive the economic system in a stable or feasible area.

In order to accomplish this goal, we consider the continuous objective function which is defined by the squares of gaps between the government expenditures and their policy values, namely $(g(t) - g^*(t))^2$. It follows that one can state the following optimal control problem:

$$\begin{cases} \min \int_0^T \left[(g(t) - g^*(t))^2\right] dt \\ \dot{a}(t) = \alpha_1 \pi^e(t)a(t) - \alpha_2 a^2(t) + \alpha_3 \pi^e(t) + \alpha_4 g(t) + \alpha \\ \dot{\pi}^e(t) = \beta_1 a(t) - \beta_2 \pi^e(t) + \beta_3 g(t) + \beta_4 \end{cases} \quad (3)$$

where $a(0)$ and $\pi^e(0)$ are given data.

Applying the Pontryagin theorem and algorithm, we obtain the following Hamiltonian dynamical system:

$$\begin{aligned} &\dot{\lambda}_1(t) = -\lambda_1(t)\alpha_1\pi^e(t) + 2\lambda_1(t)\alpha_2 a(t) - \lambda_2(t)\beta_1 \\ &\dot{\lambda}_2(t) = -\lambda_1(t)\alpha_1 a(t) - \lambda_1(t)\alpha_3 + \lambda_2(t)\beta_2 \\ &\dot{a}(t) = \alpha_1\pi^e(t))a(t) - \alpha_2 a^2(t) + \alpha_3\pi^e(t) - \\ &\frac{1}{2}\alpha_4^2 g(t)\lambda_1(t) - \frac{1}{2}\alpha_4\beta_3\lambda_2(t) + g^*(t) + \alpha_5 \\ &\dot{\pi}^e(t) = \beta_1 a(t) - \beta_2\pi^e(t) - \frac{1}{2}\beta_3\alpha_4\lambda_1(t) - \\ &\frac{1}{2}\beta_3^2\lambda_2(t) + g^*(t) + \beta_4 \end{aligned} \quad (4)$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are costate variables and the state variables $a(t)$ and $\pi^e(t)$. The values: $a(0)$, $\pi^e(0)$ and $\lambda_1(T)$, $\lambda_2(T)$ are given data.

One can notice that the nonlinear first order differential system given by the necessary optimum conditions contains mixed Cauchy conditions.

This system could be either linearized around the stationary point $(\lambda_1^*, \lambda_2^*, a^*, \pi^{e*})$ and then solved as a linear dynamic first order differential system, or numerically solved in MATLAB using Runge-Kutta algorithm.

III. EMPIRICAL EVIDENCES FOR ROMANIA

In [9] we have computed the IS-LM-SRAS equilibrium paths for real GDP and nominal interest rate. Taking into account that results, the ODE system (2) can be written as follows:

$$\begin{cases} \dot{a}(t) = 0.021479462 \;\; a(t)\pi^e(t) - 3.94444x10^{-8} \;\; a^2(t) \\ -23305.8366\pi^e(t) - 1146.7g(t) + 183817.0 \\ \dot{\pi}^e(t) = 2.53451x10^{-6} \;\; a(t) + 0.97446 \;\; \pi^e(t) \\ +0.00005632g(t) + 4.726409 \end{cases}$$

(5)

As we mentioned above, the system's path can be drive out from the unstable zone by considering the following quadratic objective function:

$$\min_{g(t)} \int_0^T (g(t) - 6870.524)^2 dt$$

(6)

where $g^*(t) = g^* = 6870.524$ is the value of government expenditures that correspond to the stationary values of the indicators, so the objective function is given by the integration of the squared differences between the actual government expenditures and its stationary value.

The necessary optimum conditions (8) become:

$$\dot{\lambda}_1(t) = -0.021479462\,\lambda_1(t)\pi^e(t) + 2x3.94444x10^{-8} \;\; \lambda_1(t)\alpha_2 a(t)$$
$$-2.53451x10^{-6} \;\; \lambda_2(t)$$
$$\dot{\lambda}_2(t) = -0.021479462\lambda_1(t)a(t) + 23305.8366\lambda_1(t)$$
$$-0.97446 \;\; \lambda_2(t)$$
$$\dot{a}(t) = 0.021479462 \;\; a(t)\pi^e(t) - 3.94444x10^{-8} \;\; a^2(t)$$
$$-23305.8366\pi^e(t) - 657460.445\lambda_1(t)$$
$$+0.032291\lambda_2(t) - 7878429.87$$
$$\dot{\pi}^e(t) = 2.53451x10^{-6} \;\; a(t) + 0.97446 \;\; \pi^e(t) +$$
$$0.032291\lambda_1(t) - 1.58597 \cdot 10^{-5}\lambda_2(t) + 5.1133569$$

(7)

The dynamic nonlinear system was solved by implementation of Runge-Kutta algorithm in MATLAB. The computation was done by considering the initial values of the real wealth and expected inflation at the level of 2016 Q1, namely $a_{2016Q1} = 152375.816$ *and* $\pi_{2016Q1} = -0.021$ and the final values of the adjunct variables were considered $\lambda_{1,2018Q3} = 0$ *and* $\lambda_{2,2018Q3} = 0$.

The optimal expected inflation and the optimal real wealth are represented in the Figure 1 and Figure 2 respectively.

The MATLAB output points out increasing path for expected inflation and parabolic path for real wealth, proving that the optimal control problem is well specified to get the economy out from the deflationary tendency and from the unstable zone.

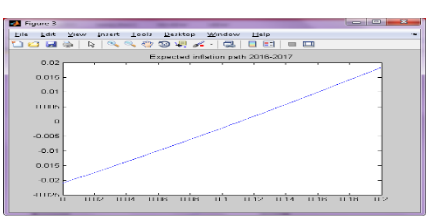

Fig. 1. Forecast of optimal expected inflation 2016Q1-2018Q3

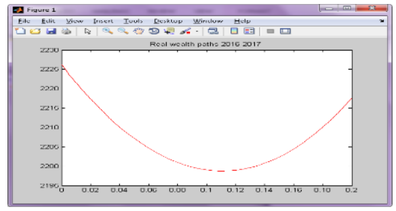

Fig. 2. Forecast of optimal real wealth 2016Q1-2018Q3

One can also notice in Figure 1 that the expected inflation has continuously increasing trend, from the initial value of $\pi^e_{2016Q1} = -2.1\%$ to $\pi^e_{2018Q3} = 2.0\%$. It proves that the policy mix assumed has the desired effect.

On the other hand, one can notice that the expected inflation gets closer to its stationary value $\pi^{e*} = 0.0237$ computed in [9].

Real wealth starts from $a_{2016Q1} = 2225.2392$ tens millions lei, and has a parabolic trend, recording a decrease down to $a_{2016Q4} = 2198{,}465$ tens of millions lei, then rising until 2018Q3, finally reaching approximate the starting value.

The minimum level of the real wealth is reached in the same period when the expected inflation passes from negative to positive values.

The uptrend of real wealth in terms of increasing current inflation reveals that budget deficits will register a slight increase, maintaining in Q3 of the year 2018, approximately the value of the 1st quarter of 2016.

The optimal actual government expenditures path has a decreasing trend from the initial value to the policy value $g^* = 6870.524$ million lei.

Next, we compute the values of the main macroeconomic indicators. In order to do this computation, we consider the equilibrium equations derived from the IS-LM-SARS model [8] and the optimal values of real wealth, expected inflation and government expenditures.

Firstly, we compute the optimal real GDP, denoted by $y^*(t)$ and defined by the following equation:

$$y^*(t) = \hat{k}_g g^*(t) + \hat{k}_a a^*(t) - \hat{k}_g i_r^{'} \pi^{e*}(t) + \hat{k}_g d^a_{ISLM}, + \hat{k}_m m(t)$$

(8)

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(8)

where $\hat{k}_g$, representing the government expenditures multiplier and $\hat{k}_a$, representing the real wealth multiplier, are computed under assumption of constant real bonds policy [9], $i_r^{'}$ denotes the sensitivity of real investments to real interest rate as estimated in [9], $g^*(t)$ representing the optimal government expenditures, $\pi^{e*}(t)$ representing the optimal expected inflation and $a^*(t)$, the optimal real wealth, are considered as optimal solution for the optimal control problem.

$d_{ISLM}^a = \underline{c} + \underline{i}^R - \hat{k}_m \underline{m}$ is the autonomous demand expressed as the sum of the autonomous consumption $\underline{c}$, the autonomous real investments $\underline{i}^R$, and the product of $\hat{k}_m$, the monetary multiplier as computed in [9] under the assumption of constant real bonds policy and $\underline{m}$ the autonomous money demand, estimated in [9]. In (8) $m(t)$ is the real monetary stock.

The computed function using statistical data for $m(t)$ is given by the following equation:

$$y^*(t) = 1.56 g^*(t) + 1.0331 m(t) + 0.02779 a^*(t) +$$
$$2284.064 \pi^{e*}(t) + 7312.5632$$

(9)

In Figure 3, actual real GDP are EXCEL forecast data and optimal real GDP are optimal data computed with the equation above. One can notice that the negative values of the optimal expected inflation, in the first quarters and the decrease of the real wealth until 2017Q3, Figure 1 and 2, give a parabolic graph to real GDP, Figure 3, surpassing in the first quarters of 2018 the values resulted from the linear forecast values. The overall tendency of the two forecasts give an increasing trend, with 5,1% in the case of optimal path forecast and 4,3% in the case of the linear forecast with respect to the corresponding value at the beginning of 2016.

Secondly, we compute the nominal interest rate using optimal values resulted from the optimal control problem. The analytical form of the equation, according to [8] is the following:

$$i^*(t) = -\frac{m_y^{'}}{m_i^{'}} y^*(t) - \frac{m_a^{'}}{m_i^{'}} a^*(t) + \frac{1}{m_i^{'}}(m(t) - \underline{m})$$

(10)

where $i^*(t)$ is the optimal nominal interest rate, $m_y^{'}$ is the sensitivity of the money demand to the real GDP, $m_i^{'}$ is the sensitivity of the money demand to nominal interest rate and $m_a^{'}$ is the sensitivity of the money demand to real wealth.

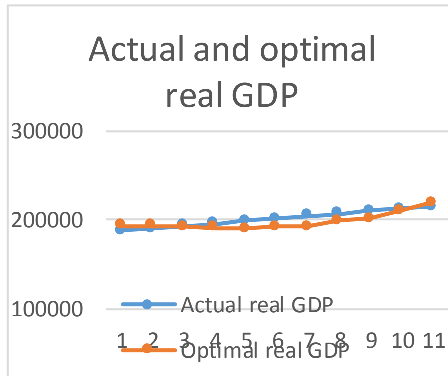

Fig. 3. Forecast of actual and optimal real GDP 20016Q1-2018Q3

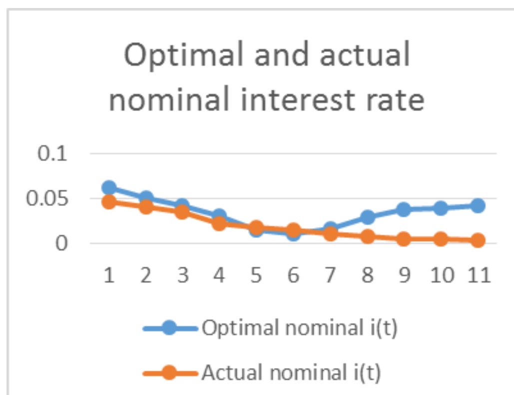

Fig.4. Forecast of actual and optimal nominal interest rate 2016Q1-2018Q3

According to the computation accomplished in [10], we can rewrite the equation (10) as it follows.

$$i^*(t) = 0.000019655 y^*(t) + 0.00002099 a^*(t)$$
$$-0.00005875 m(t) + 5.8091$$

(11)

One can notice in Figure 4 that the two forecasts have opposite trends: linear forecast has a decreasing one while the optimal path has a parabolic one. The final tendency of the optimal nominal interest rate is increasing to the value of 4%, then it stabilizes.

This is a notable result as in linear forecast, the decrease of the nominal interest rate, in a neighborhood of 0% could announce a financial crisis of liquidity trap type. Instead, the optimal paths, after a decrease, drive the nominal interest rate to an economically accepted value for this indicator and stabilizes it around 4%.

In order to compute the trajectory of real consumption, we first calculate real disposable income, using analytical expression resulted from SARS-IS-LM model, computed in [8] and the optimal values of real GDP, nominal interest rate, expected inflation, real wealth, computed above in this paper. Thus we consider the following equation:

$$y^{d*}(t)=(1-\tau)y^{*}(t)+i(t)^{*}b(t)-\pi^{*e}(t)a(t)^{*}+(\underline{\theta}-\underline{t})=(1-\tau)\tilde{y}(t)+i^{*}(t)b(t)-\pi^{e*}(t$$

(12)

The parameter $\tau = 0.179$ is the tax rate (computed as average, World Bank data).

$\underline{\theta}$ representing fixed transfers and $\underline{t}$ representing fixed taxes, are assumed to be equal.

In (17) $y^{d*}(t)$ is the optimal disposal income and $b(t)$ is the real stock of bonds, BNR data considered portfolio investment bonds nature, converted in lei and expressed in real values.

Next we considered the consumption function from IS-LM-SRAS equilibrium model as in [8]:

$c^{*}(t)=\underline{c}+c'_{y}y^{d*}(t)+c'_{a}a^{*}(t)$, where $c^{*}(t)$ is the optimal real consumption, c'_{y} is the marginal propensity to consumption in real income and c'_{a} is the marginal propensity to consumption in real wealth.

Considering the coefficients estimated in [10], $y^{d*}(t)$ and $a^{*}(t)$, we obtain:

$$c^{*}(t)=9330.47+0.593354\ y^{d*}(t)+0.130667\ a^{*}(t)$$

(13)

The real consumption path is represented in Figure 6.

One can notice in Figure 6 that the linear and optimal forecasts for real consumption have both increasing tendencies, but with a high difference in size. At the beginning of the period the consumption rate of the optimal real consumption in optimal real GDP is 0.545. Instead the consumption rate of the actual real consumption in actual real GDP is 0.3798.

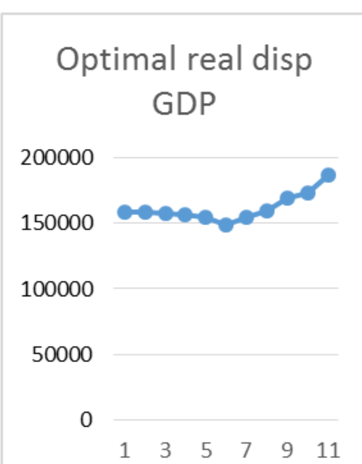

Fig. 5. Forecast of optimal real disposal GDP 2016Q1-2018Q3

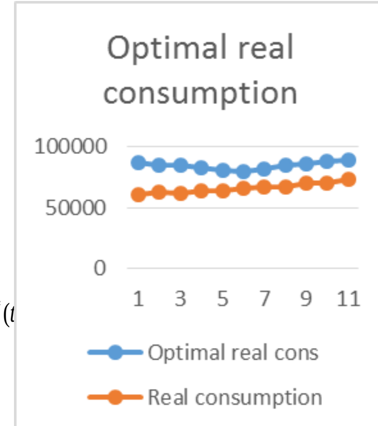

Fig. 6. Forecast of optimal and actual real consumption 2016Q1-2018Q3

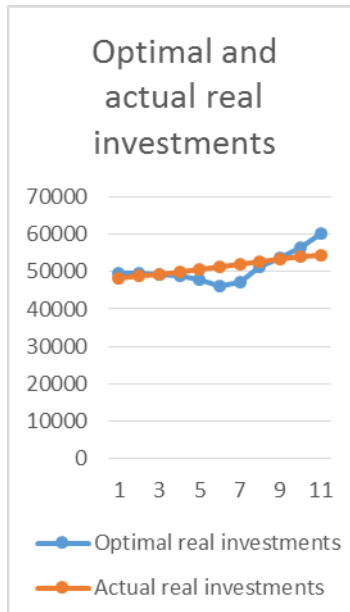

Fig. 7. Forecast of optimal and actual real investments 2016Q1-2018Q3

Next, we compute the optimal real investments, using the real investments function from IS-LM-SRAS equilibrium model [8], the regression parameters computed in [9] and the optimal paths computed above. Thus we consider the following equation

$$i^{R*}(t)=i'_{y}y^{*}(t)+i'_{r}(i^{*}(t)-\pi^{e*}(t))+\underline{i^{R}}$$

(14)

Where $i^{R*}(t)$ is the optimal real investments, i'_{r} is the sensitivity of the real investments to real interest rate and $\underline{i^{R}}$ is the autonomous real investments. The estimated equation, according to [9], is:

$$i^{R*}(t) = 0.293587y^*(t) - 1762.37(i^*(t) - \pi^{e*}(t)) \quad \textbf{(15)}$$

Both forecasts have an increasing tendency, the optimal path corresponds to a higher investments rate of 0.3193 which reveals an intensive investment process at the end of the forecasting period. Compared with the two predictions of consumption, investment predictions have closer values, the optimal forecast having a slightly parabolic curvature.

IV. Conclusions and further research

Starting from a dynamic macroeconomic model proposed by Sidrauski and Turnovski [10], [11] we initiated some years ago, the analysis of the macroeconomic Keynesian correlations in a linear form of the IS-LM-SRAS augmented with wealth model, considering the constant stock of bonds policy for deficit financing [8].

Later, in [9] we estimated the model concluding that the static model is well specified for the statistical data of Romania. Then, we constructed a dynamic differential nonlinear model with two state variables, namely real wealth and expected inflation for an approximation to the equilibrium evolution of the economy.

The dynamic equilibrium system was analyzed in [9], concluding that the paths are situated in an unstable zone with the real wealth increasing and expected inflation decreasing to negative values. To avoid this situation, we added an objective function to the dynamic equilibrium system, in order to get out the economy from the deflationist tendency and from the unstable zone. The objective function to be minimized is defined by integral during initial and final time of the squared differences between the actual and political values of government expenditures. The results reveal an improvement of the solution resulted from the equilibrium dynamic model, meaning that it pulls the economy out of deflation and out from the instable zone.

In this paper, based on the results obtained in [8], [9], we computed the main macroeconomic indicators: real income, nominal interest rate, disposal income, real consumption and real investments, using the equilibrium equations computed for IS-LM-SRAS model and estimated subsequently. In the same time, we have done a linear forecast for the statistical data and we compared it with the optimal data forecast based on the optimal control problem.

The results are notable as optimal real wealth imposes the parabolic curvature for most of the macroeconomic indicators, growth rates of the optimal indicators being higher than those obtained for the linear forecast. The overall tendencies of the two forecasts are consistent, exception making the tendency of nominal interest rate (increasing and stabilized for the optimal path forecast and decreasing for the actual data forecast). Another important difference between the two forecasts consist of the level of the values for the real consumption: the optimal forecasts values are much higher than the actual data forecast, meaning that the distribution of the optimal real income is in the favor of the consumption, the level of the investments remaining approximately the same for the two forecasts.

For further studies in this topic, we continue to search other mix of macroeconomic policies to drive the economy in a stable and feasible zone, and to study the effects on the main economic indicators on short run.

V ANNEXES

IS-LM-SRAS EQUILIBRIUM

For the closed economy with goods market, money market, government sector and wealth accumulation, the IS equation could be written as:

$$(IS): \quad (1-(1-\tau)c_y^{'} - i_y^{'})y_t - (i_r^{'} + c_y^{'} \cdot b_t)i_t = (c_a^{'} - c_y^{'}\pi_t^e) \cdot a_t + g_t - i_r^{'} \cdot \pi_t^e + d_{IS}^a \quad \text{(A1)}$$

where:

$d_{IS}^a = \underline{c} + \underline{i^R}$ is the autonomous demand for consumption and investments. We mention that relation (1) is obtained from the equilibrium between aggregate supply (y_t) and aggregate demand $d_t = c_t + i_t^R + g_t$ (A2)

computed as real values.

The equation (A1) is obtained following the usual equilibrium demand-supply requests on the goods market.

The real consumption, investments, taxes and disposal income are the following:

$$c_t = c_y^{'} \cdot y_t^d + c_a^{'} \cdot a_t + \underline{c} \quad \text{(A3)}$$

$$i_t^R = i_y^{'} \cdot y_t + i_r^{'} \cdot (i_t - \pi_t^e) + \underline{i^R} \quad \text{(A4)}$$

$$y_t^d = y_t - tx_t + i_t \cdot b_t + \theta_t - \pi_t^e \cdot a_t \quad \text{(A5)}$$

$$tx_t = \tau \cdot y_t + \underline{t} \quad \text{(A5.1)}$$

The equation (A3) is the Keynesian consumption function, extended to comprise the real wealth $a_t = \frac{A_t}{P_t}$, where $a_t = m_t + b_t$ is the sum of real money stock $m_t = \frac{M_t}{P_t}$ and real stock of bonds $b_t = \frac{B_t}{P_t}$, with P_t the consumption prices index.

The term $\underline{c}$ is autonomous consumption, $c_y^{'} \in (0,1)$ is the marginal propensity on consumption from the real disposal income y_t^d in the t-th period, $c_a^{'} \in (0,1)$ is the marginal propensity on consumption from real wealth.

In the equation (A2) in the left side, $i_t^R = i(y_t, i_t, \pi_t^e)$ is the real demand for investments,

with $i_y^{'} \in (0,1)$ is the sensitivity of investment to real income $i_y^{'} > 0$, $i_r < 0$ is the sensitivity of investment to real interest rate, $\underline{i}^R$ is the autonomous investments, $r_t = i_t - \pi_t^e$ real interest rate, i_t nominal interest rate and π_t^e expected inflation,

In the equation (A5), y_t^d is real disposal GDP, with tx_t the real taxes divided in fixed taxes ($\underline{t}$) and taxes depending on the real income ($\tau \cdot y_t$), $\tau \in (0,1)$ is the average tax rate, i_t the nominal interest rate, π_t^e is the expected inflation.

In (A5), θ_t are the transfers towards population, making initially the assumption that the transfers are constant and equal with autonomous taxes $\underline{t}$, so that, the real income becomes:

$$y_t^d = (1-\tau)y_t + i_t \cdot b_t - \pi_t^e \cdot a_t + (\theta_t - \underline{t}) = (1-\tau)y_t + i_t \cdot b_t - \pi_t^e \cdot a_t$$

(A6)

The LM function considered is:

$$(LM): m_y^{'} \cdot y_t + m_i^{'} \cdot i_t + m_a^{'} \cdot a_t + \underline{m} = m_t$$

(A7)

where in the left side of the equation is the real demand of money and in the right side, is the real supply of money $m_t = \frac{M_t}{P_t}$.

We consider the general assumptions on the propensities: $m_y^{'} > 0,\ m_i^{'} < 0,\ m_a^{'} > 0$, respectively marginal propensity of money demand with respect to real GDP, nominal interest rate, with real wealth, respectively.

The pair $(\tilde{y}_t, \tilde{i}_t)$ is the IS-LM equilibrium and is obtained by the IS-LM curves, given by the equations (A1) and (A7).

Denote by $g_t = \frac{G_t}{P_t}$ the real government spendings and $\hat{k}_{gt}$ the corresponding multiplier at time *t*.

$$\hat{k}_{gt} = \left[1-(1-\tau)c_y^{'} - i_y^{'} + \frac{m_y^{'}}{m_i^{'}}(i_r^{'} + c_y^{'} \cdot b_t)\right]^{-1}$$

(A8)

For $b_t = \overline{b}$, $\hat{k}_{gt}$ is a constant, respectively:

$$\hat{k}_g = \left[1-(1-\tau)c_y^{'} - i_y^{'} + \frac{m_y^{'}}{m_i^{'}}(i_r^{'} + c_y^{'} \cdot \overline{b})\right]^{-1}$$

(A9)

Similarly: $\hat{k}_{mt} = \frac{1}{m_i^{'}}(i_r^{'} + c_y^{'} \cdot b_t) \cdot \hat{k}_{gt}$ is the monetary multiplier at the time t. (A10)

Under the assumption of constant stock of bonds, $b_t = \overline{b}$, the monetary multiplier is also constant, given by:

$$\hat{k}_m = \frac{1}{m_i^{'}}(i_r^{'} + c_y^{'} \cdot \overline{b}) \cdot \hat{k}_g$$

(A11)

The real wealth multiplier at time *t* is:

$$\hat{k}_{at} = (c_a^{'} - c_y^{'} \cdot \pi_t^e)\hat{k}_{gt} - m_a^{'} \cdot \hat{k}_{mt}$$

(A12)

Considering the expected inflation rate constant, $\pi_t^e = \overline{\pi}^e$ and constant stock of bonds $b_t = \overline{b}$, $\hat{k}_{at}$ becomes constant in time:

$$\hat{k}_a = (c_a^{'} - c_y^{'} \cdot \overline{\pi}^e)\hat{k}_g - m_a^{'} \cdot \hat{k}_m$$

(A13)

The IS-LM equilibrium resulted is:

- equilibrium real income

$$\tilde{y}_t = \hat{k}_{gt} \cdot g_t + \hat{k}_{mt} \cdot m_t + \hat{k}_{at} \cdot a_t - \hat{k}_{gt} \cdot i_r^{'} \cdot \pi_t^e + \hat{k}_{gt} \cdot d_{ISLM}^a$$

(A14)

with: $d_{ISLM}^a = \underline{c} + \underline{i}^R - \hat{k}_{mt} \cdot \underline{m}$ autonomous demand of goods and money;

- equilibrium interest rate:

$$\tilde{i}_t = -\frac{m_y^{'}}{m_i^{'}} \cdot \tilde{y}_t - \frac{m_a^{'}}{m_i^{'}} \cdot a_t + \frac{1}{m_i^{'}}(m_t - \underline{m})$$

(A15)

Equilibrium inflation rate is obtained using initially a Phillips curve without shocks, for simplicity:

$$\pi_t - \pi_t^e = b(u_t - u_N),\ \ b < 0$$

where u_t is the actual unemployment rate and u_N is the NAIRU (Non Accelerating Inflation Rate of Unemployment) unemployment rate.

Using a simple computation, we can deduce SRAS (Short Run Aggregate Supply):

$$\pi_t = \pi_t^e + \gamma(y_t - \overline{y}),\ \ \gamma > 0$$

(A16)

with $\gamma = -\frac{b}{A\overline{L}} > 0$, $A\overline{L}$ being effective potential labor, that means the product of the potential labor and the technological progress termor labor productivity A.

In the long run, the equality $\pi_t = \pi_t^e = \pi^*$ is verified, the SRAS equilibrium becomes LRAS (Long Run Aggregate Supply) equilibrium.

$$(LARS) \qquad y_t = \overline{y}$$
(A17)

The model considers the dynamics of the economy given by two state variables: π_t^e and a_t:

$$\begin{cases} \dot{\pi}_t^e = \alpha(\pi_t - \pi_t^e), \quad \alpha > 0 & (A18) \\ \dot{a}_t = g_t - \tau \cdot y_t + i_t \cdot b_t - \pi_t \cdot a_t & (A19) \end{cases}$$

The equation (A18) emphasizes the fact that the expected inflation dynamics is given by the gap between actual and expected inflation, with the adjustment speed given by the parameter α.

The wealth dynamics $\dot{a}_t = \dot{m}_t + \dot{b}_t$ is given by the monetary policy for budget deficit $DB_t = g_t - \tau y_t + i_t b_t$ financing.

The budget deficit is set up in the primary deficit $(g_t - \tau y_t)$ and the debt service (interest payments) on government bonds $(i_t b_t)$.

In addition, the real current wealth is eroded by inflation running so it is reduced by $(-\pi_t \cdot a_t)$.

In conclusion, the economy is characterized by five endogenous variables $(y_t, i_t, \pi_t, \pi_t^e, a_t)$ from which the last two (π_t^e, a_t) are state variables.

The instrumental variables are:

1. depending on the monetary policy of the government, either - m_t the money issue, or - b_t bonds issue (so that in $a_t = m_t + b_t$ the other variable results to be endogenous);

2. g_t instrumental variable used by government.

We notice that the expected inflation dynamics $\dot{\pi}_t^e$ can be highlighted in relation to an unobservable variable "Okun gap" , i.e. the deviation between actual GDP and potential GDP $y_t - \overline{y}$:, so that, relation (A18) could be replaced with the relation :

$$\dot{\pi}_t^e = \alpha\gamma(y_t - \overline{y})$$
(A20)

The specification of the dynamic model on the short run is done replacing in the equations (A20) and (A19) the equilibrium paths $\tilde{y}_t = \tilde{y}(a_t, \pi_t^e, g_t, \overline{b})$ and $\tilde{i}_t = \tilde{i}(a_t, \pi_t^e, g_t, \overline{b})$, so that:

$$\begin{cases} \dot{a}_t = g_t - (\gamma \cdot a_t + \tau)\tilde{y}_t + (\gamma \cdot \overline{y} - \pi_t^e)a_t + \tilde{i}_t \cdot \overline{b} \\ \dot{\pi}_t^e = \alpha\gamma(\tilde{y}_t - \overline{y}) \end{cases}$$
(A21)

Taking into account the equilibrium paths (A14), (A15) we obtain the differential equations system that give the dynamics of the economy in equilibrium:

$$\begin{cases} \dot{a}_t = \alpha_1(\pi_t^e) \cdot a_t - \alpha_2 \cdot a_t^2 + \alpha_3\pi_t^e + \alpha_4 \cdot g_t + \alpha_5 & (A22) \\ \dot{\pi}_t^e = \beta_1 \cdot a_t - \beta_2\pi_t^e + \beta_3 \cdot g_t + \beta_4 & (A23) \end{cases}$$

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