

Multi-attribute Risk Assessment using Stochastic Dominance

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Abstract—Risk assessment became a crucial factor in the process of decision making regarding the management of both physical and intangible assets. However, when making these decisions, several aspects have to be considered simultaneously, very often in an uncertain environment, what makes standard risk assessment techniques insufficient. Moreover, for safety critical assets, usual measure of risk – the expected value cannot be the unique quantification measure. This paper introduces a new methodology for the multi-attribute risk assessment of critical assets, seen as the combination of different risk factors. The methodology is based on numerical convolution of probability distribution functions of multi-attribute utilities. After the new, aggregated probability distribution is built for every alternative, the ranking of alternative is performed by the stochastic dominance rules. This method allows the usage of both additive and multiplicative multi-attribute utility function, which enables to incorporate the risk attitude of the decision maker in the initial stage. Finally, numerical example is given to illustrate the proposed methodology.

Index Terms — Convolution, Multi-attribute utility function, Stochastic dominance, Risk Assessment

I. INTRODUCTION

Risk Assessment is integral part of Risk management process, representing the necessary step before proceeding to the treatment of risk attempting to answer the following questions:

- What can happen and why?
- What are the consequences?
- What is the probability of their future occurrence?
- Are there any factors that mitigate the consequence of the risk or that reduce the probability of the risk?

Most often, the risk associated with an event E is defined as the product of probability of event (failure of component or group of components) and consequences of this event:

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$$Risk(E)=p(E) \times Con(E) \quad (1)$$

However, this simple relationship does not take into account the factors such as nonlinear relationships between the utility and the value of consequences, which requires more complex relationships between values:

$$Risk(E)=p(E)^y \times Con(E)^x \quad (2)$$

With this definition of risk widely adopted by many disciplines, its translation into quantitative terms has been a major source of conflicts and misguided use. The most common quantification of risk as the expected value is probably the dominant reason for this misunderstanding in the quantification of risk. In both discrete and continuous universe of probability space, the expected value of risk is an operation that multiplies each event by its probability of occurrence and sums (or integrates) all these products over the entire universe of events. However, this operation commensurate adverse events of high consequences and low probabilities of exceedance with events of low consequences and high probabilities of exceedance.

Another problem with such data analysis models lies in extreme events, which are rare by nature and may have not yet occurred in the recent history of a financial institution [1, 2]. Therefore, accurate prediction of the frequency of extreme events is of primary importance in many financial applications such as Value-at-Risk (VaR) analysis [3].

The third problem in the Risk assessment process is the simplistic risk evaluation that result in lists of risks or sets of scenarios not providing sufficient information to prioritize requirements when faced with resource constraints (e.g., time, money) [4, 5]. These models are also incomplete due to the need to take into consideration the aggregation of multiple objectives, such as: human, environmental and financial consequences that an accident on physical asset may cause.

For these reasons, multi-attribute risk assessments and stochastic dominance are combined in this methodology as possible answer to problems imposed by the complex requirements of the risk assessment process. Multi-attribute risk assessment provides a convenient framework for systematically developing quantitative risk assessments that one can use to prioritize security requirements. On the other hand - Stochastic multiple criteria decision making (SMCDM) refers to the problem of selecting alternatives associated with multiple attributes/criteria, where consequences of alternatives with respect to criteria are in the form of random variables.

In the remainder of the paper, we consider a decision problem consisting of n alternatives denoted by a_i , $i = \{1, \dots, n\}$, each evaluated on m criteria denoted by c_j , $j = \{1, \dots, m\}$. Let z_{ij} be the evaluation of a_i in terms of criterion c_j , according to some suitable performance measure. Our concern is with decision making situations in which the values of z_{ij} for each i are not known with certainty for all j . This formulation is known as ACE (or AEE) model (Alternatives, Attributes/Criteria, Evaluators).

The proposed methodology is based on the numerical convolution of probability distribution functions of attributes evaluations. The result of this convolution is used for the derivation of aggregated utility function probability distribution. This method of aggregation enables the use of more complex utility functions, besides the simple additive form. After the new, aggregated probability distribution is built for every alternative, the ranking of alternatives is performed by the stochastic dominance rules. This step greatly reduces and makes more practical the procedure of the ranking of different alternatives, what will be demonstrated through the illustrative example.

The paper is organized in the following way. In the next section, the literature review, following the problem formulation is presented. In the following section, the new methodology for the stochastic multiple criteria decision making, describing each step of the methodology: composition of multiplicative multi-attribute utility function, aggregation of utility function based on probability convolution and the stochastic dominance rules for the ranking of alternatives is explained. Finally, the methodology is illustrated through the example of finding the investment alternative with the minimum risk.

II. LITERATURE REVIEW

The problem of uncertainty in decision making is related to two types of uncertainties: internal uncertainty, relating to the process of problem structuring and analysis, and external uncertainty, regarding the nature of the environment and the consequences of particular course of action, which may be outside of the control of the decision maker. The taxonomy of different uncertainty types and the approaches to solve them is given in [6, 7]. The most common type of external uncertainty is related to the stochastic nature of outcomes of proposed actions, where multivariate probability distribution governs the joint realization of performance outcomes across all alternatives and all attributes.

Since the early works in this field, multi-attribute utility theory (MAUT), a structured methodology designed to handle the tradeoffs among multiple objectives has proven to be very useful when dealing with different problems regarding multi-agent and communication systems [8, 9]. The purpose of using MAUT was to produce a function such that an alternative is preferred to another if and only if its expected utility is greater. Practically, this requires the construction, for each criterion c_j , of a marginal utility function u_j satisfying the Von Neumann–Morgenstern axioms [10], and some way of aggregating the marginal utility functions into a global utility function U such that the expected utility hypothesis is still satisfied. For an additive aggregation, preferences for lotteries defined over

multiple attributes must be “additively independent”, i.e. depend only on the marginal distributions and not on any interactions between attributes. More complex aggregation forms are available (multiplicative and multi-linear aggregations) if additive independence does not hold [11, 12].

In recent years, some methods using stochastic dominance (SD) rules have been proposed to solve SMCDM problems [13-29]. In a multi-criteria framework, stochastic dominance conditions are checked for each individual criterion, using the marginal distributions. These methods generally include two processes: comparison and selection. The former is to identify whether there exists a SD relation for comparison of any pair of alternatives using SD rules. The latter is to rank alternatives based on the determined SD relations using outranking methods [20], interactive procedures [21, 22] or Rough Set Theory [24, 25]. Huang et al. [31] have shown that a necessary condition for multi-attribute stochastic dominance is stochastic dominance on each individual criterion. However, the conflicting nature of multi-criteria problems means that these simultaneous conditions are unlikely to occur often.

Practical applications of SMCDM problems real-world situations where consequences with respect to criteria such as the traffic component of passenger are random variables are elaborated in [9, 26]. Besides, the method based on stochastic dominance degree (SDD) has been also proposed [29, 30]. In this method, SD relations for pair-wise comparisons of alternatives with respect to all criteria are identified using SD rules, and then, in the situation that SD relations are determined, the SDDs corresponding to the SD relations are calculated to build SDD matrices. Furthermore, an overall SDD matrix is constructed. Finally, based on the obtained overall SDD matrix, an outranking method is given to obtain the ranking result of alternatives.

III. PROBLEM FORMULATION

A discrete multi-attribute decision making problem may be conceived as a model (A, C, E), where A is finite set of actions a_i , $i = \{1, \dots, n\}$, C is a finite set of attributes c_j , $j = \{1, \dots, m\}$, and Z is the set of evaluation of action with respect to attributes z_{ik} , $i = 1, 2, \dots, n$, $k = 1, 2, \dots, m$. In decision analysis based on probabilities, for each alternative a_i , the z_{ij} are viewed as random variables with an associated (m -dimensional) multivariate probability distribution function F_i . Let F_{ij} denote the corresponding marginal distribution function for criterion c_j if alternative a_i is selected, and f_{ij} the associated probability density function.

Huang et al. [31] showed in the case of probability independence and the additive multi-attributes utility function, that the necessary condition for the multi-attributes stochastic dominance is to verify stochastic dominance on the level of each attribute. In practice, the essential characteristic of a multi-attributes problem is that the attributes are conflicting, and consequently, multi-attributes stochastic dominance relations results poor and useless to decision maker.

In the proposed methodology, stochastic dominance of aggregated probability distribution functions of alternatives is used. The problem of correlation between attributes is solved by multiplicative utility function. The aggregated utility

function is composed with the convolution of marginal probability distributions. The stochastic dominance is evaluated only on aggregated probability functions, instead of pair-wise comparison of alternatives for individual criterion, which makes the application of methodology more practical.

The decision process is performed in four steps. The first step is the building of multiplicative utility function. In the second step, using the numerical calculation of convolution of individual criterion probability distribution functions, the aggregated probability distribution is composed. In the third step, using stochastic dominance rules, a dominance matrix is formed, based on which, in the final step, a ranking of alternatives is performed.

The decision process is performed in four steps, schematically represented on figure 1.

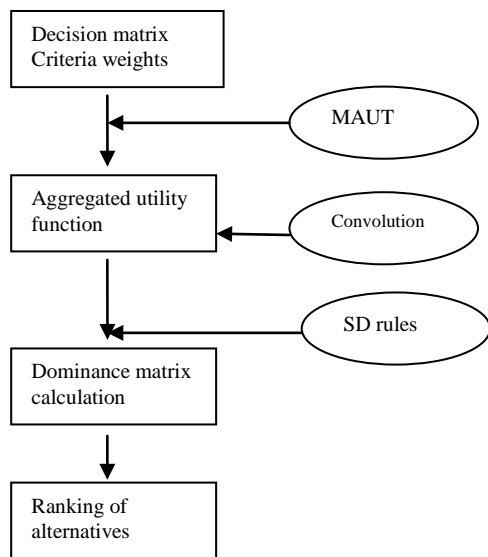


Figure 1. Schematical representation of decision making procedure

The first step is the building of multiplicative utility function. In the second step, using the numerical calculation of convolution of individual criterion probability distribution functions, an aggregated probability distribution is derived. In the third step, using SD rules, a dominance matrix is formed, based on which, in the final step, a ranking of alternatives is performed.

A. Aggregated multi-attribute utility function

Multi-attribute utility theory is concerned with expressing the utilities of multiple-attribute outcomes or consequences as a function of the utilities of each attribute taken singly. The theory specifies several possible functions (additive, multiplicative and multi-linear) and the conditions to be met under which each would be appropriate. If additive independence exists, the multi-attribute utility function is additive (3), where w_i represents the weighting factor.

$$U(x) = \sum_{i=1}^n w_i \cdot u_i(x_i)$$

$$\sum_{i=1}^n w_i = 1 \tag{3}$$

If mutual utility independence exists, the multi-attribute utility function is additive or multiplicative of the form

$$U(x_1, x_2, \dots, x_n) = \frac{\prod_i (1 + K k_i u_i(x_i)) - 1}{K} \tag{4}$$

Here,

$u_i(x_i)$ = the single-attribute utility value for attribute i with score x_i (ranges from 0 to 1),

k_i = a parameter from the tradeoff for component i , for all i , and

K = a normalization constant, ensuring that the utility values are scaled over the component range space between 0 and 1.

One method to determine the function is to measure each $u(x)$, determine the k_j values, and find the K value by iteratively solving (5).

$$1 + K = \prod_{i=1}^n (1 + K \cdot k_i) \tag{5}$$

So far, SMCMDM problems were exclusively related to additive form of utility functions. However, the three cases in expression (6) can be distinguished in terms of the multivariate risk posture, which they represent. As the value of K ranges from negative, to 0, and to positive, the overall utility function can reflect three different types of interactions between individual components. In the compensatory case, performance of one component can make up for the lack of performance by other components. In the complementary case, a good performance by one component is less important than balanced performance across the components. In the additive case, the performance of one component does not interact with the value of the other components.

$$\text{if } \sum_{i=1}^n k_i > 1, \text{ then } -1 < K < 0 \tag{6.a}$$

$$\sum_{i=1}^n k_i = 1, \text{ then } K = 0, \text{ and the additive model holds } \tag{6.b}$$

$$\sum_{i=1}^n k_i < 1, \text{ then } K > 0 \tag{6.c}$$

The case a) represents multivariate risk aversion, second case risk neutrality and the c) case risk seeking behavior. The attributes in the first case can be characterized as compensatory, or substitutes, while those in the third case are complements. The intuitive interpretation is that substitute attributes are such that an improvement in one is relatively satisfying, while an improvement on two or more is not much better. With complementary attributes, an improvement on any one alone is not very useful, while a simultaneous improvement on several is much better.

The useful representation of the function is obtained by setting $c_i = K k_i$, for all i , which leads to the following form:

$$U = \frac{\prod_i (1 + c_i u_i(x_i)) - 1}{\prod_i (1 + c_i) - 1} \quad (7)$$

A compensatory relationship means that a high utility on one component can partially compensate for a low utility on the other. The strong compensatory case can be thought of as a strong OR, where the overall utility evaluates to 1 if any of the components utility functions evaluate to 1. Algebraically, this interaction is obtained when $c_i = -1$ for all components i .

The Archetypal Weak Compensatory case is obtained when $c_i = -0.5$ for all components i .

Components have a complementary relationship when they reinforce each other, or when all are needed to perform a function. In a strong complementary case, the worst performance by one component entirely cancels out the performance of the other components. This can be thought of as a strong AND, where the overall utility evaluates to 0 if any of the components utility functions evaluate to 0. Algebraically, this kind of interaction is obtained when $c = \infty$ for all components i .

In previous methodologies, the decision maker risk attitude is taken into account only at individual level of criterion comparison, while with the multiplicative function; this attitude can be directly incorporated in the model itself. Having in mind that the additive form is just a special case of multiplicative function, the later is adopted as general utility function form in this methodology.

In this model, $u_i(x_i)$ - the single-attribute utility value for attribute i with score x_i (ranges from 0 to 1), has been represented by the evaluation z_i , following some probability distribution.

After the appropriate model has been adopted for the utility function representation, the second step is the aggregation of marginal probability distributions. The problem of this aggregation can be formulated in a following way. If we know the probability distributions of individual evaluations – random variables z_i , what is the probability distribution of combination of random variables given by expression (4)?

B. Aggregation of utility distribution functions

For the four basic arithmetic operations on random variables, the answer to the question what is the probability distribution of combination of random variables given by expression (4) can be obtained by the convolution. The convolution of probability distributions arises in probability theory and statistics as the operation in terms of probability distributions that corresponds to the addition of independent random variables and, by extension, to forming linear combinations of random variables. For two functions, f_X and f_Y , the convolution of basic operation of functions is given [32]:

$$\begin{aligned} Z = X + Y : f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-x) \cdot f_Y(x) du \\ Z = X - Y : f_Z(z) &= \int_{-\infty}^{\infty} f_X(z+x) \cdot f_Y(x) du \\ Z = X \times Y : f_Z(z) &= \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(z/x) \cdot f_Y(x) du \\ Z = X / Y : f_Z(z) &= \int_{-\infty}^{\infty} |x| f_X(zx) \cdot f_Y(x) du \end{aligned} \quad (8)$$

The implementation of this result is relying mostly on Laplace, Mellin, Fourier or other transformation techniques. The convolution of n distributions can be obtained by the inverse Fourier transform, or other asymptotic expansion [33]. However, these transformations are not straightforward for general X and Y .

The definition of the convolution of vectors is similar to the above definition of the convolution of continuous functions. Let X and Y be two independent integer-valued random variables, with distribution functions f_X and f_Y respectively. Then the convolution of f_X and f_Y is the distribution function f_Z given by:

$$f_Z(j) = \sum_k f_X(k) \cdot f_Y(j-k) \quad (9)$$

for $j = -\infty \dots \infty$. The function $f_Z(j)$ is the distribution function of the random variable $Z = X + Y$.

Evans and Leemis [34] presented an efficient algorithm for computing the distributions of sums of discrete random variables. However, the main problem of using regular convolutions or some form of transformation (Discrete Fourier, Fast Fourier or Edgeworth expansion) is their unique relation to the summation of individual distributions. The more general, multiplicative form of utility function requires other convolution type. In the proposed methodology, the numerical solving of discrete convolution of vectors is used, allowing more complicated utility function forms, directly computing the cumulative distribution function which will be used in stochastic dominance matrix calculation.

The simplest methodology used to compute the PDF of the convolution of the sum of two independent discrete random variables PDFs is by the "brute force method". Let X have values of x_1, x_2, \dots, x_n and Y have values of y_1, y_2, \dots, y_m . The method computes all possible sums between the values of X and the values of Y by brute force, e.g., $x_1 + y_1, x_1 + y_2, \dots, x_1 + y_m, x_2 + y_1, x_2 + y_2, \dots, x_n + y_m$. The sums are placed in the one-dimensional array S of length $n \cdot m$. The corresponding probabilities for each of these sums, $f_X(x_1) \cdot f_Y(y_1), f_X(x_1) \cdot f_Y(y_2), \dots, f_X(x_n) \cdot f_Y(y_m)$, are stored in an one-dimensional array P of length $n \cdot m$. The probability in position P_i corresponds to the sum in position $S_i, i = 1, 2, \dots, n \cdot m$.

In the proposed methodology, the computational procedure is speeded up by the reduction of dimensions of arrays P and S to the number of evaluation grades. For the l grade evaluation scale,

$$p_j = \sum_i^{n-m} p_i, i \in (j-l, j) \tag{10}$$

The same method is used for any combination of two or more random variables.

For the sake of illustration, suppose that seven experts evaluate alternatives over the set of three criteria (C1, C2, C3) on a scale of ten (1, the worst, 10, the best), and evaluations of i-th alternative are expressed in the form of the discrete probability distribution as shown in Table I.

For three evaluation distributions, results of 4 different combinations of the type of utility function and appropriate weight factors are presented in table II. The first two columns represent simple additive model (3) with different and equal weights, while the third and fourth column represent the multiplicative model. The third case, with equal coefficient represents “archetypal” compensatory case, for the risk averse behavior, while the U_4 represents the risk seeking attitude.

Using expressions (9) and (10), the aggregated probability and evaluations are obtained. Weighting factors for all cases together with the probability distributions of evaluations are presented in the table II.

$$U = w_1 \cdot e_1 + w_2 \cdot e_2 + w_3 \cdot e_3 \tag{11}$$

$$U = \frac{(1 + K \cdot k_1 \cdot E_1) \cdot (1 + K \cdot k_2 \cdot E_2) \cdot (1 + K \cdot k_3 \cdot E_3)}{K} \tag{12}$$

TABLE I. EVALUATION DISTRIBUTION OF THREE CRITERIA

Scores e	Criteria		
	C_1	C_2	C_3
1		2/7	
2			1/7
3	3/7		
4		1/7	1/7
5	2/7	2/7	
6		1/7	3/7
7			
8	1/7	1/7	1/7
9			
10	1/7		1/7

TABLE II. AGGREGATED UTILITY VALUES

i	e_1	e_2	e_3	P	U_1	U_2	U_3	U_4
					$w_1=0.5$	$w_1=0.33$	$k_1=0.5$	$k_1=0.2$
					$w_2=0.3$	$w_2=0.33$	$k_2=0.5$	$k_2=0.3$
					$w_3=0.2$	$w_3=0.33$	$k_3=0.5$	$k_3=0.2$
							$K = 0.76$	$K = 0.309$
1	1	1	1	0	1	1	1	1
2	1	1	2	0	1.2	1.32	1.6	1.25
3	1	1	3	0	1.4	1.65	2.19	2.50
...								
302	3	1	2	0.017	2.2	1.98	2.50	1.77
...								
878	8	8	8	0.003	8	7.92	9.97	7.45
...								
1000	10	10	10	0	10	10	10	10

After the calculation of probability distribution, for every real number x , the cumulative distribution function of a real-valued random variable U is given by

$$F_x(x) = P(X \leq x) = \sum_{u \leq x} P(X = x) = \sum_{u \leq x} f_x(u) \tag{13}$$

Probability distribution functions and their appropriate cumulative distributions are represented on figure 1.

The comparison of different CDFs corresponding to aggregated utility function is now possible with the stochastic dominance principle.

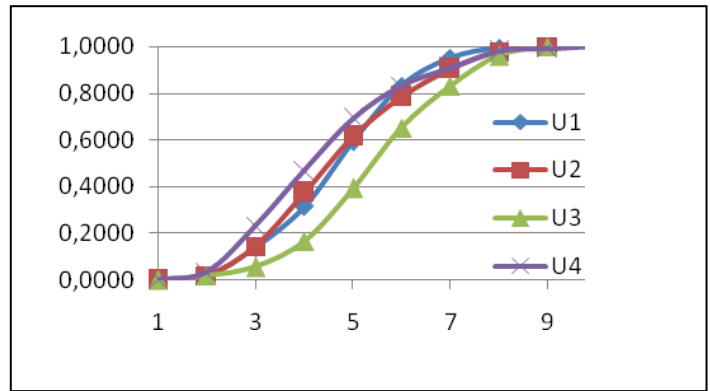


Fig. 1. Cumulative probability distribution function of aggregated utility

C. Stochastic dominance

In order to determine whether a relation of stochastic dominance holds between two distributions, the distributions are characterized by their cumulative distribution functions, or CDFs. Suppose that we consider two distributions A and B , characterized respectively by CDFs F_A and F_B . Then distribution B dominates distribution A stochastically at first order if, for any argument y , $F_A(y) \geq F_B(y)$.

The stochastic dominance rules can be fundamentally classified in two groups for two classes of utility functions [13]. The first group is for increasing concave utility function and includes first degree stochastic dominance, second degree stochastic dominance and third degree stochastic dominance. These rules can be applied for modeling risk averse preferences.

Definition 1.

Let a and b ($a < b$) be two real numbers, X and Y be two random variables, $F(x)$ and $G(x)$ be cumulative distribution functions of X and Y , respectively. Let U_1 include all the utility functions u for which $u' \geq 0$, U_2 include all the functions u for which $u' \geq 0$ and $u'' \leq 0$, U_3 include all the functions u for which $u' \geq 0$, $u'' \leq 0$ and $u''' \geq 0$.

Let EF and EG be the two expectations or the means, respectively. Let $SD1$, $SD2$ and $SD3$ denote first, second and third degree stochastic dominance, respectively. The SD rules are:

$$F(x) SD1 G(x) \text{ if and only if}$$

$EF(u(X)) \geq EG(u(Y))$ for all $u \in U_1$ with strict inequality for some u , or

$F(x) \leq G(x)$ for all $x \in [a, b]$ with strict inequality for some x ;

$F(x)$ SD2 $G(x)$ if and only if

$EF(u(X)) \geq EG(u(Y))$ for all $u \in U_2$ with strict inequality for some u , or

$$\int_a^x F(t) dt \leq \int_a^x G(t) dt$$

for all $x \in [a, b]$ with strict inequality for some x ;

$F(x)$ SD3 $G(x)$ if and only if $E_F(X) \geq E_G(Y)$ and

$EF(u(X)) \geq EG(u(Y))$ for all $u \in U_3$ with strict inequality for some u , or

$$\int_a^x \int_a^t F(z) dz dt \leq \int_a^x \int_a^t G(z) dz dt \leq \text{for all } x \in [a, b] \text{ with}$$

strict inequality for some x ;

The second group of SD rules is for increasing convex utility function and includes first degree stochastic dominance, second inverse stochastic dominance, third inverse stochastic dominance of the first type and third inverse stochastic dominance of second type. These rules are equivalent to expected utility maximization rule for risk-seeking preferences. The first group is used in the domain of gains, whereas the second group is used in the domain of losses. As larger values of criteria are better than smaller ones and the criteria can be defined in the domain of gains, it is assumed that the decision maker's utility function is concave utility function.

Definition 2

In [28], a SD degree is defined, in the following way:

If $F(x)$ SD $_h$ $G(x)$, $h \in \{1, 2, 3\}$, then the stochastic dominance degree SDD of $F(x)$ SD $_h$ $G(x)$ is given by:

$$\psi(F(x)SD_h G(x)) = \frac{-\int_{\Omega} [F(x) - G(x)] dx}{\int_{\Omega} G(x) dx}, h \in \{1, 2, 3\} \quad (14)$$

$$\Omega = \{x | x \in [a, b]\}$$

Both SD rules and SD degrees are used in the proposed methodology.

The final step in this methodology is the alternative ranking based on the results of the dominance matrix. Two types of dominance matrices will be used in this methodology: the first one obtained by the three types of stochastic dominance from the Definition 1. Using the first, second or third degree stochastic dominance rule, the appropriate type of the dominance matrix is obtained, where the elements of the dominance matrix are defined in the following way:

$$sd_{ij} = 1, \text{ if } F_{Ai} \text{ SD}_h F_{Aj}, \text{ otherwise, } sd_{ij} = 0, h \in \{1, 2, 3\}$$

In the second dominance matrix type, the matrix elements are obtained by the expression (14). The use of both types of dominance matrices is equal in this methodology, but the choice of appropriate dominance type depends on the required level of distinction and grading between alternatives.

IV. CASE STUDY

This example is investigated in [28] and is related to the possibility of investment for an investment company to a most potential industry. There are six alternatives to be considered: the car industry A_1 , the pharmaceutical industry A_2 , the food industry A_3 , the logistics industry A_4 , the clothing industry A_5 and the computer industry A_6 . When making a decision, the criteria considered includes: the profit C_1 , the growth C_2 and the environment C_3 . The criterion weight vector provided by the investment company is $w = (0.3; 0.5; 0.2)$. The five experts provide their preference evaluations on alternatives in the form of scores 1-5 (1, the worst; 5, the best), as shown in Table III. From the data in Table III, it can be seen that experts' evaluations are in the form of probability distributions. To select the best alternative(s), the proposed method is used and some computation results are presented as follows.

TABLE III. DISTRIBUTION OF EVALUATION FOR THREE CRITERIA

Criteria	Scores	Alternatives					
		A_1	A_2	A_3	A_4	A_5	A_6
C_1	1	1/5	0	2/5	1/5	0	0
	2	1/5	0	3/5	0	0	0
	3	1/5	1/5	0	3/5	0	2/5
	4	2/5	1/5	0	1/5	2/5	0
	5	0	3/5	0	0	3/5	3/5
C_2	1	0	2/5	1/5	0	3/5	0
	2	0	2/5	0	0	0	1/5
	3	1/5	1/5	2/5	1/5	1/5	1/5
	4	1/5	0	2/5	2/5	1/5	2/5
	5	3/5	0	0	2/5	0	1/5
C_3	1	0	0	0	3/5	3/5	1/5
	2	0	1/5	0	0	2/5	1/5
	3	0	2/5	1/5	2/5	0	2/5
	4	4/5	2/5	1/5	0	0	1/5
	5	1/5	0	3/5	0	0	0

The overall investment risk is calculated for two cases: using the simple additive form and for the "archetypal" compensatory case.

Using the simple additive form (3) and numerically convoluting probabilities represented in table III, we are getting six different probability distributions represented on figure 2.

Using the first degree stochastic dominance rule, the dominance matrix (15) is obtained, where the elements of the dominance matrix are defined in the following way:

$$sd_{ij} = 1, \text{ if } F_{Ai} \text{ SD1 } F_{Aj}, \text{ otherwise, } sd_{ij} = 0$$

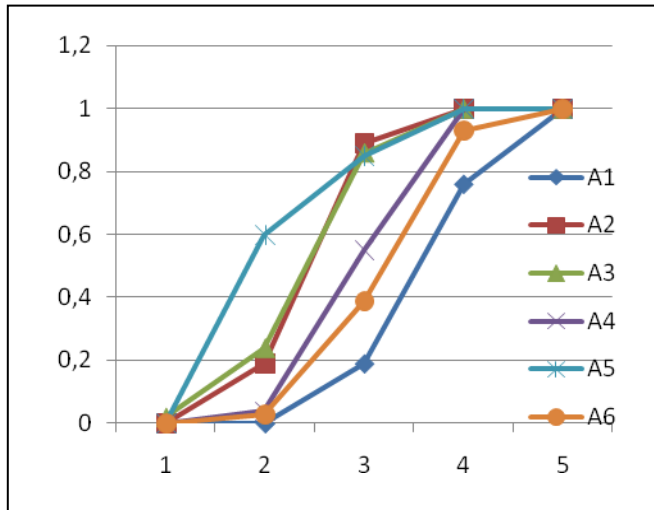


Fig. 2. Aggregated probability distribution

$$SD = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad (15)$$

As the final step, the ranking of alternatives from matrix (15) is obtained, with the following order:

$A_1 \succ A_6 \succ A_4 \succ A_2 \succ A_3, A_5$, what is the same result as in [28]. In the proposed methodology, the result is obtained in only four steps.

For the case of multiplicative function, the same procedure is performed, and for the “archetypal” compensatory case, the ranking of alternatives is as follows:

$A_1 \succ A_6 \succ A_2 \succ A_4 \succ A_3 \succ A_5$.

V. CONCLUSION

Models based on traditional methods for risk assessment do not incorporate the decision-makers behavior regarding risk, such as prone or aversion to risk. These models are also incomplete due to the need to take into consideration the aggregation of multiple objectives, such as: human, environmental and financial consequences that an accident on physical asset may cause. Nowadays, risk management must reconcile the concerns of society, the state and companies in relation to the operation and safety of assets, which is one of the most difficult tasks in risk management. Usual approach for solving the stochastic multiple criteria decision making (SMCDM) problem is based on pair-wise comparison with respect to each criterion, allowing only the additive form of aggregation of individual criteria.

Therefore, so as to tackle these problems presented by traditional methods, this decision model for risk assessment was based on MAUT, which provides a risk assessment

process under a multi-dimensional perspective. Utility theory also translates the decision-makers behavior regarding risk, by making a quantitative measurement of this behavior. In this paper, a new method for solving the SMCDM problem, based on numerical convolution of criteria probability distribution functions, according to both additive and multiplicative multi-attribute utility function is proposed. After the new, aggregated probability distribution is built for every alternative, the ranking of alternative is performed by the stochastic dominance. The use of the proposed method is illustrated on numerical example. The application of this methodology is not restricted to probability distributions represented as vectors, because of possible discretization of available CDF.

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