An application of the COG technique to assessment of student understanding of polar coordinates

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Abstract— According to the popular in fuzzy mathematics Centre of Gravity (COG) technique the defuzzification of a fuzzy set representing a problem's solution is obtained through the calculation of the coordinates of the COG of the level's section contained between the graph of its membership function and the OX axis. The COG technique is applied in this paper for assessing the results of the application of the APOS/ACE instructional treatment for teaching and learning mathematics on student understanding of the Polar Coordinates in the plane.

Keywords— Polar Coordinates (PCO), Center of Gravity (COG) Defuzzification Technique, APOS/ACE Instructional Treatment of Mathematics, Student Assessment.

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I. Introduction

There are several coordinate systems used in mathematics, physics, engineering and other applied sciences, all of them based on the same idea, i.e. as being a rule for mapping pairs of numbers to points of the plane (or the space).

Descartes (1596-1650) introduced in 1637 the *Cartesian coordinates* (x, y), which opened the door to the development of Analytic Geometry.

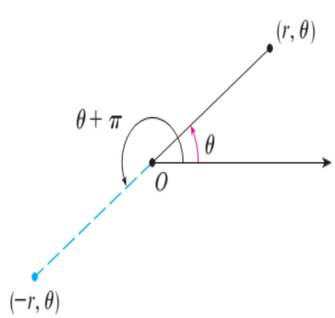

Fig. 1: Polar coordinates of a point in the plane

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Another popular coordinate system with many applications to differential and integral calculus, to complex numbers, to other branches of pure and applied mathematics, to physics, etc. is the system of *polar coordinates (PCO).* It is recalled that the PCO of a point P of the plane are defined by a pair of numbers *(r, θ)*, where *r* is the algebraic distance of P from a fixed point O of the plane, called the origin, and *θ* is the angle formed by the polar semi-axis OX and the straight line segment OP. The numbers *r* and *θ* can be positive, negative or zero (Figure 1). It becomes evident that, in contrast to its Cartesian representation, a point of the plane has not a unique polar representation. Note that polar coordinates are also defined for the points of the three-dimensional space, but this is out of the study of the present paper.

In an earlier work [1], applying the *APOS* instructional treatment of mathematics [2, 3] and based on the results of a written test and on oral interviews taken from students of an Iranian University, we developed a *genetic decomposition (GD)* for teaching and learning the PCO in the plane. More recently [4] we have extended this research by designing an ACE circle [3, 5] for teaching the PCO with the help of computers.

Here, using methods of the *Fuzzy Logic (FL)* we test the effectiveness of our ACE design on an experimental group of university students. The rest of the paper is organized as follows: In Section II we develop a special form of the *Center of Gravity (COG)* technique, the most popular defuzzification method of fuzzy mathematics, for assessment purposes. In Section III we apply the COG technique for assessing the understanding of PCO by the students of the experimental group and of a control group, for which the PCO were taught in the traditional, lecture-based way. The results of the two groups are compared in Section IV and interesting conclusions are drawn, which give some hints for further research on the subject.

II. The COG Defuzzification Technique as an Assessment Method

FL is based on the notion of *fuzzy set (FS)* initiated by Zadeh [6] in 1965 as follows:

Let U denote the universal set of the discourse. Then a FS A on U (or otherwise a fuzzy subset of U), , is defined in terms of the *membership function* m_A that assigns to each element of U a real value from the interval [0,1]. More explicitly, A is a set of ordered pairs of the form

$$A = \{(x, m_A(x)): x \in U\}.$$

The value $m_A(x)$ is called the *membership degree* of x in A. The greater is $m_A(x)$, the better x satisfies the characteristic property of A. The definition of the membership function is not unique depending on the user's subjective criteria, which are usually based on statistical or empirical observations. However, a necessary condition for a FS to give a reliable description of the corresponding real situation is that its membership function's definition is compatible to the common sense. For general facts on FSs we refer to the book [7] of Klir & Folger.

Since FL is based on the notion of FS, it yields the property of characterizing the frequently appearing in everyday life ambiguous situations with multiple values, thus being closer to our natural language and offering more realistic resources for assessment purposes than the classical bi-valued logic does [8, 9]. The process of reasoning with fuzzy rules involves:

- Fuzzification of the problem's crisp data by choosing the suitable membership functions to define the corresponding to those data FSs.
- Application of FL operators on the defined FSs and combination of them to evaluate the fuzzy data in order to obtain the problem's solution in the form of a unique FS.
- Defuzzification of the final FS to return to a crisp output value expressing the problem's solution in our natural language.

According to the popular in FL *Centre of Gravity* (COG) technique the defuzzification of the FS representing the corresponding problem's solution is obtained through the calculation of the coordinates of the COG of the level's section contained between the graph of the FS's membership function and the OX axis [10].

In an earlier work Subbotin et al. [11], based on a Voskoglou's fuzzy model for the process of learning a subject matter in the classroom [12], utilized the COG technique for assessing student learning skills. Since then Voskoglou and Subboting, working either jointly or independently have used the COG technique for assessing several human or computer activities; e.g. [9, 10, 12 – 16], etc. Here we shall adapt the COG technique for assessing student understanding of the PCO in the plane.

For this, we consider as set of the discourse the set U = {A, B, C, D, F} of the *linguistic labels (grades)* A= excellent, B = very good, C = good, D = fair and F = unsatisfactory of the student performance. Then, a student group G can be represented as a FS in U by defining the membership function $m: U \to [0, 1]$ in terms of the frequencies, i.e. by $y = m(x) = \frac{n_x}{n}$, where n is the total number of the students of G and n_x is the number of students of G whose performance is characterized by the grade x in U. Mathematically speaking, we can write

$$G = \{(x, \frac{n_x}{n}): x \in U\}.$$

Next, we correspond to each $x \in U$ an interval of values from a prefixed numerical distribution as follows: F $\to$ [0, 1), D $\to$ [1, 2), C $\to$ [2, 3), B $\to$ [3, 4), A $\to$ [4, 5]. This actually means that we replace U with a set of real intervals. Consequently, we have that $y_1 = m(x) = m(F)$ for all x in [0,1), $y_2 = m(x) = m(D)$ for all x in [1,2), $y_3 = m(x) = m(C)$ for all x in [2, 3), $y_4 = m(x) = m(B)$ for all x in [3, 4) and $y_5 = m(x) = m(A)$ for all x in [4,5). Since the membership values of the elements of U in G have been defined in terms of the corresponding frequencies, we obviously have that $\sum_{i=1}^{5} y_i = m(A) + m(B) + m(C) + m(D) + m(F) = 1$.

We are now in position to construct the graph of the membership function $y = m(x)$, which has the form of the bar graph of Figure 2. From Figure 2 one can easily observe that the level's area, say S, contained between the graph of $y = m(x)$ and the OX axis is equal to the sum of the areas of the five rectangles F_i , i =1, 2, 3, 4, 5. The one side of each of these rectangles has length 1 unit and lies on the OX axis.

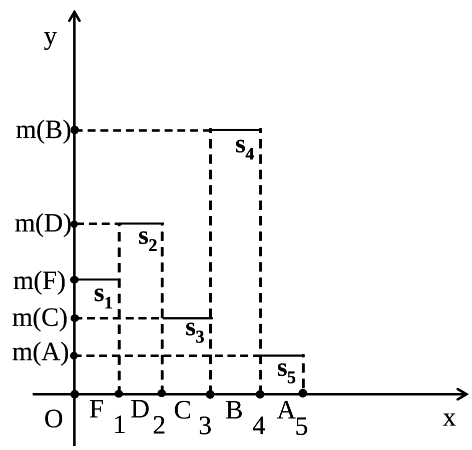

Fig. 2: The graph of the COG technique

The COG coordinates (x_c, y_c) are calculated by the well known from Mechanics [17] formulas:

$$x_c = \frac{\iint_S x\,dxdy}{\iint_S dxdy}, \quad y_c = \frac{\iint_S y\,dxdy}{\iint_S dxdy} \quad (1).$$

In our case we have that $\iint_S dxdy = \sum_{i=1}^{5} y_i = 1$. Also

$$\iint_S x\,dxdy = \sum_{i=1}^{5} \iint_{F_i} x\,dxdy = \sum_{i=1}^{5} \int_0^{y_i} dy \int_{i-1}^{i} x\,dx = \sum_{i=1}^{5} y_i \int_{i-1}^{i} x\,dx = \frac{1}{2}\sum_{i=1}^{5}(2i-1)y_i$$, and

$$\iint_S ydxdy = \sum_{i=1}^{5}\iint_{F_i} ydxdy = \sum_{i=1}^{5}\int_0^{y_i} ydy \int_{i-1}^{i} dx =$$
$$\sum_{i=1}^{n}\int_0^{y_i} ydy = \frac{1}{2}\sum_{i=1}^{n} y_i^{\,2}$$

Replacing the above values of the double integrals to equations (1) one finds that

$$x_c = \frac{1}{2}\left(y_1 + 3y_2 + 5y_3 + 7y_4 + 9y_5\right),$$
$$y_c = \frac{1}{2}\left(y_1^{\,2} + y_2^{\,2} + y_3^{\,2} + y_4^{\,2} + y_5^{\,2}\right) \quad (2)$$

But $(y_i - y_j)^2 = y_i^2 + y_j^2 - 2\,yij \geq 0$, or $y_i^2 + y_j^2 \geq 2\,yij$ for i, j = 1, 2,..,5, with the equality holding if, and only if, $y_1=y_2=y_3=y_4=y_5$. Therefore,

$$1 = (\sum_{i=1}^{5} y_i)^2 = \sum_{i=1}^{n} y_i^{\,2} + 2(y_1y_2 + y_1y_3 + ... + y_4y_5) \geq 5\sum_{i=1}^{n} y_i^{\,2}$$

Therefore, $y_1^2+y_2^2+y_3^2+y_4^2+y_5^2 \leq \frac{1}{5}$, with the equality holding if and only if $y_1=y_2=y_3=y_4=y_5=\frac{1}{5}$. Thus equations (2) show that the unique minimum value $y_c = \frac{1}{10}$ corresponds to the COG F_m $(\frac{5}{2},\frac{1}{10})$.

The ideal case is when $y_1=y_2=y_3=y_4=0$ and $y_5=1$. Then from equations (2) one finds that $x_c = \frac{9}{2}$ and $y_c = \frac{1}{2}$. Therefore the COG in this case is the point F_I $(\frac{9}{2},\frac{1}{2})$.

On the other hand, the worst case is when $y_1=1$ and $y_2=y_3=y_4= y_5=0$. Therefore, from equations (2) one finds that the COG is the point F_X $(\frac{1}{2}, \frac{1}{2})$.

Consequently, the area where COG F_c lies is the triangle $F_w F_m F_i$ of Figure 3.

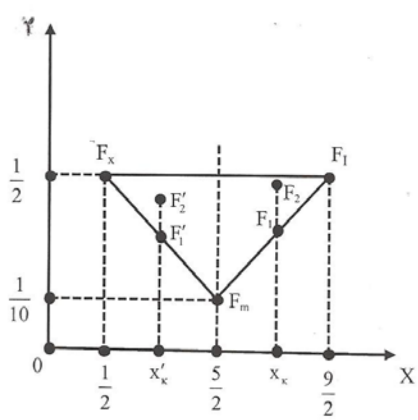

Fig. 3: The area where the COG lies

In Figure 3 one observes that the greater is the value of x_c the closer is the COG to F_I . Also, for equal values of x_c , if $x_c \geq 2.5$, then the greater is the value of y_c the closer is the COG to F_I , whereas, if $x_c < 2.5$, then the greater is the value of y_c the closer is the COG to F_x . With the help of the above observations one obtains the following criterion for comparing the performance of two (or more) student groups:

- *Between two groups the group with the greater value of x_c performs better.*
- *If two groups have the same value of x_c greater than 2.5, then the group with the greater value of y_c performs better.*
- *If two groups have the same value of x_c less than 2.5, then the group with the lower value of y_c performs better.*

According to the above criterion a group's performance depends mainly on the value of the x-coordinate of the corresponding COG. Further, the first of equations (2) shows that for calculating this value greater coefficients are assigned to the higher scores. Therefore, the COG method, in contrast to the traditional calculation of the mean value of the student scores measuring the group's *mean performance*, focuses on the group's *quality performance*.

Finally note that, since the ideal group's performance corresponds to the value $x_c = \frac{9}{2}$, values greater than half of this value, i.e. greater than $\frac{9}{4}$ =2.25, demonstrate a more than satisfactory group's performance..

III. THE CLASSROOM EXPERIMENT

Methodology

Two groups of university students participated in this research. The first group (*control group*) was enrolled in a calculus course in the fall semester of 2016. Eighteen students participated in the class and the PCO were taught in the traditional, lecture - based way.

The second group (*experimental group*) was enrolled in a calculus course in another university at the same semester. The PCO were taught in this group by an instructor who used the ACE cycle designed in [4] with the help of the *Maple* software. Twenty students participated in this class.

The Pre-Test

Before starting the teaching of PCO both groups completed a written pre-test. Since angles, trigonometric functions and Cartesian equations are fundamental prerequisites for learning polar coordinates and, as we have found in our earlier research [1], students face usually problems when dealing with them, the pre-test's questions listed below were based on these topics:

1. *Sketch the angles* $\frac{\pi}{4}, -\frac{2\pi}{3}, \frac{27\pi}{4}, -\frac{15\pi}{7}$ *and find in which quadrant ends each one of them.*

2. *Compute the values:* $\cos\frac{7\pi}{3}$, $\sin(\frac{-\pi}{3})$, $\tan\frac{11\pi}{4}$

3. *Find all values of* θ *such that* $\tan\theta = -1$ *and* $0 \leq \theta \leq 2\pi$.

4. *Write the Cartesian equation of the below curves:*
a) A vertical line through the point (3, 2).
b) A circle with radius 4 and center (1, 2)

5. *Sketch the graphs of the functions y= sin2x and y=cos3x*

6. *Find the center and radius of the circle with equation* $x^2 + y^2 = 2x - 4y$

The student answers were marked in a range from 0 to 100 and the scores obtained were the following:

Control group (G_1): 84, 70, 60, 59, 58, 40, 39, 38, 35, 32, 26, 25, 25, 15, 15, 10, 7, 5.

Experimental group (G_2): 83, 65, 65, 55, 50, 40, 40, 40, 38, 35, 35, 28, 25, 22, 22, 15, 15, 10, 7, 5.

Assigning to the student scores the *linguistic labels (grades)* A (100-85), B (84-75), C (74-60), D (59-50) and F (49-0) of excellent, very good, good, fair and unsatisfactory performance respectively, the results of the two groups are depicted in Table 1 as follows:

Table 1: Pre-Test Results

Grades	G_1	G_2
B	1	1
C	2	2
D	2	2
F	13	15
Total	18	20

The mean value of the student scores for G_1 is 35.72 and for G_2 is 34.75 demonstrating an unsatisfactory *mean performance* for both groups with the control group demonstrating a better performance.

From Table 1 one finds the frequencies $y_1 = \frac{13}{18}$, $y_2 = y_3 = \frac{2}{18}$, $y_4 = \frac{1}{18}$, $y_5 = 0$ for G_1 and $y_1 = \frac{15}{20}$, $y_2 = y_3 = \frac{2}{20}$, $y_4 = \frac{1}{20}$, $y_5 = 0$ for G_2. Replacing these values in the first of equations (2) one finds that $x_c = \frac{36}{36} = 1$ for G_1 and $x_c = \frac{38}{40} = 0.95$ for G_2. Therefore both groups demonstrated an unsatisfactory quality performance, with the performance of the control group being slightly better.

The Post-Test

One week after the end of the lectures on PCO a post-test was performed for both groups. For reasons of justice the questions of the post-test were designed by other mathematics faculties who are familiar with polar coordinates and none of the two instructors participated in their design. The questions of the post-test were the following [4]:

1. *Plot the points whose polar coordinates are:* (2, $-\frac{5\pi}{8}$), (2, $\frac{91\pi}{4}$) and (-1, $\frac{3\pi}{4}$).

2. *Find the Cartesian coordinates of the point* (3, $\frac{9\pi}{4}$)

3. *Find the polar coordinates of the point* $(-2, -2)$

4. *Find the polar equation for the curve* $x^2 + y^2 = 3x$

5. *Identify the curve* $r = 4\csc\theta$ *by finding the Cartesian equation for it.*

6. *Sketch the curve with polar equation* $r = -\cos 2\theta$

7. *The figure below shows a graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.*

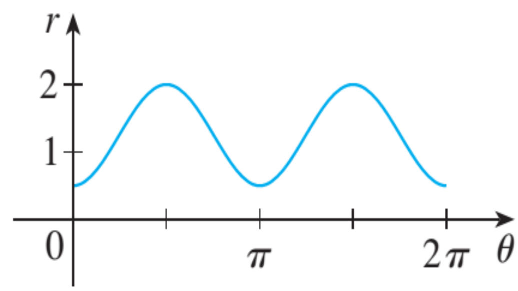

Fig. 4: The Cartesian graph of r = f(θ)

The student papers were marked together by both instructors for the control and the experimental group with scores ranging from 0–100. The student scores were the following:

Control group (G_1): 88, 75, 62, 55, 55, 45, 40, 36, 32, 30, 27, 27, 25, 20, 18, 15, 15, 10.

Experimental group (G_2): 100, 85, 80, 76, 76 65, 60, 60, 55, 48, 45, 42, 35, 30, 28, 25, 20, 20, 15, 15. The results of the two groups are depicted in Table 2 as follows:

Table 2: Post - Test Results

Grades	G_1	G_2
A	1	2
B	1	3
C	1	3
D	2	1
F	13	11
Total	18	20

The mean value of the student scores is 37.5 for the control and 49 for the experimental group showing that both groups demonstrated again an unsatisfactory mean performance. However, the performance of the experimental group was clearly better this time than that of the control group.

From Table 2 one finds the frequencies $y_1 = \frac{13}{18}$, $y_2 = \frac{2}{18}$, $y_3 = y_4 = y_5 = \frac{1}{18}$ for G_1 and $y_1 = \frac{11}{20}$, $y_2 = \frac{1}{20}$, $y_3 = y_4 = \frac{3}{20}$, $y_5 = \frac{2}{20}$ for G_2. Replacing these values in the first of equations (2) one finds that $x_c = \frac{42}{36} \approx 1.17$ for G_1 and $x_c = \frac{68}{40} = 1.7$ for G_2. Therefore both groups demonstrated an unsatisfactory quality performance, with the performance 20112011of the experimental group being better.

VI. DISCUSSION AND CONCLUSIONS

On comparing the performances of the two groups in the pre-test and the post-test one observes that both groups demonstrated a better performance in the post-test, although the questions of it were more difficult. However, the experimental group succeeded a much greater progress, which means that its students benefited better by the ACE instructional treatment for learning the PCO than the students of the control did by the traditional lecture-based approach.

On the other hand, taking into account the poor student performance in the pre-test, one concludes that there is a need for further experimental research for studying the effect of the ACE instruction on students with a higher mathematical background than those of the present experiment.

Finally, since the COG technique has the potential of a general fuzzy assessment method, further research could be done in future on applications of this technique to a variety of other human and machine activities.

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