Algorithmic thinking and mathematical competences supported via entertaining problems

Eva Milkova, Andrea Sevcikova

***Abstract*—**The essential part of studies at the faculties instructing students in the field of computer science is not only the development of student's ability to think algorithmically, but also to enhance their mathematical competences - the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations. Besides a deep analysis of the topic and discussion on mutual relationships within the problem solving, the involvement of students into the subject is a very important educational factor. Practical and entertaining problems attract students not only to know more about the explained subject matter, but also and more importantly, to apply gained knowledge. If an interesting task is assigned as prototype to a new topic using a picture and resulting explanation of the task solution, both spatial and verbal working memory is required and students recall the explained subject matter much better. In the paper we pay attention to the role of entertaining problems as a useful study material supporting development of both, algorithmic thinking and mathematical competences.

***Keywords*—**Algorithmic thinking, entertaining problems, learning style preferences, mathematical competences, visualization.

I. Introduction

COMPUTERS have become the most empowering tool humankind have ever created. They are the tools of communication, the tools of creativity, and they can be shaped by their user.

An essential part of studies at faculties preparing students in the area of computer science is the development of their mathematical competences and ability to think algorithmically. Students must be able to create various algorithms solving given problems starting with easy ones and consecutively increase their knowledge and shifts during studies till the level where they deeply understand much more complex problems. Subjects dealing with graph theory and combinatorial optimization serve very well for development and deepening of students' capacity for the mentioned skills and knowledge.

People learn and process information in different ways, thus to recognize student's strengths and weaknesses in a subject better, it is valuable to learn how to respond under different circumstances and how to approach information in a way that best addresses students' particular needs.

One of the suitable means to approach the instruction to the students is considered multimedia, respectively suitable implementations of different types of media in the instruction with regard to didactic knowledge of content and students' needs.

Very important educational factor is students' involvement into a subject. If an appropriate task is introduced and examined within a topic, students recall the explained subject matter much better and their involvement progresses when looking for similar examples.

The aim of the paper is to discuss the role of entertaining tasks provided in the subjects dealing with graph theory and combinatorial optimization, as a useful study material supporting the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations with regard to preferred learning styles of students. The paper is loosely based on papers [1], [2].

II. Learning styles preferences

There are plenty of learning style models. At our university we have chosen the Felder-Silverman learning style model and on this model based questionnaire called the Index of Learning Styles. Let us point out here at least the following remarks (cf. [3], [4])

Felder [5] defines individual learning styles as follows "The ways in which an individual characteristically acquires, retains, and retrieves information are collectively termed the individual's learning style".

The Felder-Silverman learning style model was created by Richard M. Felder on Dr. Silverman's expertise in educational psychology and his experience in engineering education.

Felder´s model includes four dichotomous learning style dimensions which indicate students´ preferences for certain poles of the dimensions:

Sensing or *Intuitive* - this spectrum determines how we perceive or take in information: sensory (external) – sights, sounds, physical sensations, or intuitively (internal) – possibilities, insights, hunches.

Visual or *Verbal*, this spectrum determines how we prefer

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the information to be presented: visual – pictures, diagrams, graphs, demonstrations, or auditory – through words or sounds.

Active or *Reflective* - this spectrum determines how we prefer to process the information: actively – through engagement in physical activity or discussion, or reflectively – through introspection.

Global or *Sequential* - this spectrum determines how we prefer to organize and progress understanding information: sequentially – in continual steps, or globally – in large jumps, holistically.

Felder and Silverman [6] claim that a balance of the two dimensions is desirable, i.e. it is valuable to be able to function both ways.

The research conducted in last years at our university (see e.g. [7], [8] and [9]) shows that most of our students are visual learners. The results show that in fact 98% are *strong visual learners*.

More students belong to the *moderate till strong* learners belonging to the *sensing* dimension (about 72%).

Concerning the other two dimensions, active/reflective and sequential/ global, most students belong to the mild level of these dimensions more of them to the „left side", which means to the *mild active* and *mild sequential dimension*.

Therefore, it is important that we prepared such educational materials, which support a student's preferences, and at the same time emphasize the less preferred dichotomous dimension. With regard to the above results the main aim of our study materials is:

- To visualize each new concept at first, and then to use oral explanation of its meaning, property, application and related theorems.
- To develop the student's ability to form images in mind.
- To learn students how to describe various situations with the aid of graphs, solve the given problem expressed by the graph, and translate the solution back into the initial situation.
- To examine each concept and problem from more than one point of view and to discuss various approaches to the given problem solution with respect to the previously explained subject matter.
- To let students thoroughly practice the explained concept and its properties, and let them add their own applications concerning the topic.

III. VISUALIZATION

According Felder and Silverman [6] "Visual learners remember best what they see--pictures, diagrams, flow charts, time lines, films, and demonstrations. Verbal learners get more out of words--written and spoken explanations. Everyone learns more when information is presented both visually and verbally." This claim is in accordance with advocates of dual coding theory who argue that people retain information best when it is encoded in both visual and verbal codes [10]. Memory for some verbal information is enhanced if a relevant visual is also presented or if the learner can imagine a visual image to go with the verbal information. Likewise, visual information can often be enhanced when paired with relevant verbal information, whether real-world or imagined. [11]

The dual coding theory has been applied to the use of multimedia presentations. As multimedia presentations require both, spatial and verbal working memory, individuals dually code information presented and are more likely to recall the information when tested at a later date [12].

Multimedia applications included into the learning process is undoubtedly one of the alternatives to convey to students often very sophisticated subject matter in the way which is familiar to them at present. Multimedia is a suitable tool for linking theory and practice [13]. (cf. [14])

There are several multimedia applications used as a useful support of various subjects at our university. For the subjects dealing with the graph theory and combinatorial optimization was created GrAlg program, whose main purpose is a visual representation of basic graph-concepts and graph-algorithms.

A. GrAlg Program

The program GrAlg was developed by our student within his thesis [15] and it can be downloaded from the page http://lide.uhk.cz/prf/ucitel/milkoev1/en_index.htm in the section GRAFALG > Lecture. The main possibilities of the program are as follows. (cf. [2])

The program enables the creation of a new graph represented by figure, editing it, saving graph in the program, in its matrix representation and *also saving graph in bmp format*.

The ability to save graphs in bmp format allows teachers and students easy creation of needed graphs for their tasks (texts and/or presentations) where they describe various practical situations with the aid of graphs and solve the given problem.

It also makes it possible to *display some graph properties* of the given graph.

Students can use not only graphs prepared by the teacher but also graphs created by themselves and explore the properties of these graphs.

The program enables to *add colour to vertices and edges*, to *add text next vertices, and to change positions of vertices and edges* by "drop and draw a vertex (an edge respectively)".

In this way the teacher can complete her/his in-class explanation in such a way that makes topic more comprehensible.

The program allows the user to *open more than one window* so that two (or more) objects or algorithms can be compared at once. Window size can be adjusted as needed.

This option enables the teacher to explain the problem from more points of view and show mutual relations among used graph-concepts and algorithms on graphs. In such way students can follow these mutual relations among concepts and algorithms, and remember those relations when using this option also by their self-study.

In the GrAlg program there is the option to *run step by step programs* visualizing all of the subjects explained algorithms in a way from which the whole process and used data structures can clearly be seen.

Step by step the solution of an algorithm allows students to follow the whole algorithm in their speed and compare it or its result only with their own solution.

Remark: Most of figures presented in this paper were created in the GrAlg program.

IV. Entertaining Tasks

Practical and entertaining tasks encourage students to know more about the explained subject matter and to apply acquired knowledge. If an interesting task is assigned as a prototype to a new topic by means of a picture and subsequent explanation of the task solution, both spatial and verbal working memory are required and students recollect the explained subject matter much better. Their engagement progresses when looking for similar examples. (cf. section III)

Motivation means literally the desire to do things. It's the crucial element in setting and attaining goals. Motivation is regarded as a crucial drive by which learning behavior can be stimulated.

Usage of puzzles can be compared with gamification, which is the use of game-like elements in non-game context. The main goal of gamification is to motivate participants and encourage expected behaviors in a meaningful way [16]. The similar approach for motivating and engaging participants can be used by means of puzzles. Integrating puzzles elements into educational context coupled with effective pedagogy has the potential to increase students' motivation and improve their learning outcome.

In the section on a particular topic from the graph theory, namely on Hamiltonian circles, we present examples of appropriate tasks enhancing algorithmic thinking and mathematical competences with regard to above discussed approach to the preferred learning style preferences of students. We also introduce here in specific subsections some areas used as valuable source of entertaining tasks.

A. *Entertaining tasks based on the historical background*

The history serves as a good source of practical examples and puzzles. In the area of graph theory there is a very interesting book Graph Theory 1736 – 1936 [17]. The most important problems from 1736 till 1936 are introduced there. In this book is also explained how the study of various graphs arose from the consideration of recreational problems.

As an example let us briefly illustrate a problem called "The Icosian Game" created by the famous Irish mathematician William Rowan Hamilton. This problem can serve as an appropriate prototype of Hamiltonian graphs, i.e. graphs containing a Hamiltonian circle - a circle containing all the vertices of a graph.

The Icosian Game

The object of the game was to find paths and circuits on the following dodecahedron represented by Fig. 1, satisfying certain specified conditions. In particular, the first problem was that of finding a circuit passing just once through each vertex of the graph [17].

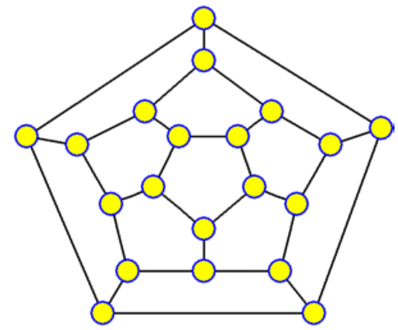

Fig. 1 dodecahedron

One possible solution of the mentioned problem is presented in the Fig. 2.

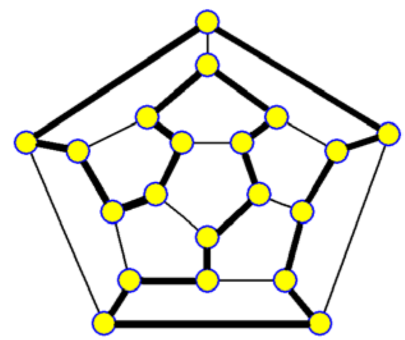

Fig. 2 hamiltonian circle in the given dodecahedron

B. *Enjoyable tasks based on puzzles*

There are an endless number of puzzles and logic problems in books like "Mathematics is Fun", in riddles magazines and on the Internet.

Let us go back to Hamiltonian graphs mentioned above.

To determine whether a given graph is or is not Hamiltonian belongs to the NP complete problems. However, there are four basic rules that students can apply when discussing small graphs and through them to better acknowledge the characteristics (qualities) of Hamiltonian circle concept.

1. If a given graph has n vertices, then a Hamiltonian circle has exactly n edges.
2. If vertex v has degree k, then a Hamiltonian circle has to contain exactly two edges incident with vertex v.
3. When constructing a Hamiltonian circle in a graph with n vertices, no circle containing less than n vertices is allowed to be created (closed) during the process.
4. Once a constructed Hamiltonian circle contains two edges incident with vertex v, the remaining edges incident with vertex v are excluded.

To deepen imagination and enhance students' facility to find out a Hamiltonian circle in the given graph we offer students not only graph represented by a figure, but also a puzzle similar to the following puzzle "Beads", to be solved.

The puzzle Beads

Join all the beads on the figure Fig. 3 into a closed, non-crossing path (i.e. circle from the graph theory point of view). The beads have various shapes. Each bead can be entered and left once only. The line joining the beads must follow the lines of the existing grid. However, the paths between each pair of vertices have to be direct, i.e. the paths cannot be bent.

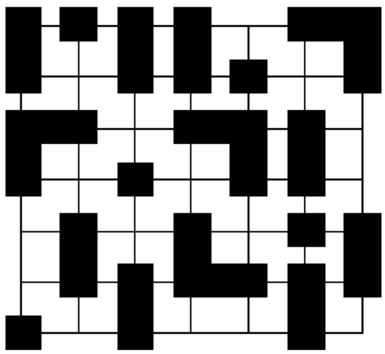

Fig. 3 given grid with beads

To solve the puzzle, we proceed in the following steps.

1. Finding a graph-representation of the given puzzle:
   - We remove unsuitable lines from the given grid.
   - We add letter to each bead. In this way we get a simple undirected graph (Fig. 4- beads represent vertices, lines represent edges), a graph-representation of the puzzle.

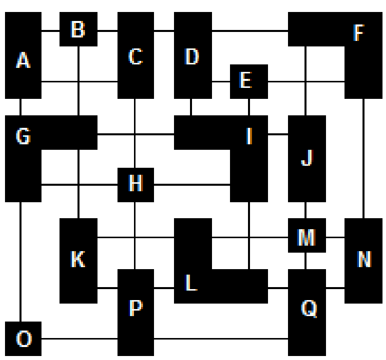

Fig. 4 graph representation of the given grid with beads

2. Solving the problem expressed by the graph: We use the rules to find a Hamiltonian circle in the gained graph (Fig. 5 – bold edges).

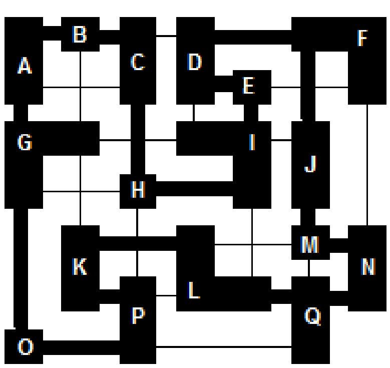

Fig. 5 bold edges denoting a Hamiltonian circle (A, B, C, H, I, E, D, F, J, M, N, Q, L, K, P, O, G, A)

3. Translation of the solution into the initial situation: We denote the found Hamiltonian circle in the initial gird - a solution of the puzzle (Fig. 6).

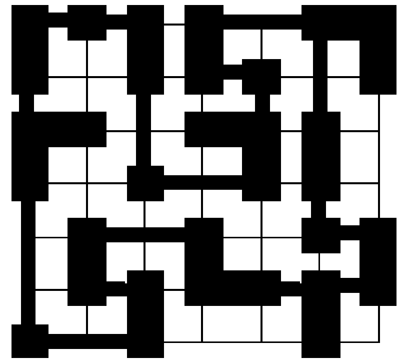

Fig. 6 solution of the puzzle

The puzzle is chosen from the Czech semi-monthly magazine “Hádanka a Křížovka” (Riddle and Crossword puzzle in English). It allows a deeper analysis when trying to find a Hamiltonian circle having further special properties which further improves understanding of the subject matter.

Another example of puzzles seeking Hamiltonian circle having further special properties is introduced in the following section.

C. Enjoyable tasks inspired by another subject

With regard to the fact that our students of Computer Science study both, Graph Theory and English language, we have prepared various puzzles combining the acquired knowledge from both study-areas and thereby we indicate further application possibilities of the discussed concept. Let us present very simple one.

English sentence on a Hamiltonian graph

In the following graph (Fig. 7) find a sequence of vertices that creates a Hamiltonian path and at the same time a correct English sentence.

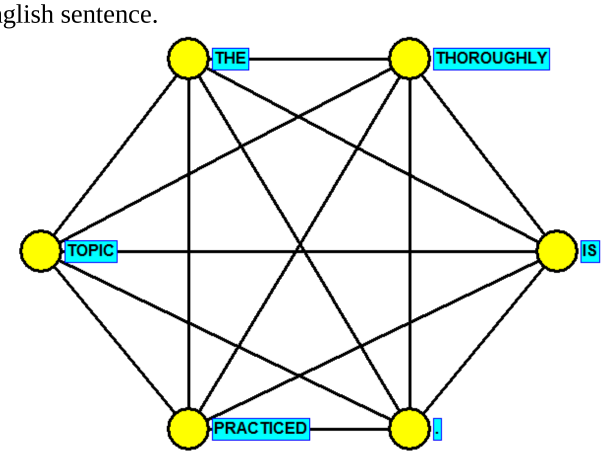

Fig. 7 given graph with English words

Obviously, the given graph is complete graph. Hence, each sequence of all vertices creates a Hamiltonian path. However, although there are 6! = 720 various Hamiltonian paths in the given complete graph, the solution of the task is simple. It is the sequence “THE TOPIC IS THOROUGHLY PRACTICED.”.

To enhance awareness of the given concept properties, in relation to this example, it is appropriate to discuss not only number of all Hamiltonian paths in a complete graph but also the number of its Hamiltonian circles, i.e. (n-1)! if we distinguish their orientation and ((n-1)/2)! if we do not distinguish their orientation.

D. Enjoyable tasks based on practical situation

One of our basic teaching principles that we have been applying in our teaching for many years is thorough practicing of the explained topic on various examples, discuss students' own examples describing the topic and encourage them to solve similar examples.

The following task laying on the intersection of a practical example and entertaining example is excellent connection of two topics, Hamiltonian graphs and bipartite graphs. It also helps to enhance student's ability to form images in mind.

Arrangement of guests around the table

Imagine that you would like to invite to your house seven guests, represented by letters *a, b, c, d, e, f, g* with whom you (denoted *h*) are going to dine at a round table. Among the guests, however, are also those who would appreciate if you do not sit them next to certain people, above all the guests *c* and *g*, who have problem almost with all of the invited persons. The overview of the people who "are not on friendly relations" is as follows

c … a, d, e, f, g

g … b, c, d, e, f

a … b

Will you find such guests seating arrangement at the table that everyone is feeling well?

To solve the puzzle, we proceed in the following steps.

1. Finding a graph-representation of the given task:
   a. We create a graph *G* with vertices representing people *a, b, c, d, e, f, g, h* and edges corresponding with their relation "not on friendly relation" (Fig. 8).
   b. We create complement to the graph *G* that represents relation "on friendly relation" (Fig. 9).

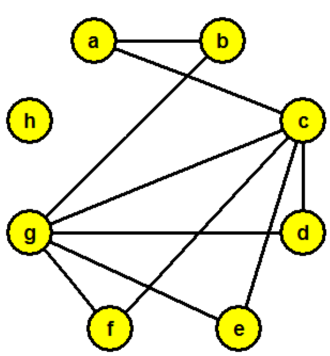

Fig. 8 graph *G*

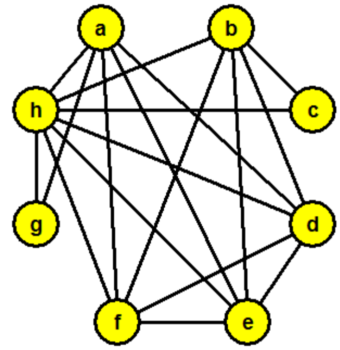

Fig. 9 complement to the graph *G*

2. Solving the problem expressed by the graph:
   - We seek a Hamiltonian circle in the complement of *G*.
   - If there is a Hamiltonian circle the task is successfully solved – a found Hamiltonian circle represents result

One of possible solutions of our task (a guests seating arrangement at a round table) is *h, g, a, d, f, e, b, c, h*. (see Fig. 10 – green edges).

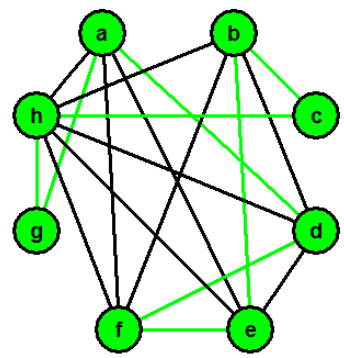

Fig. 10 a solution of the task

In relation to this example, it is desirable to discuss another task, namely:

Arrangement of guests to two tables

Imagine that you would like to invite to your house seven guests, represented by letters *a, b, c, d, e, f, g* with whom you (denoted *h*) are going to dine at two tables. Among the guests, however, are those who would appreciate if they are not sitting next to certain people at the same table, above all the guests *c* and *g*, who are troubled by almost all of the invited people. The overview of the people who "are not on friendly relations" is as follows

c … a, d, e, f, g

g … b, c, d, e, f

a … b

Will you find such guests seating arrangement at the given two tables that everyone is feeling well?

Although the assignment of this task is very similar to the previous one, we solve it using the concept application of bipartite graph. Instead a Hamiltonian circle we need to determine if the graph representing "not on friendly relation" is or is not bipartite. *Solution* of this task "There is no guests seating arrangement at the two given tables that everyone is feeling well." is obvious. It is evident from the graph *G*, Fig. 8, regarding the known theorem "graph *G* is bipartite if and only if graph *G* does not contain a circle of odd length".

V. Discussion

Algorithmic complexity is often viewed as a difficult part of IT education.

Let us add to the topic discussion concerning a puzzle "Pilgrimage" found on the Internet in the following form.

The puzzle Pilgrimage

A traveller wants to travel from a castle *A* to castle *B*. The traveller has a map with the two castles and cities between the castles. The cities are structured in n layers according to their distance to the castles. Given an arbitrary layer *k*, all cities in *k* have a direct path to every city in neighbouring layers (*k*-1) and (*k*+1). Cities in the first and last layer are connected directly to castle *A* and *B* respectively. Each city on the map belongs to one of the castles, *A* or *B*. Some pairs of the cities are in war and when the traveller visits a city that is in war with some previously visited city, he will be imprisoned and won't reach his goal. Thankfully, political power of the castles ensures that cities belonging to the same castle are never in war. The traveller has a list of cities that are in war, allowing him to find a safe path (if such exists). Because of a large number of cities on the map, it's not possible to check all possible paths, as there are exponentially many of them. Devise a fast (**polynomial time**) algorithm to find a safe path between the two castles.

The puzzle "Pilgrimage" corresponds with known NP-complete problem of finding paths avoiding forbidden pairs of vertices which originates in graph theory. This problem is known to be NP-complete in general, but some restricted versions lay in P.

The puzzle "Pilgrimage" can be described as the following restricted version of the problem [18]:

Given a graph $G^* = (V, E)$, so called *free-path-problem graph*, with two fixed vertices $s, t \in V$, vertex *s* is called the source vertex and vertex *t* is called the sink vertex. All vertices are divided into layers i, $i = 1, \ldots, n$, in a way that vertices of each layer *i* are fully connected to vertices of their surrounding layers $i-1$, $i+1$, $i = 2, \ldots, n-1$, whereas the source vertex *s* is the only vertex belonging to the first layer and the sink vertex *t* is the only vertex belonging to the last layer *n*. Additionally, the vertices are divided into two non-empty disjoint sets *A* and *B*. We define a set of pairs of vertices $S \subset (A \times B)$. The pairs in the set *S* are called forbidden pairs and the paths containing at most one vertex from each pair in *S* are called *free paths*. The goal is to find a free path from *s* to *t*, or to recognize that no such path exists.

The source and sink vertices can be assigned arbitrarily to *A* or *B* and they cannot form a forbidden pair – there would trivially be no *free path* in G^* if the source and sink formed a forbidden pair. Figure 11 displays the vertices from *A* in white and vertices from *B* in grey.

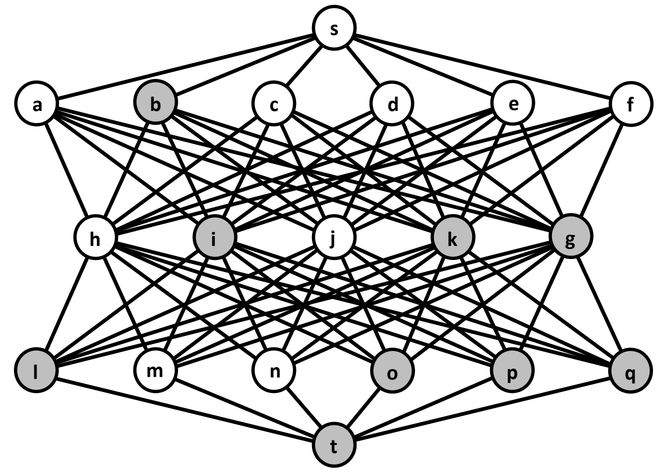

Fig. 11 an example of a free-path-problem graph with the source vertex *s* and the sink vertex *t*

However, contrary to the intuition that the puzzle "Pilgrimage" is solvable by a polynomial algorithm (see the task formulation), we can prove (see in press [18]) that even this special case is NP-complete and thus a fast algorithm is unlikely to exist.

The puzzle serves as an excellent educational example when explaining NP-complete problems. Its formulation is interesting and attractive for students and the NP-completeness proof is straightforward enough to further improve student motivation to the matter.

VI. Conclusion and Future Work

Technology related to teaching/learning plays a vital role in 21st century education [19]. The needs to serve the learners become urgent to make learning activities more motivating, funny and engaging for the students who are continuously surrounded with every form of new technology [20].

In the paper we discussed the role of entertaining tasks used as suitable complement of educational materials developing algorithmic thinking and mathematical competences and enhancing student's aptitude to solve everyday life practical situations. We also discussed a particular learning style preferences model, namely Felder´s model directed on four dichotomous learning style dimensions, and we also emphasized an importance to enhance student's ability to function both ways.

Future work is focused on search for more entertaining tasks to each explained subject topic, which deals with graph theory and combinatorial optimization. We will focus mainly on the preparation of the tasks that require within the solution process the application of not only one, but more graph-concepts and graph-algorithms. To them we will add tasks that have similar wording, but a different approach to a solution process - as demonstrated in the example tasks described as *Arrangement of guests around the table* and *Arrangement of guests to two tables*.

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