Algorithmic approach to improve democratic systems

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Abstract— The democratic system is understood and accepted as the fairest form of government in western countries. This is subject to little discussion. Nevertheless, citizens tend to be very critique with their rulers that ironically have been democratically elected. This paper firstly introduces and proves the problem that subtly relies in democratic systems presently by applying statistical concepts; then it proposes an algorithmic solution that provides an improvement and ultimately a fix to the previously detected problem. The right output, correctness and universal character of the algorithm are proven. In other words, our proposal increase the quality of the elected governments regardless their specific circumstances.

The proposal is based in a reasoning framework, axioms, and other political scientist's work. By applying the resulting solution in the current model, the democratic systems would clearly improve and it will be feasible to define democratic systems as the optimal way of ruling under any circumstance.

Keywords— Keywords Governance quality, trained voters, Educational algorithms

I. INTRODUCTION

Democratic elected governments are often criticized by the citizens . It is common to see protests and to hear statements of citizens complaining about the lack of commitment of the elected governments, their corruption and the inability of fulfilling the promises that they did during the campaign. Bipartisanship is a reality in most democratic countries. In fact, consolidated bipartisanship is the first sign that something in the system is locked; and this was not the purpose of the democracy when it was created [1]. Clearly, democracy is accepted as the fairest and cleanest way of having a government and that is why many parties can use that concept on their own benefit [2]. The political class, especially the major parties use the concept of "democracy" for its own perpetuation knowing that while this democratic system exists,

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they will rule a country for 4, 8 or even 12 years, or in the worst case they will be involved in politics for long time [2].

When searching the reason for this to reoccur, the need of analyzing the political decision-making arises. As the political world grows from different ideas and definitions , decision-making is an inner value that remains unchanged. [3]. This shows that policies related to decisions are the manifestations of values and personal evaluations that are interestingly enough made with little rational scientific rigor. [3]

Conflict resolution and decision-making are always personalized by individuals who make the decisions they need; then rational approach would correctly measure the impact of the political decisions. [3].

According to [5], governments controls the population voting in three different ways:

1. Through the power of decision-making.
2. Through the power of not making decisions.
3. Through the ideological power.

[5]also sets the power to make decisions as the most public face of the three and as the way governments almost always want to be seen. In fact, this power is the one that makes a government exist. It conforms to the definition found in books mainly written by political leaders and former presidents. The decisions that governments make appear to be the result of consensus and views from different experts.

The second power control technique occurs through information dissemination and it is precisely what the government wants to keep hidden. This includes controlling issues in public environments and not exposing the matters that could potentially damage the government's image. This is known as the power of not making decisions

In [5] the ideological power is considered the most important facet. The use of this power is able to influence the thoughts and desires of society, including making the mass desire an action that benefits the government's personal interest. For example, the way that citizens are encouraged to support a war or any other destructive events. By providing specific information through the media, an entire country could support these kind of actions.

[6] proves that governments keep on managing and maintaining information opacity, which often crashes against all democratic values that the same government defends publicly.

The literature referenced in this work establishes that the distance between governments and citizens has proven to be large. Nevertheless, [6] propose ideas to bring government closer to the citizens in order to create a truly democratic

government and make people more participant in government decisions. However this does not seem to be a feasible option as the elected government usually hides information to the citizens it represents.

In fact, [7] mentions the elites as the reason why citizens and government are separated. The elites remain secure and preserve their status, regardless of their professionalism or preparation. They are able to maintain their positions, again, through their or union and by distributing convenient information to the population.

As previously mentioned, there is a common factor shown in the cites of this article: The relationship between elected governments and the citizens they represent is damaged.

This work focuses in this gap; how to detect and solve it by using an algorithmic process.

The next sections are organized as follows.

1. Introducing the major issue that current elected governments and citizens presently face.
2. Defining the necessary concepts to understand our proposal.
3. Establishing axioms to build the reasoning model of the proposal.
4. Running the algorithm that detects and proposes the solution to the problem.
5. Proving the correctness of algorithms by applying statistical principles.
6. Conclusions and further work

II. DEFINITIONS

Some concepts need to be properly defined to provide with a better understanding of our proposal:

1. *Opt(g)*

Let us define Opt(g) as an optimal government. This is a government who is the best (among all possible candidates) in terms of corruption, preparation, professionalism and competence.

2. *W(g)*

W(g) is defined as any government that is not optimal. In other words $Opt(g) \rightarrow \neg W(g)$

3. *Government Quality Function* Let us define a
Government Quality Function (GQF) as a function that measures the government's quality in terms of the corruption, devotion, professionalism and competence. Some function proposals can be found in [4]

4. *Educational algorithm*

The term algorithmic process is used to describe a set of instructions that allows us to calculate a result based on given inputs. In this paper we work on an algorithmc approach that feeds from quality parameters of elected governments and returns the output (Opt(g)).

5. *Algorithm correctness*

This is defined as the proof of the right functionality of this algorithm. Proving its correctness means there are no flaws on its implementation and indeed returns the desired output Opt(g)).

III. OPTIMAL AND WRONG GOVERNMENTS

Let us see what an optimal government consist of. An optimal government must have the following characteristics:

1. Transparency [6]
2. Low corruption degree [9]
3. Professionalism [Murray (1994)]

When a country suffer either financial or social or any other crisis the rulers must be professional enough to handle it properly, as the mediocre ones may not work and may make situation worse. The citizens clearly needs the best possible governments; the most prepared one [[10]and the least corrupt one [9].

According to the cites below, the reasons why most governments are not optimal are:

1. Media influence [11]
2. Strong bipartisanship [1]
3. Citizens lack political knowledge [12]

The last one is the most important for our analysis. The paper shows that the system rulers generally pushes democratic societies to elect non optimal governments: this means that chances for the government to be a wrong one are close to 100%, settling for the mediocre that has enough contacts, influence and power to influence and control the media [8], [11]. Thus, when a country struggle with social, financial or any other kind of crisis , poor quality governments can clearly make them worse, by, for example, generating unnecessary wars, or even making financial disaster bigger.R12

The proposed algorithmic process will focus in overcoming the cause of keeping on choosing the wrong rules. Thus, the target will be changing the conditions for the democratic societies to choose optimal governments and therefore empowering the citizens to follow this path. [13], [14],[15].

IV. EDUCATION SYSTEM AN PERFORMANCE LINK

This section introduces the idea of relating the education models to the government performance: Below the list of the top 23 countries countries with the best educational system rates during years 1980-1995. We choose this range because citizens who went through these educational systems during those years are the a big part of the voters presently.

These values are calculated by using the education index which is based on the Mean Years of Schooling and Expected Years of Schooling.[16]

Country	1980	1985	1990	1995
Norway	0.666	0.699	0.75	0.805
Australia	0.87	0.87	0.874	0.894
Switzerland	0.678	0.675	0.695	0.726
Netherlands	0.667	0.686	0.744	0.811
United States	0.79	0.805	0.834	0.861
Germany	0.59	0.6	0.646	0.739
New Zealand	0.76	0.772	0.794	0.864
Canada	0.749	0.763	0.809	0.826
Singapore	..	..	0.545	0.58
Denmark	0.672	0.704	0.711	0.751
Ireland	0.633	0.666	0.697	0.743
Sweden	0.659	0.665	0.692	0.792
Iceland	0.598	0.639	0.673	0.732
United Kingdom	0.608	0.622	0.642	0.791
Hong Kong	0.53	0.589	0.626	0.639
Korea (Republic of)	0.565	0.638	0.679	0.743
Japan	0.663	0.681	0.699	0.74

Table 1: Education systems index from 1980 to 1995

Table2 shows the list of the 10 top countries in terms of government performance, according to a combined index of the Forbes list and EQI index [17] (if exist), during the same years.

Our study also considers interesting indicators the prosperity index [18] and the democratic index [19] shown in table 3 and table 4. These indicators measure the economic Fundamentals, Entrepreneurship and Innovation, Democratic Institutions, Education Health, Governance and Personal freedom among others. Most of the factors included in these indexes are directly linked to democratic standards.

The graph shown in figure 1 suggests a correlation between the quality of the educational system, the standards of the elected government and the prosperity of the country.
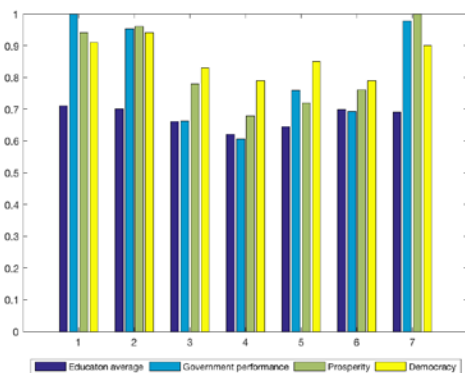

Figure 1: Democratic indexes comparison

In figure 1 every block represents an index and every aggregation of indexes represents democratic country. Figure 2

displays the dispersion of the education index with the rest and Figure 3 displays the correlation between the indexes.
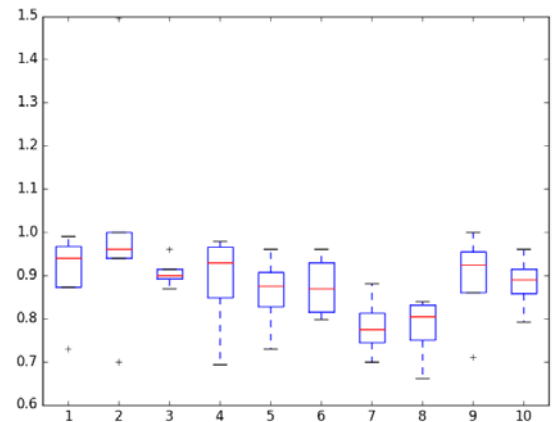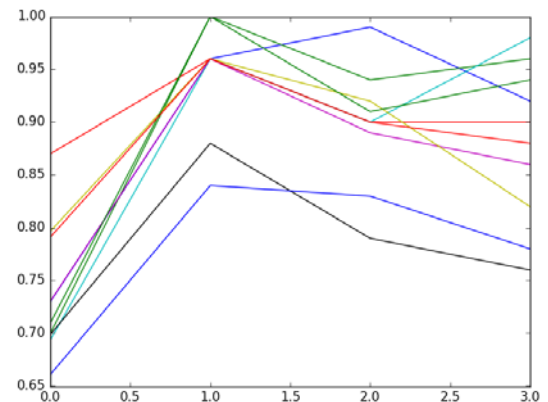

Figure 2: Indexes dispersion

Figure 3: Indexes patterns

Country	Index
Norway	0.96
Australia	0.96
Switzerland	0.96
Netherlands	0.96
Germany	0.92
New Zealand	0.96
Canada	0.96
Singapore	0.84
Denmark	1
Sweden	1
Iceland	0.92
United Kingdom	0.84

Hong Kong, China (SAR)	0.88
Liechtenstein	0.95
Israel	
France	0.84
Austria	0.96
Belgium	0.88
Luxembourg	0.95

Table 2: Government performance index

Country	Index
Finland	1
Switzerland	0.98
Sweden	0.96
Denmark	0.94
Norway	0.92
Australia	0.90
Canada	0.88
Netherlands	0.86
United States	0.84
New Zealand	0.82
Ireland	0.80
United Kingdom	0.78
Belgium	0.76
Austria	0.72

Table 3: Prosperity index ranking

Country	Index
Finland	0.90
Switzerland	0.90
Sweden	0.94
Iceland	0.95
Denmark	0.91
Norway	0.99
Australia	0.90
Canada	0.90
Netherlands	0.89
United States	0.80
New Zealand	0.92
Germany	0.86
Ireland	0.88
United Kingdom	0.83
Japan	0.79
Austria	0.85
Belgium	0.79
Spain	0.83

Table 4: Democratic index ranking

As the indexes come from different sources a normalization process has been applied to extrapolate the values in the [0,1] interval. This has made to the comparative analysis clearer. Data suggest there is a strong correlation between the education indicators the years 1980 and 1995, the quality governance indicators and democratic quality. Based on these results, our proposal includes a simple algorithm to increase the quality of governance of any democratic country.

V. THE ALGORITHM

This section propose an algorithm based on the results of the previous section.

The algorithm consider a function GQF that measures the quality of a government in terms of devotion, professionalism , competence of corruption. For this particular case, we choose a function that comes from the governance measurements found in [4] although any other function proposal could be used and the output would be the same. An optimal government based on the parameters set within the GQF.

The algorithm also uses an OPTIMAL benchmark (constant value) to establish the expression that satisfies the optimal character of the government and makes the iterative process possible.

Let it g be the current elected ruler.

1. BeginAlgorithm
2. OPTIMAL=0.9
3. CURRENT= GQF(g)
4. while CURRENT < OPTIMAL Do
5. $Education_level + +$
6. Wait (Next election)
7. EndWhile
8. EndAlgorithm

The output of this algorithmic process linearly depends on the number of elections happening until the education level makes the government reach the optimal value.

Based on the indicators and the strong correlation between education level and government quality, the process suggest that countries should check their education systems to ensure that citizens are well educated, specially in government affairs. The elected government should be checked according to the the standards the evaluation function is built on. This value is stored in CURRENT parameter. Optimal is arbitrarily set to 0.9. Education level is a variable that represents the education index of a given society, ++operator means increasing the index by investing more resources on improving the quality of the education that future voters receive.

Regardless the evaluation function is used and the value OPTIMAL parameter is set, the algorithmical process establishes that a country should invest in improving the quality of their education systems as long as the government is not performing as an optimal one Opt(g).

Next section proofs the correctness of the algorithm by using logic and statistic principles.

5.1 Correctness of the process

The correctness of the process is proven as follows.

1. Define the axiomatic statements.
2. Assigning probabilistic values.
3. Building the statistical model
4. Display the results.

Before creating the reasoning model, its necessary to accept two statements as axioms in our research field

1. *Axiom 1: "In any election for government, a person with little or no proper political training has a low probability of voting for the best one of all possible candidates."*
2. *Axiom 2: "Presently, most voters do not receive a proper specialized political training"*

These 2 statements are the basis for proving the correctness of our method.

5.2 Assigning probabilistic values

[20] classifies the risk in decision making into 3 main categories (Low, Medium, High) . For this particular problem, we select these values that fall into the following standards (see table 5):

Very high	0.9-0.99
High very high	0.8-0.89
High	0.7-0.79
Medium High	0.6-0.69
Medium slight high	0.51-0.59
Medium	0.5
Medium slight low	0.4-0.49
Medium Low	0.3-0.39
Low	0.2-0.29
Low very Low	0.1-0.19
Very low	0.01-0.09

Table 5: Probability assignations

According to our definitions, the first axiom states that the chances for a person with no political training to choose the optimal government ts lower that 0.3.
Now, let us focus on the possibility for a random person to vote for the best candidate party.
The *Law total probability theorem* states:

If $A_1, A_2, ... , A_n$ are mutually exclusive events in pairs whose union is the sample space E.

Let C be whatever event occurring in the space E.
It turns out that:

$$p(C) = P(A_1) \cdot p(C/A_1) + P(A_2) \cdot p(C/A_2) + ... + P(A_n) \cdot p(C/A_n)$$

According to the these principle our scenario is defined as follows: Event C = *"Voter votes for the optimal candidate to rule a nation"*

Event (-C) = *"Voter does not vote for the best candidate to rule a nation"*

Event A = *"Voter is untrained"*

Event B =(-A) *"Voter is properly trained"*

Event C / A = *"Voter votes for the best candidate to rule knowing that the voter is untrained"*

Event C / B = *"Voter votes for the best candidate to rule knowing that citizen is properly trained "*

The following calculations are based in the previously defined axioms and values .

• P(C/A)= *P("voter votes for the best candidate to rule knowing that the citizen is untrained")*=0.2 (low or very low)

• P(A)= *P("voter is politically untrained")*=0.8 (high or very high)

• P(C/B)= *P("voter votes for the test candidate to rule knowing that the citizen is properly trained")* =0.8 (high or veny high)

• P(B)= P*(citizen is properly trained in politics")=0.2 (low or very low). This assignation considers that 10% of the population receives a good and a proper political training before voting*

After replacing the values en the formula the result is:

$$p(C) = P(A) \cdot p(C/A) + P(B) \cdot p(C/B) = 0.8 \cdot 0.2 + 0.2 \cdot 0.8 = 0.32.$$

This is to say that a citizen who is randomly selected who does not show whether he's been politically trained or not, has about 32% chances of voting for the best possible government; which is to say around 68% of choosing a wrong one.

By accepting the axioms and the definition of low medium and high probability the study suggest that there are in fact a probability around to 0.68 for a random voter to choose a wrong government in the election. This figure, although representative, is not a valid measure in small samples. However, democratic system is implemented in spaces with millions of voters. In other words, the samples where to apply our method are big enough to extract conclusive results just by using *the laws of large numbers and the central Theorem of the limit.* [21] and [22]

The law of large numbers states that the sample expected value converges to the distribution mean as the sample size increases. There are different versions of the law (weak or strong) [22], depending on the convergence type.

Let us $X_1, X_2, X_3,...$ be defined as an infinite series of independent variables with the same expected value μ and variance σ^2. Thus, the average:

$$\overline{X}_n = (X_1 + \cdots + X_n)/n$$

converges to μ. In particular, The central theorem of the Limit states:

_n   P (| X_n- |<)=1.

For our study, Let us see an application of this law by looking at this example: Let us choose the event :

$X_i = \{$ Obtaining heads when tossing a coin $\}$. We know that by tossing a coin there are 50% chances to obtain heads and same for tails. Thus, by tossing 2 coins, there is a 50% chance to obtain half tails and half heads and 25% for getting 2 heads and 25% for obtaining 2 tails.

Then, by increasing the number of coins (let us say several billions), the results converges to a normal distribution, uniform and balanced set of heads and tails with a marginal error ε calculated from σ^2. The error becomes smaller as the sample grows bigger [21]

Let us suppose this experiment requires n coins, n $\mathbb{N}$.

The probability of getting exactly n / 2 heads and n / 2 tails follows this combinatorial formula:

$$P(A) = \frac{\binom{n}{n/2}}{2^n} = \frac{\frac{n!}{(n/2)!\cdot(n/2)!}}{2^n} = \frac{\frac{n\cdot(n-1)..((n/2)+1)}{(n/2\cdot(n/2-1)...1)}}{2^n} \frac{n\cdot(n-1).(n-2)..((n/2)+1).}{2^n\cdot(n/2\cdot(n/2-1)...1)} \quad (1)$$

As we increase the sample of coins, the chances to obtain a more balanced number of heads and tails increases too. Logically, if 1000 well constructed coins are tossed, it is virtually impossible to obtain 1000 heads or 1000 tails. The weak law of large numbers demonstrates that the proportion of heads and tails becomes more balanced as the number of coins grows.

According to the Central Theorem of the limit Tossing an infinite number of coins would exactly show half heads and half tails.

Several authors illustrate the example of tossing n coins [23] to demonstrate an application of the law of large numbers and the central theorem of the limit.

For our study, these laws can be applied to m small-medium country of 10,000,000 voters. According to the principles of total probability and the laws of large numbers, the number of people who would choose a wrong government is estimated to be 6,800,000 with a marginal error ε calculated from σ^2.

Our study shows that those votes are being given to wrong a government which indeed can make virtually impossible for the optimal candidate to be elected.

VI.   TOWARDS THE OPTIMAL

By assuming that democratic societies are not properly trained we have been able to prove the statement: "Voters generally choose wrong governments" Now the question is: what is the best method to ensure optimal elected governments?

Our algorithmic proposal involves training the population to increase the quality of the elected rulers. This section is going to prove that trained voters will indeed choose the optimal governments.

Let us define the following events h and Q:
Event P={Voters are not properly trained }
Event Q={The elected government is a wrong government}

This is what our study has shown:

$P \rightarrow Q$. In other words, voters who are politically untrained generally choose wrong governments.

Now, let us demonstrate the following:

$Q \rightarrow P$

If that is proven, it is precise to state not only that wrong governments are chosen because untrained voters but also it would be theoretically impossible for a proper trained society to elect a wrong government.

Let us define the following axiom:

Axiom 3: "A well trained person in politics has a high (or medium high) probability of voting for the best of all possible candidates."

This is similar to say that a well trained person in any field has a high chance of performing well on their job. Thus, by following the standards predefined by the risk management in the previous section, the probability of electing the optimal government by a trained voter is assigned to (0.6 or 0.7):

By applying the Law of large numbers a country with a 1,000,000 of voters , all of them well trained. The laws state that the number of votes to the optimal government are indeed more than 600,000 and the error in the approximation is bench-marked by the variance of the population distribution. These figures can enforce a greater choice of a high quality elected government.

In the end, our study has been able to prove $P \leftrightarrow Q$

VII.   CONCLUSION

An analysis has been performed to study the current problems that democratic societies seem to struggle with . Some axiomatic principles have been defined to prove the correctness of the proposed algorithmic process included in our proposal. The terms wrong and optimal government are defined in section 2 in terms of devotion, professionalism and competence. Any possible candidate of all possible ones, who runs for election and who is not the best (optimal) in those terms, is considered a wrong government for our study.

Our paper not only introduces and proves the existence of a problem that democratic societies suffer but also proposes a solution by applying an algorithmic process and proves its functionality and correctness. The solution involves applying a method that consist of gradually training the

potential voters. The method is introduced as an algorithmic process due to the nature of its application. As the democratic societies need to understand basic concepts of economics, fund management, international relations among others to become more knowledgeable, our proposal satisfies these needs and enforce an improvement that is proven in section 5.

In particular, our work demonstrate two facts:
• Democratic societies generally choose wrong governments. This applies to middle-big size countries as this is essential to apply the weak law of large numbers.
• Trained voters would choose optimal governments.

This clearly has an impact in the improvement of the quality of life of the societies

The results are clear. They reveal the existence of a necessary and sufficient condition to improve the quality of the elected governments. This condition is to educate the citizens to become knowledgeable voters. Therefore the equivalence between education and government quality is proven. In other words,there is no another way to ensure high quality standards of democratically elected governments . If democratic societies are willing to ensure the existence of optimal leaders, education standards must improve. As the equivalence is proven in chapter 5 we can also conclude that any other solution to improve the quality of the elected governments without considering to educate the potential voters, will not work. In that regard, the results look conclusive.

The way to introduce this training to democratic societies will require further study. Either by including it in the education system as part of the basic education or with the creation of additional programs for potential voters can be a good starting point. The contents of that training should also be subject of deep study. Nevertheless, the need of training democratic populations is beyond doubt and should be strongly taken into account by the democratic societies.

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