# Visual representation of proofs in graph theory

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***Abstract***— Mathematical proof is a basic concept characterizing mathematics and its application is inevitable in the practice of mathematicians and computer scientists. However, in the area of informatics we meet very often students who are not keen on the proof and thus they have problem to pass exams. Teachers are constantly designing tools and methods to explain complicated matters to their students in an attractive way. Many researchers claim that one of the suitable means of approaching difficult matters is the implementation of multimedia into teaching and learning. The aim of the paper is to introduce a research dealing with the influence of multimedia applications employed to enhance comprehensibility and attractiveness of mathematical proofs. Three research questions are discussed: Can visualization help to improve the instruction of proofs of theorems from the area of graph theory? What is the student's attitude towards proving? What is the attitude towards the use of visualizations in instruction of the proofs of theorems from the area of graph theory? According to gained results students see visualization as very beneficial and confirm that visualization of the instruction helps them to understand the subject matter.

***Keywords***— graph theory, multimedia application, visualization of mathematical proofs.

## I. Introduction

PROOF is the concept that we use in everyday life, it is the argument used to convince others that something is valid and true. Mathematical proof is a part of mathematics and belongs to one of the basic stones. It is a series of logical arguments and statements that clearly show that a particular statement is true. The significance, the role of proof has been dealt with by a lot of researchers, [1], [2], [3], [4] and [5]. Despite the importance of proof, the research [6], [7], [8] confirmed that students at all levels have problems with the construction of proofs. The primary problem is that students do not have a precise understanding of what mathematical proof consists of. Many students believe that verifying the validity of a theorem in a particular case is a sufficient proof [8].

Teachers around the world are constantly designing and inventing applications, tools, and methods to explain complicated matters, including mathematical proofs, to their students in an attractive way. One of the suitable means of approaching difficult or complicated matters to students is the implementation of multimedia, or application of different types of media in the classroom, see e.g. [9], [10] and [11]. The implementation of multimedia into the learning process is undoubtedly one of the possibilities how to provide students with a subject matter difficult to grasp in the way that is close to them nowadays [12]. It turns out that the appropriate visualization or animation of the proofs of mathematical theorems could help students not only to better understand the subject matter, but also to construct proofs of a simple statement, see [13], [14], [15], [16] and [17]. "*Vision is central to our biological and socio-cultural being. Thus, the biological aspect is described well in the following [18]: The faculty of vision is our most important source of information about the world. The largest part of the cerebrum is involved in vision and in the visual control of movement, the perception and the elaboration of words, and the form and color of objects. The optic nerve contains over 1 million fibers, compared to 50,000 in the auditory nerve.* [19]" McCormick, DeFanti and Brown even claims, that visualization provides a way of seeing the invisible [20].

The paper discusses multimedia applications designed and programmed by our students in collaboration with teachers as part of their diploma theses used at our university within subjects dealing with graph and graph algorithms at first. Secondly, a research study examining the effect of visual representation of proofs on its comprehensibility together with gained results is briefly presented.

## II. Graph Theory Visualization

The subjects dealing with graph and graph algorithms give us an elegant tool to describe different situations or a real life problems and provide guidance on how to deal with them. We are convinced that these subjects are beneficial to our IT students in their future life, and that is why it is important for them to understand the topics and relevant processes explained and discussed within these subjects. In addition to graph algorithms, which make up a considerable content of the subject, we place great emphasis on the understanding of mathematical theorems and their proofs, as well as on the derivation and proving of certain properties of different graphs and graph structures. To enhance the effectiveness of the courses, specific support multimedia applications have been developed:

- GrAlg – the main purpose is to easily create and modify graphs and visualize graph algorithms on graphs given by

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an image ([21], [22]),

- A-DIMA – application is suitable for demonstration of graph algorithms on graphs given by matrix ([22]),
- Visualization of proof – proof is presented step by step using PowerPoint presentations,
- GraPro – the multimedia tool visualizes certain proofs of mathematical theorems from the graph theory discussed in graph theory subjects at our university.

Multimedia applications GrAlg, A-DIMA and GraPro were designed and programmed by our students in collaboration with teachers as part of their diploma theses. They are user friendly and positively evaluated by our students.

GrAlg and A-DIMA applications have been already discussed in [24]. While the biggest advantage of both tools is to visualize graph algorithms, GrAlg is also very helpful in explaining graph concepts and graph properties by assigning colors to vertices and edges, so it can also be used to visualize certain proof steps.

### *A. GraPro*

The GraPro Multimedia Tool was created in a source code editor called Visual Studio Code by Microsoft [21]. Students described the application as follows:

- easy operation and clarity,
- interactive environment and animations,
- the ability to move in time,
- quality design of visualization.

The application consists of three content components:

a) Propositional Logic - This section contains a basic theoretical introduction to this area, which is necessary to understand the principle of mathematical proof. It is possible to interactively browse the visualization of negation here.

b) Types of proofs - here are presented the basic types of mathematical proofs.

c) Visualization of proof - this part contains the visualization of selected proofs from graph theory discussed in the subject DIMA (Remark: DIMA course is one of the courses devoted to graph theory and graph algorithms).

The player of proof visualization includes written steps of proof verbally using the propositional logic on the left side, the main rendering area on the right side, and the "..." button in the lower right corner to display the definitions and other auxiliary information that appear below the main screen. The actual step of the written proof is always automatically increased and highlighted. The application allows to draw by an orange pen on the rendering area. It is possible to mark different parts, to illustrate the elements in the graph and then delete them, see Fig. 1a-c.

The tool has several controls:

- Button for repeating animation in the given proof step,

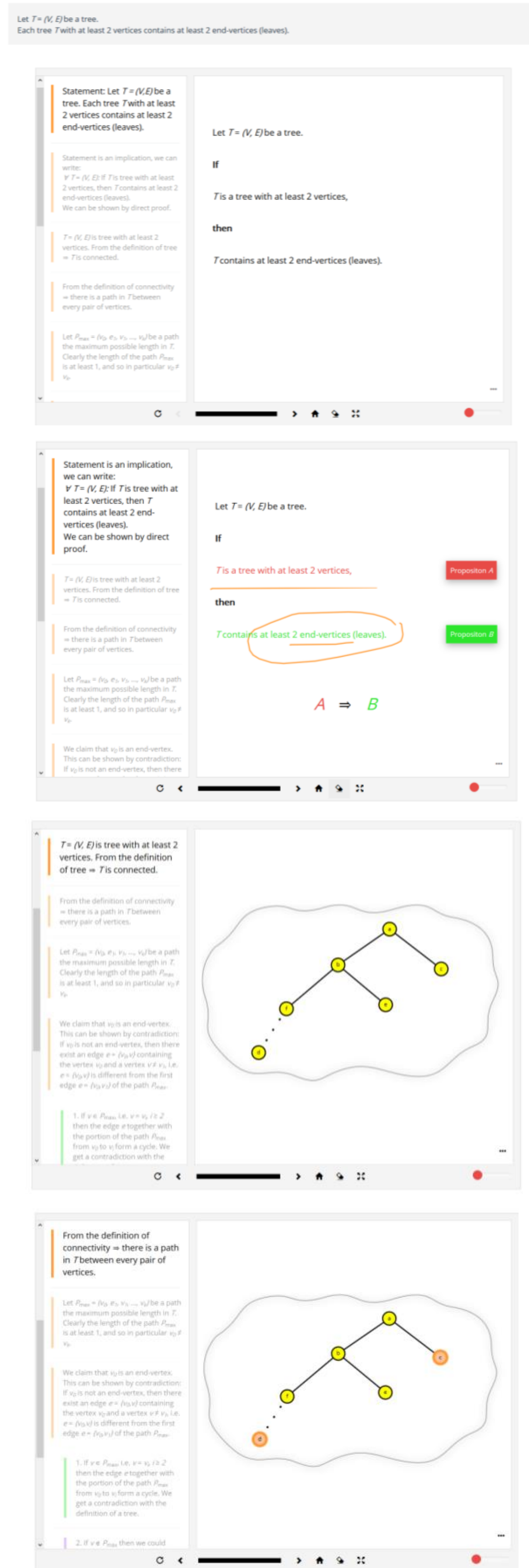


Fig. 1a illustration of proof in application GraPro

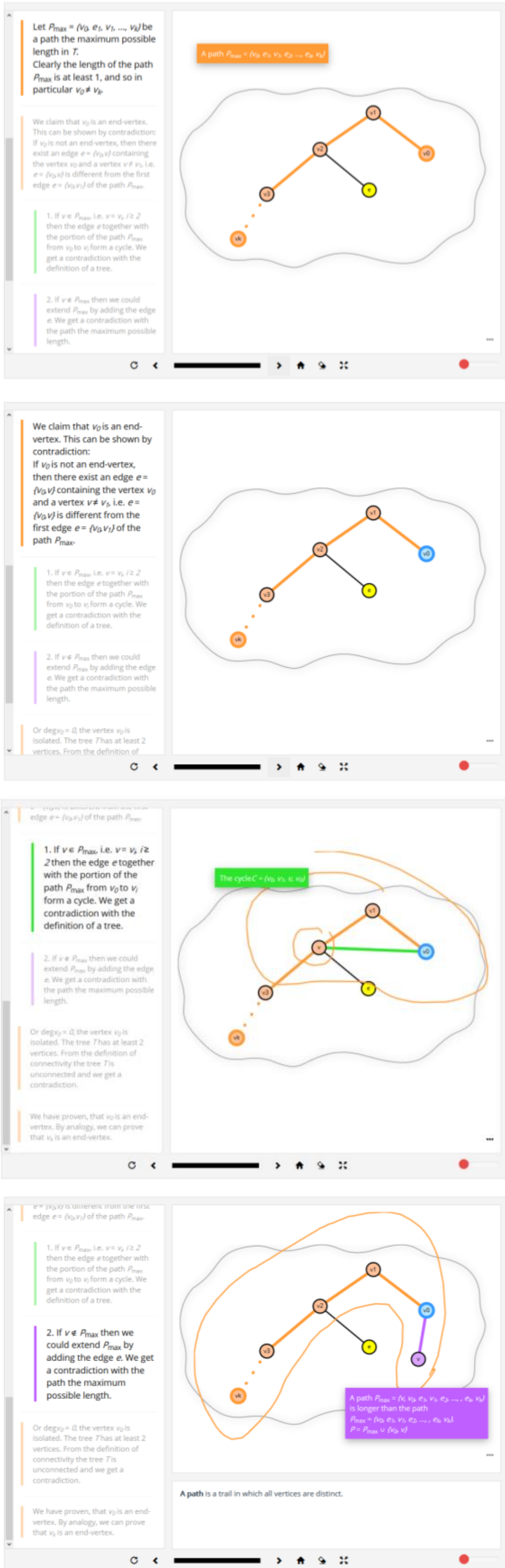

Fig. 1b illustration of proof in application GraPro

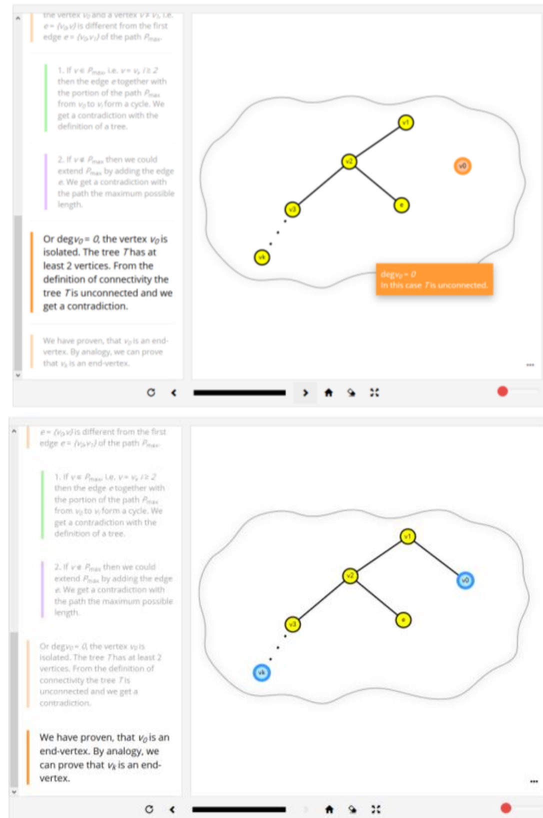

Fig. 1c illustration of proof in application GraPro

- Move back button,
- Animation progress indicator,
- Forward button,
- The first step button, the button for deleting the drawn shapes,
- Full screen zoom button,
- Slider to slow animation.

### *B. Benefits of GraPro*

The GraPro benefits for both, teachers and the students, are as follows:

- The possibility to draw into visualization allows the teacher to emphasize certain parts, sections in proof during interpretation.
- Written proof using the propositional logic on the left side and visualization on the right side helps the teacher to present proof at the same time formally - it is presented by using propositional logic and visualization, which helps the student understand the construction of proof in propositional logic.
- Using the button to display the definitions and terms used in the proof helps the teacher to interpret with a quick reminder of the concepts and definitions used in the proof.
- Moving back and forth allows to return to an incomprehensible part of the proof, or to speed up clear steps.

- The section on propositional logic helps students to repeat and practice the fundamentals of negation that are necessary for indirect proof or for proof of argument.
- The Type of proofs section provides students with a comprehensive overview of the types of proofs and their practices.

## III. Research Study

The study is part of a quasi-experiment in which the effect of visual representation of proofs on its comprehensibility is examined.

### *A. Problem statement*

While mathematical proofs are very important, our IT students consider them complicated and even unnecessary (authors´ unpublished research, questionnaire survey, 2015). To enhance student's motivation, we decided to introduce visual means into the DIMA course - one of the courses devoted to graph theory and graph algorithms, for teaching proofs, because we are aware that most of our students prefer the visual style of learning (Remark: in February 2016, 2017 and 2018, students attending the DIMA course completed a questionnaire Index of Learning Styles, which was developed by Richard M. Felder and Barbara A. Soloman [25]. It was found out that 85% of students prefer the visual style of learning (30% - strong visuals, 34% - moderate visuals, 21% - slight visuals), see Fig. 2.

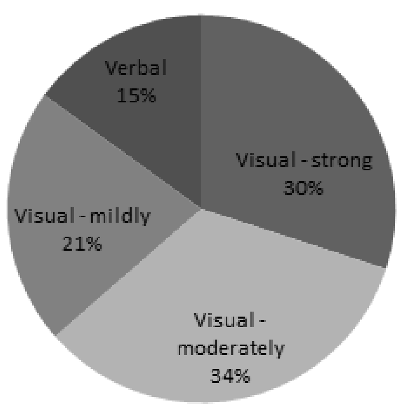


Fig. 2 preferred style of learning: Visual - Verbal

### *B. Research questions*

a) Can visualization help to improve the instruction of proofs of theorems from the area of graph theory?
b) What is the student's attitude towards proving?
c) What is the attitude towards the use of visualizations in instruction of the proofs of theorems from the area of graph theory?

### *C. Purpose of the study*

The aim of this study was to examine available visualization tools at the University of Hradec Kralove as a supplement to clarify the proofs of mathematical theorems taught in the graph theory, to analyze students' projects and their views on visualization of proofs and proving in general, and to present the results of the evaluation of students who used visual tools and those who did not.

### *D. Research sample*

Research sample included students who attended the DIMA course in the academic years 2015/16, 2016/17 and 2017/18. The control group consisted of 128 students and an experimental group involved 134 students. No visualization was used in the instruction of the control group, while presentations with visualizations were used in the instruction of the experimental group.

### *E. Research methods*

The research was conducted in the real learning conditions. Participants of this study at the beginning of the course wrote a pre-test and after the DIMA course completion, students wrote a post-test. More precisely, the pre-test included tasks related to the negation of the theorem, the variance of the theorem and the proofs from the compulsory subject Basics of Mathematics taught before the DIMA course. Post-test assignments, which were written at the end of the course, included negation of statements, the creation of variations, and proof assigments from the area of graph theory.

Students also created a credit project in which they had to visualize a selected simple mathematical statement from the area of graph theory. They were also asked to write an essay about their work, their attitudes towards proving and visualization, and fill in a questionnaire.

### *F. Findings*

The results of the pre-test and post-test in groups are shown in Table I. The results are given in percentage considering the assessment. The evaluation was carried out according to the Study and Examination Rules of the University of Hradec Králové, which uses the European Credit Transfer Scale. The evaluation was set according to the following scale, see Table I:

Table I: evaluation

| Grade | Points |
|---|---|
| A | 91-100% |
| B | 81–90% |
| C | 71-80% |
| D | 61-70% |
| E | 51-60% |
| F | 0-50% |

Thanks to the fact that during the whole term students are pushed to work with mathematical proofs, there is significant improvement in evaluation of post-test with regard to pre-test. In addition, according to our hypothesis, thanks to the implementation of multimedia application visualizing the subject matter into instruction, more significant improvement we can see in the experimental group, see Fig. 3.

At the end of the DIMA course, the students were divided

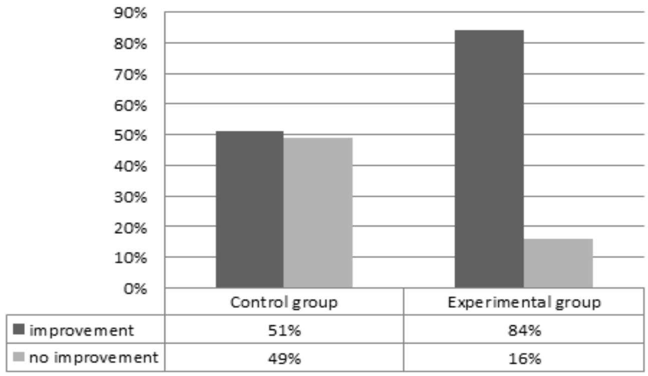


Fig. 3 post-test results

into teams (1-5 members), each team chose a simple statement of the teacher's list and created proof for the statement, as well as its visualization. Consequently, each team had to present its project in clear and understandable way. 49 projects were created in the control group and 53 projects in experimental group. Based on presentation and visualization, projects were evaluated according to the following parameters: Proof right, Proof with minor errors and Proof wrong. Results are illustrated in the graph, see Fig 4.

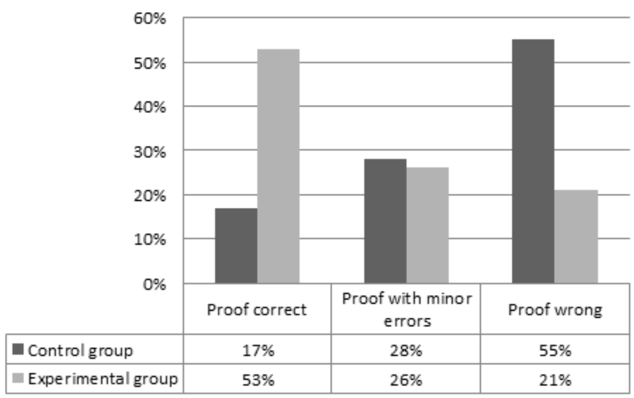


Fig. 4 results of final projects

At the end of the project, each student wrote an essay in which they commented on their own project, what troubled them, and what relation they had to proving and visualizations. Overall, we received 230 essays in which 23 percent of students admitted a negative relationship to proof and proving in general, and 14 percent had problems with proving. The biggest problems see our students in how to start proof (19%). Despite the fact that students have a negative attitude towards proving, 19% admit the importance of proving, and 26% see visualization as very beneficial.

We observed that the most common errors in the construction of proof include:

1) They do not know how to start, which type of proof to use. They do not realize that sometimes they have to try to proof it in some way, and if it goes nowhere, try another way. Selecting the right type of proof is sometimes not easy even for experienced mathematicians.
2) They do not know the exact procedure for certain types of proofs. When the type of proof is precisely specified, they do not follow the procedure, e.g. in indirect proof, students negate the statement, or in the case of proving the controversy, they did not start from the neglected part, but from the assumption with which they should have been contradictory.
3) Wrong negation using proving by contradiction.
4) Instead of general proof, they present the validity of a theorem using a specific example.

## IV. Conclusion

The results of our research confirmed the effectiveness of using visualization in teaching proofs. The visualization of the instruction is very welcomed by the students and they confirm that it helps them to understand the subject matter (cf. [26]). We learned that engaging students in visualizing proof, when they could use their potential and their programming knowledge within creation of their projects, was enjoyable for them and led to better understanding.

At present, a new multimedia application is developed. It focuses on graph theory proofs visualization, where emphasis on the specific educational needs of our IT students is placed. Our effort is to devise additional tool for better comprehensibility of the proofs, to make this topic more attractive to students, and get rid of the fear to construct proof; proving helps to deepen students' logical thinking.

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