On the comparison of the implicit advanced and hybrid methods and their application to solving Volterra inteqro-differential equations

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Abstract— The mathematical models for many problems of natural sciences are constructed by using the Volterra integral and the Volterra integro-differential equations. Usually, for solving the initial-value problem for Volterra integro-differential equations have applied the methods which are used in solving of the initial-value problem for ODE. But for the calculation of the integral participated in the integro-differential equations are used the quadrature methods which have some disadvantages. Therefore, here to solving named problem proposed to apply the multistep methods of the advanced type which are freed from above-mentioned disadvantages. For the shown advantages of this method, we have used the comparison of some known methods with the proposed. And also have constructed some algorithms for solving the initial-value problem for the Volterra integro-differential equations. For the illustration of the received here, results have considered the application of these algorithms to solving of the initial-value problem for some Volterra integro-differential equations.

Keywords— Volterra integral equations, Volterra integro-differential equations, advanced methods, hybrid methods.

I. Introduction

As is known in the construction of the mathematical models for many practical problems, by Vito Volterra have used the integro-differential equations with the variable bounders which were fundamentally investigated by him (see [1, p. 177-195]). The initial value problem for the Volterra integro-differential equation has investigated by some known scientists (see for example [2]-[13]) in results of which the theory of Volterra integro-differential equations very extended and has applied to the investigation of many processes to take place in the different industry of the natural science. In basically these works are devoted to the construction and application of approximate methods to solving initial-value problems for integro-differential equations of the Volterra type. For the finding of the numerical solution of this problem are used the wide class of numerical methods which has been proposed to solving the initial-value problem for ODE where to calculate of the integral participated in the integro-differential equations have used the quadrature formulas, spline functions, methods of collocation and etc. Here, we also for solving the above-mentioned problem have used some relation between of the solution of the initial-value problem for ODE and also for the Volterra integro-differential equation. Note that the methods which have recommended to solving of initial-value problem for Volterra integro-differential equations are the same with the methods have application to solving initial-value problem for the ODE (see for example [14]-[24]). It follows that the method proposed here guarantees the constant volume for the computational works at each step.

Let us consider the following problems, which are usually is called the initial-value problem for the Volterra integro-differential equation of the first order:

$$y' = F(x, y(x), v(x)), y(x_0) = y_0; x \in [x_0, X], \quad (1)$$

here the function $v(x)$ is defined in the following form:

$$v(x) = \lambda \int_{x_0}^{x} K(x, s, y(s)ds, x_0 \le s \le x \le X. \quad (2)$$

If the parameter $\lambda = 0$, then from the problem (1) we obtain the ordinary initial-value problem for the ODE of the first-order. Therefore, we assume that $\lambda \ne 0$ and put $\lambda = 1$. In this case, the problem (1) in one version can be rewritten in the form:

$$y' = f(x, y) + v(x), y(x_0) = y_0, x_0 \le x \le X. \quad (3)$$

To find the numerical solution of the problem (3), we assume that the problem (3) has a unique continuous solution defined on the segment $[x_0, X]$. To find the approximate values of the solution of the problem (3) at some mesh points, let us divide the segment $[x_0, X]$ into N equal parts by the mesh points $x_i = x_0 + ih$ $(i = 0,1,..., N)$. Here $0 < h$ - is a step size. Let us also denote by the y_i approximately values, but by the $y(x_i)$ through, the exact values of the solution of the problem (3) at the mesh points x_i $(i = 0,1,..., N)$,

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respectively. As is known in many works of different authors, have constructed the numerical methods for solving of the problem (3) by replacement the integral to some integral sum in the result of which the volume of computational works increases in the transition from the current point to the next. For example, let us consider the following quadrature method:

$$\int_{x_0}^{x_m} K(x,s,y(s))ds = \sum_{i=0}^{m} K(x_m, x_i, y_i) + R_m. \quad (4)$$

As follows from equality (4), with increasing the values of m, the number of terms which are used in the sum of the equality of (4) also increases. The proposed here methods are freed from the above-mentioned shortcomings, and they are effective in applying them to solving some special cases of the first-order Volterra integro-differential equations.

As is known one of the numerical methods for solving of the problem (3) can be constructed in the following form (in one variant see e.g. [4] - [13])

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+1} + h\sum_{i=0}^{k}\beta_i v_{n+i}, \quad (5)$$

$$\sum_{i=0}^{k}\alpha_i v_{n+i} = h\sum_{i=0}^{k}\sum_{j=0}^{k}\beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}). \quad (6)$$

Here $f_m = f(x_m, y_m), v_m = v(x_m)(m = 0,1,2,..)$

Note that the method (5) is an ordinary multistep method with the constant coefficients, and the method (6) is a multistep method which has been proposed to solving the Volterra integral equation. Thus, we obtain a system of difference methods for solving of the problem (3). It is obvious that the accuracy of the methods determined by the proposed system of difference methods does not exceed the accuracy for each of these methods. Therefore, some scientists are constructed the methods consisting of the single formula.

We attempt to construct here the methods similar to the above proposed, which are used to solve of the problem (3), in special cases.

II. An investigation of the problem (3) in the special case

Here we will investigate the following problem, which received from the problem (3) in special particular case:

$$y' = f(x) + \int_{x_0}^{x} K(s, y(s))ds, y(x_0) = y_0, x \in [x_0, X]. \quad (7)$$

If we apply methods (5) and (6) to the solving of the problem (7), then we get:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + $$
$$+ h\sum_{i=0}^{k-1}(\beta_i - \beta_k\alpha_i)v_{n+i} + h^2\beta_k\sum_{i=0}^{k}\beta_i K_{n+i}, (\alpha_k = 1), \quad (8)$$

here: $K_m = K(x_m, y_m), (m = 0,1,2,..)$.

Remark that by the Dahlqvist theorem one can obtain that the order of accuracy of the method (8) satisfies the condition $p \le k+2$ (see for example [14]). To construct more exact methods, the specialists proposed different ways. One of them is the use of the higher derivatives of the function $y(x)$, which is a solution of the problem (3) (see [15] - [19]). Euler himself proposed to use the calculation of the subsequent terms in the expansion for a sufficiently smooth function in a Taylor series to increase the order of accuracy for his famous method (see [20], [21]). One such method is the following:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+i} + h^2\sum_{i=0}^{k}\gamma_i y''_{n+i}. \quad (9)$$

If the method (9) has the degree of p and is stable, then its degree is determined by the formula: $p \le 2k+2$.

It is said that the method (9) is stable if the roots of the following polynomial

$$\rho(\lambda) = \alpha_k\lambda^k + \alpha_{k-1}\lambda^{k-1} + ... + \alpha_1\lambda + \alpha_0$$

lie inside of the unit circle, on the boundary of which there are no multiple roots. And the integer variable of p is the degree of the method (9) if the following holds:

$$\sum_{i=0}^{k}(\alpha_i y(x+ih) - h\beta_i y'(x+ih) - h^2\gamma y''(x+ih))$$
$$= O(h^{p+1}), h \to 0.$$

It is easy to understand that to increase the values of the degree for the methods defined by the formula of (5), it is sufficient to add to its right-hand side new sums using the values of the higher derivatives of the function $y(x)$. It is known that there exist stable multistep methods of Obreshkov`s type with degree $p = r(k+1)+1$ for even k and odd r, and $p = r(k+1)$ in other cases. Here, the order of the method is denoted by the k and the order of the differentiation by r (see for example [22]). Consequently, the method (9) is a method of Obreshkov type with the second derivative and therefore $p_{\max} = 2k+2$.

Let us consider to the application of method (9) to solving of the problem (7). Then we have:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+i} + h^2\sum_{i=0}^{k}\gamma_i(f'_{n+i} + K_{n+i}). \quad (10)$$

If we take into account that the derivative of the function $y(x)$ does not exist on the right-hand side of the integro-differential equation, then (10) can be replaced by the following:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h^2\sum_{i=0}^{k}\gamma_i(f'_{n+i} + K_{n+i}). \quad (11)$$

Note that, the use of the method (11) is simpler than using the method (10). However, the maximum value for the degree of the method (11) is less than the value of the degree of the method (10). It does not follow from this that if a method is used to calculate the values of a quantity $y'_m (m \geq k)$, then a stable method of type (10) will have the degree $p = 2k+2$. For example, if to calculate the values $y'_m \ (m \geq k)$, proposed to use the following method:

$$\sum_{i=0}^{k} \alpha_i y'_{n+i} = h \sum_{i=0}^{k} \beta_i (f'_{n+i} + K_{n+i}) \qquad (12)$$

then the degree of the stable method obtained from the formula (12) will be satisfy the condition: $p \leq 2[k/2]+2$. Thus, if we construct a method based on the formulas (10), (12) and apply it to solving of the problem (7), then its degree will satisfy the condition $p \leq k+3$. Consequently, the use of the method (11) is preferable than using the method constructed on the basis of the formulas (10) and (12).

And now let us to compares the above proposed methods with the stable methods, constructed on the basis of the methods (5) and (6). If the methods defined by the formulas (5) and (6) are applicable to solving of the problem (3) then, as a result, we obtain method (8). But in this case, the order of accuracy of the methods constructed according to the scheme described above will coincide with the order of accuracy of the methods constructed by the formula (5). Consequently, the degree of the stable methods obtained from the formula (8) satisfies the condition $p \leq 2[k/2]+2$. Thus, we obtain that the order of accuracy of all methods defined by one of the formulas (8), (11) and (12), satisfies the condition $p \leq 2[k/2]+2$. Therefore, to construct methods with the higher order of accuracy, we can use formula (10). However, the order of accuracy of the method (10) depends from the orders of accuracy for the methods proposed for calculating the values of the quantity $y'_m (m = 1,2,..,)$. If for the calculation of the values of y'_m we use the stable methods determined by formula (12), then their degree satisfies the condition $p \leq k+2$. But the degree of a stable method, defined by the formula (10), satisfies the condition $p \leq 2k+2$. Hence, we need to construct more exact methods for calculation of the values y'_m participated in the formula (10) having the degree $p \leq 2k+1$. Those methods are in the class of hybrid methods. For the purpose of constructing such methods consider the following formula:

$$\sum_{i=0}^{k} \alpha_i y'_{n+i} = h \sum_{i=0}^{k} \beta_i y''_{n+i} + h \sum_{i=0}^{k} \gamma_i y''_{n+i+v_i}, \qquad (13)$$
$$(|v_i| < 1, i = 0,1,..,k).$$

In the class of methods (13), there exist stable methods with degree $p \geq 2k+2$ (see, for example [22], [25]). If we apply formula (13) to solving of the problem (7), then receive the following:

$$\sum_{i=0}^{k} \alpha_i y'_{n+i} = h \sum_{i=0}^{k} \beta_i (f'_{n+i} + K_{n+i}) + h \sum_{i=0}^{k} \gamma_i (f'_{n+i+v_i} + K_{n+i+v_i}). \qquad (14)$$

Note that the calculation of the values of the function $f'(x)$ does not cause any difficulties. But when calculating the values of the function $K(x, y)$, some difficulties arise related to the calculation of the values $y_{n+i+v_i} (0 \leq i \leq k)$. Thus, we get that the order of order of accuracy of the methods obtained from the formula (14), depends on the accuracy of approximate values of the solution of problem (7), calculated at hybrid points. We have not constructed a general method for calculating such values. However, constructed some specific methods for using formula (14). For this aim have proposed specific methods for calculating the values of y_{n+i+v_i}, the order of accuracy of which agrees with the order of accuracy of the proposed methods. Below are some specific methods are proposed.

Let us put $k = 1$ in the formula (14). Then from that one can receive the following methods:

$$y'_{n+1} = y'_n + h(f'_{n+1} + +K_{n+1} + f'_n + K_n)/12 + 5h(f'_{n+\frac{1}{2}-\alpha} +$$
$$+ K_{n+\frac{1}{2}-\alpha} + f'_{n+\frac{1}{2}+\alpha} + K_{n+\frac{1}{2}+\alpha})/12, (\alpha = \sqrt{5}/10), \qquad (15)$$

$$y'_{n+1} = y'_n + h(f'_n + K_n)/9 + h((16+\sqrt{6})(f'_{n+\gamma_0} + K_{n+\gamma_0}) +$$
$$+ (16-\sqrt{6})(f'_{n+\gamma_1} + K_{n+\gamma_1}))/36; \gamma_0 = (6-\sqrt{6})/10; \gamma_0 = (6+\sqrt{6})/10. \qquad (16)$$

Note that these methods are stable. Method (15) has the degree $p = 6$, and the method (16) has degree $p = 5$. The following method is also stable and has the degree $p = 4$:

$$y'_{n+1} = y'_n + h(f'_{n+l_0} + K_{n+l_0} + f'_{n+1+l_1} + K_{n+1+l_1})/2,$$
$$(l_0 = -l_1; l_0 = \frac{3-\sqrt{3}}{6}; 1+l_1 = \frac{3+\sqrt{3}}{6}). \qquad (17)$$

Note that if the function $K(s, y(s))$ depends on x, i.e. $K(s, y(s)) = \varphi(x, s, y(s))$, then the above proposed methods are written in the different form. For example, method (17) in one variant can be rewritten in the form:

$$y'_{n+1} = y'_n + h(2 f'_{n+l_0} + \varphi(x_{n+1}, x_{n+l_0}, y_{n+l_0}) +$$
$$+ \varphi(x_{n+l_0}, x_{n+l_0}, y_{n+l_0}) + 2 f'_{n+1-l_0} + \qquad (18)$$
$$+ \varphi(x_{n+1}, x_{n+1-l_0}, y_{n+1-l_0}) + \varphi(x_{n+1-l_0}, x_{n+1-l_0}, y_{n+1-l_0}))/4.$$

Note that the hybrid methods, generally speaking, are not implicit. For example, methods (16) and (17) are not implicit. The method (15) is implicit, since the term in the form $K(x_{n+1}, y_{n+1})$ participates in it. The hybrid method can be constructed so that the term in the form $K(x_{n+v}, y_{n+v})(|v| < 1)$ don't participate in it. If $v = 1$, then the point x_{n+v} is not a hybrid point. Therefore, for the construction of implicit hybrid methods, the condition $1 < v < 2$ must be satisfied. But in this case, we get the forward-jumping methods. Therefore, the hybrid methods which have the following form:

$$\sum_{i=0}^{k}\alpha_i y'_{n+1} = h\sum_{i=0}^{k}\beta_i y''_{n+i+v_i}, (|v_i|<1; i=0,1,..,k) \qquad (19)$$

are not implicit because the conditions $v_k < 0$ are usually hold. However, they can't be called explicit. Usually, the method is called explicit if it can be used to solve proposed problems without using other methods. As follows from here, for using hybrid methods, it is necessary to find the values of the quantities $y_{n+i+v_i}, (|v_i|<1; i \geq 0)$, which are the values of the solution of the considered problem at the hybrid points. Note that it is not easy to calculate these values.

Thus, we get that a lot of hybrid methods constitute a separate class of the methods. Usually, they are called a class of hybrid methods. Note that the methods, which have composed from the multi-step and hybrid methods also constitute the new classes of numerical methods, which include the methods constructed at the junction of multistep methods with constant coefficients (of types: explicit, implicit and forward-jumping methods) and hybrid methods. Note that these classes of methods include in itself the methods of both type (14) and (10) (methods of type (10) are constructed with the addition of the quantity of type:

y'_{n+i+v_i}, and $y_{n+i+v_i}, (|v_i|<1; i=0,1,...,k)$).

Now let us consider the construction of an algorithm for calculating the values of the solutions of the problem (7) by using the methods of the type (5) and (6):

Algorithm I:

$$y_{n+1/2} = y_n + h(f_n + v_n)/2, p=1; \qquad (20)$$

$$v_{n+1/2} = v_n + h(K_n + K_{n+1/2})/4, p=2; \qquad (21)$$

$$y_{n+1/2} = y_n + h(f_n + v_n + f_{n+1/2} + v_{n+1/2})/4, p=2; \qquad (22)$$

$$y_{n+1} = y_n + h(f_{n+1/2} + v_{n+1/2}), p=2; \qquad (23)$$

$$v_{n+1} = v_n + hK_{n+1/2}, p=2; \qquad (24)$$

$$v_{n+1} = v_n + h(K_n + 4K_{n+1/2} + K_n)/6, p=4; \qquad (25)$$

$$\begin{aligned} y_{n+1} &= y_n + h(f_{n+1} + 4f_{n+1/2} + f_n)/6 + \\ &+ h(v_{n+1} + 4v_{n+1/2} + v_n)/6, p=4 \end{aligned}; \qquad (26)$$

$$\begin{aligned} y_{n+1/2} &= y_n + h(5f_n + 8f_{n+1/2} - f_{n+1})/24 + \\ &+ h(5v_n + 8v_{n+1/2} - v_{n+1})/24, p=3 \end{aligned}; \qquad (27)$$

$$v_{n+1/2} = v_n + h(5K_n + 8K_{n+1/2} - K_{n+1})/24, p=3; \qquad (28)$$

$$v_{n+1} = v_n + h(K_{n+1} + 4K_{n+1/2} + K_n)/6, p=4; \qquad (29)$$

$$\begin{aligned} y_{n+1} &= y_n + h(f_{n+1} + 4f_{n+1/2} + f_n)/6 + \\ &+ h(v_{n+1} + 4v_{n+1/2} + v_n)/6, p=4 \end{aligned}. \qquad (30)$$

In this algorithm, the basic methods are the methods (29) and (30), and the methods (20) - (28) are provide the calculation of the values of the quantities $v_{n+1/2}, v_{n+1}, y_{n+1}$ and y_{n+1} with the required accuracy. However, in the sequence of methods (20) - (28), the number of these methods can be reduced by assuming that the values of the quantities $v_{1/2}, v_1, y_{1/2}$, y_1 are known. In this case, more precise methods can be used in this sequence.

Now consider the construction of algorithm for using the methods constructed by the help of the formula (11).

Algorithm II:

$$y_1 = y_0 + hy'_0;\ y'_1 = y'_0 + hy''_0;\ y_1 = y_0 + h(y'_0 + y'_1)/2;$$

$$y'_1 = y'_0 + h(y''_0 + y''_1)/2,$$

$$y_2 = y_1 + hy'_1;\ y'_2 = y'_1 + hy''_1;\ y_2 = y_1 + h(y'_1 + y'_2)/2,$$

$$y'_2 = y'_1 + h(y''_1 + y''_2)/2,;$$

$$y_1 = y_0 + h(5y'_0 + 8y'_1 - y'_2)/12;\ y_3 = y_2 + hy'_2;\ y'_3 = y'_2 + hy''_2;$$

$$y'_3 = y'_2 + h(y''_2 + y''_3)/2;$$

$$y_2 = y_1 + h(5y'_1 + 8y'_2 - y'_3)/12;\ y_2 = 2y_1 - y_0 + h^2(y''_2 + 10y''_1 + y''_0)/12;$$

$$y'_1 = y'_0 + h(5y''_0 + 8y''_1 - y''_2)/12;$$

$$y_1 = y_0 + h(y'_0 + y'_1)/2 + h^2(-y''_0 + y''_1)/12;$$

$$y_{n+2} = 2y_{n+1} - y_n + h^2(13y''_{n+1} - 2y''_n + y''_{n-1})/12; \qquad (31)$$

$$y_{n+2} = 2y_{n+1} - y_n + h^2(y''_{n+2} + 10y''_{n+1} + y''_n)/12, (n \geq 1). \qquad (32)$$

Note that the sequence of the methods proposed before method (31) are ancillaries and therefore used once, and methods (31) and (32) are basic and are used as much as necessary. Suppose that y_1 is found in some way. Then, to find the approximate solution of the problem (7), we can use the following sequence of methods, also constructed by using the formulas (5) and (6).

Algorithm III:

$$y_{n+2} = y_n + hf_{n+1} + hv_{n+1},$$

$$v_{n+2} = v_{n+1} + h(5K_{n+2} + 8K_{n+1} - K_n)/12,$$

$$y_{n+2} = y_{n+1} + h(5f_{n+2} + 8f_{n+1} - f_n)/12 + h(5v_{n+2} + 8v_{n+1} - v_n)/12,$$

$$v_{n+2} = v_n + h(K_{n+2} + 4K_{n+1} + K_n)/3,$$

$$y_{n+2} = y_n + h(f_{n+2} + 4f_{n+1} + f_n)/3 + h(v_{n+2} + 4v_{n+1} + v_n)/3.$$

In order to construct an algorithm for using the methods of the type (10), we propose the following sequence of methods:

Algorithm IV:

$$y_{n+2} = y_{n+1} + h(3y'_n - y'_{n+1})/2 + h^2(7(f'_n + K_n) + 17(f'_{n+1} + K_{n+1}))/12,$$

$$y'_{n+2} = y'_n + 2h(f'_{n+1} + K_{n+1}),$$

$$y'_{n+2} = y'_{n+1} + h(5(f'_{n+2} + K_{n+2}) + 8(f'_{n+1} + K_{n+1}) - f'_n - K_n)/12,$$

$$y'_{n+2} = y'_n + h((f'_{n+2} + K_{n+2}) + 4(f'_{n+1} + K_{n+1}) + f'_n + K_n)/3,$$

$$\begin{aligned} y_{n+2} = y_{n+1} &+ h(101y'_{n+2} + 128y'_{n+1} + 11y'_n)/240 + h^2(-13(f'_{n+2} + K_{n+2}) + \\ &+ 40(f'_{n+1} + K_{n+1}) + 3(f'_n + K_n))/240. \end{aligned}$$

If the values of $y_n, y'_n, y_{n+1}, y'_{n+1}$ are known, then by using the above described sequence of methods, one can calculate the values of the quantities $y_m, y'_m (m \geq 2)$.

From the above proposed algorithms it follows that to using the method (10) with the maximum accuracy, it is necessary to construct more precise methods for calculating the quantity $y'_m (m = 2,3,...)$. To this end, hybrid methods can be used, since these methods are more exact. To illustrate the foregoing, proposed here the algorithm by using which the values of the quantity of $y'_m (m = 2,3,...)$ are calculated if are known the values

$$y_n, y'_n, y'_{n+1}, y'_{n+1/2}, y'_{n+3/2}, y'_{n+1/2}, y'_{n+1/2}$$

let us consider the following sequence of methods for calculation of the values $y_m (m \geq 2)$:

Algorithm V:

$$y_{n+2} = y_{n+1} + h(-y'_{n+1} + 3y'_n)/2 + h^2(17(f'_{n+1} + K_{n+1}) + \\ +7(f'_n + K_n))/12,$$

$$y'_{n+2} = y'_n + 2h(f'_{n+1} + K_{n+1}),$$

$$y'_{n+2} = y'_{n+1} + h(3f'_{n+1+1/3} + 3K(x_{n+4/3}, \frac{4}{9}y'_{n+1} + \frac{5}{9}y'_{n+2} - \\ -\frac{2h}{9}(f'_{n+2} + K_{n+2})) + f'_{n+2} + K_{n+2})/4,$$

$$y'_{n+2} = y'_{n+1} + h(f'_{n+2} + K_{n+2} + 4(f'_{n+1} + K_{n+1}) + f'_n + K_n)/3,$$

$$y_{n+1+3/2} = y_{n+3/2} + h(7y'_{n+2} - 2y'_{n+3/2} + y'_{n+1})/6,$$

$$y'_{n+1+3/2} = y'_{n+1+1/2} + h(7(f'_{n+2} + K_{n+2}) - \\ 2(f'_{n+1+1/2} + K_{n+3/2}) + f'_{n+1} + K_{n+1})/6,$$

$$y_{n+1+\alpha} = y_{n+1} + \alpha h y'_{n+1} + \alpha^2 h((\alpha^2 - 12\alpha + 6)y'_{n+1+3/2} + \\ (3\alpha^2 - 48\alpha + 27)y'_{n+2} + \\ +(3\alpha^2 - 60\alpha + 54)y'_{n+1+1/2} - (\alpha^2 - 24\alpha + 33)y'_{n+1})/18,$$

(For the value $\alpha = (6-\sqrt{6})/10$ and $\alpha = (6+\sqrt{6})/10$),

$$y'_{n+2} = y'_{n+1} + h(f'_{n+1} + K_{n+1})/9 + \\ +h((16+\sqrt{6})(f'_{n+1+\gamma_0} + K_{n+1+\gamma_0}) + \\ +(16-\sqrt{6})(f'_{n+1+\gamma_1} + K_{n+1+\gamma_1}))/36,$$

(For the value $\gamma_0 = (6-\sqrt{6})/10$ and $\gamma_1 = (6+\sqrt{6})/10$),

$$y_{n+2} = y_{n+1} + h(101y'_{n+2} + 128y'_{n+1} + 11y'_n)/240 + \\ +h^2(-13(f'_{n+2} + K_{n+2}) + \\ +40(f'_{n+1} + K_{n+1}) + 3(f'_n + K_n))/240.$$

Thus, we have constructed five algorithms for solving the problem (7). Each of these algorithms has advantages and disadvantages. Let us compare them with the help of the model problem.

For this aim to consider the application of the above descried algorithms to the solving of the following problem:

1. $$y'(x) = 2\lambda\exp(\lambda x) - \lambda - \lambda^2\int_0^x y(s)ds, \qquad y(0) = 1, \\ 0 \leq x \leq 1.$$

Here λ is a certain parameter. The exact solution of the problem can be represented in the form: $y(x) = \exp(\lambda x)$.

$$y'(x) = 2\exp(x) - 1 + \int_0^x y(s)ds, \qquad y(0) = 0, \\ 0 \leq x \leq 1.$$

The exact solution of the problem can be represented in the form: $y(x) = x\exp(x)$.

The results of solving these problems using three algorithms are placed in the following tables:

Table 1. Results of the example 1.

Step size	Variable x	Error of the Algorithm 1	Error of the Algorithm 2	Error of the Algorithm 3
$h = 0.01$	0.02	1.11E-12	4.77E-15	3.99E-15
	0.25	1.09E-11	1.48E-12	0.25E-12
	0.50	1.68E-11	5.55E-12	5.95E-12
	0.75	1.30E-11	1.13E-11	1.44E-11
	1.00	7.23E-13	1.80E-11	2.76E-11

Table 2. Results of the example 2.

Step size	Variable x	Error of the Algorithm 1	Error of the Algorithm 2	Error of the Algorithm 3
$h = 0.01$	0.02	5.65E-12	2.81E-14	2.52E-14
	0.25	9.58E-11	1.09E-11	8.35E-12
	0.50	2.73E-10	5.03E-11	3.83E-11
	0.75	5.60E-10	1.30E-10	9.90E-11
	1.00	9.92E-10	2.67E-10	2.03E-10

Remark: Note that the above described way can be applied to a more general integro-differential equation. To this end, consider the following equation:

$$y'(x) = f(x,y) + \int_{x_0}^{x} K(s, y(s))ds \qquad (33)$$

Suppose that in some way an exact solution of this equation is found, after taking that into account and by the differentiating of the obtained equality, we have:

$$y''(x) = f'_x(x, y(x)) + f'_y(x, y(x))y'(x) + K(x, y(x)). \qquad (34)$$

Obviously, that to find the numerical solution of the resulting equation, one can use the method (9). Then depending on the used method to calculate the values of the function $y'(x)$ at the points $x_m (m > 0)$, receive different problems. If we use the equation (33) to find the values of the function $y'(x)$ then obtain an integro-differential equation. But, if we use the following method to calculate the values of $y'_m (m > 0)$:

$$\sum_{i=0}^{k}\alpha_i y'_{n+i} = h\sum_{i=0}^{k}\beta_i(f'_x(x_{n+i}, y_{n+i}) + \\ + f'_y(x_{n+i}, y_{n+i})y'_{n+i} + K(x_{n+i}, y_{n+i})),$$

then we obtain a system of difference methods, which have applied to solve the initial-value problem of the ODE of the second order.

Thus, we have reduced the solving of the initial-value problem for the integro-differential equations to solving of the initial-value problem for an ODE. By the same way, the solving of the initial-value problem for the equation (33) can be reduced to solving of the initial-value problem for higher order ODEs. Some authors by using this phenomenon proposed to replace the finding of the solution of the initial-value problem for the ODEs of the first order to the finding of the solution of initial-value problem for higher order ODEs and then use the appropriate method of the original problem.

III. CONCLUSION

Here have constructed 5 algorithms having the same accuracy. In three of them, the degree for the basic methods coincide and are equal to $p = 4$. In the other two algorithms, the basic method applied to calculate the values of y_{n+2} has a higher accuracy, namely $p = 6$. As follows from the above reduced tables, the accuracy of the results obtained by the algorithms II, III and IV are the same. In algorithm III, the methods with the degree $p = 4$ are not used. In the result of which, received that the maximum value for the error of algorithm III in application to solving of the first problem is $6{,}3E-0{,}8$ and in application to solving of the second problem is $3{,}62E-0{,}7$, which corresponds to the accuracy of the proposed algorithms. If in algorithm III replace h by $h/2$, then the using methods turn into hybrid methods. In this case, the accuracy of the algorithm does not increase, but the area of its stability increases and the algorithm becomes more correct Note that the above described scheme cannot be applied to all methods. For example, if the indicated scheme is applied to algorithm II, then the expected effect will not be obtained. And algorithm II is the simplest. However, application of it to solving of the problem (3) is not easy. But, Algorithm III can be easily applied to solving of the problem (3). Thus, receive that each method has its own field of application, the use of which improves its application properties.

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