General conditioned and aimed information on fuzzy setting

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Abstract: In this paper our investigation on aimed information, started in 2011, will be completed on fuzzy setting. Here will be given a form of information for fuzzy sets, when it is conditioned and aimed. This information is called *general,* because it is defined without using probability or fuzzy measure.

KeyWords: Fuzzy sets, information, conditioning information, aimed information

I. INTRODUCTION

By using the concept of general information (i.e. information without probability or fuzzy measure [1, 2, 3]), the definition of conditional information [4, 5] and aimed information [6] have been introduced for crisp sets.

It is possible to move to fuzzy setting. In fact the goal of this paper is to introduce a form of general information J conditioned and aimed by two different sets, independent of each other with respect to J ($J-$independence).

This measure can be useful when we want to measure information of a set of people with different levels of the same illness, treated with different dose of a medicament.

The paper is organized in the following way. Sect.2 contains some preliminaries. In Sect.3 in fuzzy setting will be introduced the definition of general conditional information with a given aim, by means of axioms. The properties of this information are traslated in a system of functional equations [7, 8]. In Sect.4 the problem is solved, finding a class of solutions and a particular solution in $J-$independent case. Sect.5 is devoted to the conclusion.

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II. PRELIMINAIRES

Let X be an abstact space and $\mathcal{F}$ the $\sigma-$algebra of all fuzzy sets of X, such that $(X, \mathcal{F})$ is measurable. Basic notions, notations and operation on fuzzy sets can be found in [9, 10]. Now, the definition of measure of general information for fuzzy sets is recalled [11].

Definition n.1

Measure of general information $J(\cdot)$ is a mapping $J(\cdot) : \mathcal{F} \to [0, +\infty]$ such that $\forall F, F' \in \mathcal{F}$:

$(i) F \supset F' \to J(F) \leq J(F')$,

$(ii) J(\emptyset) = +\infty, \quad J(X) = 0$.

Given a measure of general information J and $K, K' \in \mathcal{F}$ with $K \neq K', K \cap K' \neq \emptyset$, K and K' are said $J-$ independent (i.e. independent of each other with respect to J) if

$(iii)\ J(K \cap K') = J(K) + J(K')$.

III. STATEMENT OF THE PROBLEM

In this paragraph will be introduced measure of general information when it is conditioned by a given event H and it is aimed by a different event S.

A. Definition

From now on, the following assumption is considered:

$$\text{let } H, S \in \mathcal{F}, \mathrm{H} \neq \mathrm{S}, \tag{1}$$

$$J(H) \neq +\infty, J(S) \neq +\infty,$$

H and S are calling *conditioning* and *aiming* events, respectively. Now, given a conditioning and aiming sets as in (1), it is introduced the definition of general information of the set $F \in \mathcal{F}$ conditioned by H with the aim S: this information will be denoted by $J_H(F \to S)$.

Definition n.2

Given H and S as in (1), measure of general information conditioned by H with the aim S is a mapping

$$J_H(\cdot \to S) : \mathcal{F} \to [0, +\infty]$$

such that $\forall F, F' \in \mathcal{F}$:

(l) $F \supset F' \to J_H(F \to S) \leq J_H(F' \to S)$,
(ll) $J_H(\emptyset \to S) = +\infty, \quad J_H(X \to S) = 0$.

Given a measure $J_H(\cdot \to S)$ as in Def.2, $K, K' \in \mathcal{F}$ with $K \neq K', K \cap K' \neq \emptyset$, K and K' are said $J-$ *conditional independent with the aim* S (i.e. independent of each other with respect to J conditioned by H with the aim S) if

$$(lll)\ J_H((K \cap K') \to S) =$$

$$J_H(K \to S) + J_H(K' \to S).$$

B. The function Φ

With the assumption (1), our study considers that measure $J_H(\cdot \to S)$ of $F \in \mathcal{F}$ depends on $J(F), J(H), J(S), J(F \cap H), J(F \cap S)$. So, one will find a function Φ such that:

$$J_H(F \to S) = \tag{2}$$

$$\Phi\Big(J(F), J(H), J(S), J(F \cap H), J(F \cap S)\Big),$$

with $\Phi : T \to [0, +\infty]$ and T will be specified later. Putting: $x = J(F), y = J(H), z = J(S), u = J(F \cap H), v = J(F \cap S)$, with $x, u, v \in [0, +\infty], y, z \in [0, +\infty), x \leq u, y \leq u, x \leq v, z \leq v$, from (2) it is

$$J_H(F \to S) = \Phi\Big(x, y, z, u, v\Big) \tag{3}$$

and $T = \{(x, y, z, u, v)/x, u, v \in [0, +\infty], y, z \in [0, +\infty), x \leq u, y \leq u, x \leq v, z \leq v\}$.

Moreover, setting $x' = J(F'), u' = J(F' \cap H), v' = J(F' \cap S)$, with $x', u', v' \in [0, +\infty], x' \leq u', x' \leq v'$, the properties $[(l)-(ll)]$ of $J_H(\cdot \to S)$ are traslated in the following system of functional equations:

$$\begin{cases} (e_1)\ \ \Phi(x, y, z, u, v) \leq \Phi(x', y, z, u', v') \\ \ \text{if} \quad x \leq x', u \leq u', v \leq v'\ , \\ (e_2)\ \ \Phi(+\infty, y, z, +\infty, +\infty) = +\infty\ , \\ (e_3)\ \ \Phi(0, y, z, y, z) = 0\ . \end{cases}$$

IV. SOLUTION OF THE PROBLEM

C. General case

For the system $[(e_1) - (e_3)]$ it is
Proposition n.1
A class of solution of the system $[(e_1) - (e_3)]$ is

$$\Phi_h(x, y, z, u, v) = \tag{4}$$

$$h^{-1}\Big(h(x) - h(y) - h(z) + h(u) + h(v)\Big)$$

where h is any continuous, strictly increasing function $h : [0, +\infty] \to [0, +\infty]$ with $h(0) = 0, h(+\infty) = +\infty$.

Proof: The prof follows easily from the properties of the function h.

From (3) and (4), given H and S as in (1), measure of general information of any fuzzy set F conditioned by H with the aim S is

$$J_H(F \to S) = h^{-1}\Big(h(J(F)) - h(J(H)) - \tag{5}$$

$$h(J(S)) + h(J(F \cap H) + h(J(F \cap S)\Big)$$

where h is any continuous, strictly increasing function $h : [0, +\infty] \to [0, +\infty]$ with $h(0) = 0, h(+\infty) = +\infty$.

D. $J-$independence

In the case of $J-$independence the system $[(e_1)-(e_3)]$ must be completed with an extra equation deduced by the property (lll) :

$$(e_4)\ \ \Phi\Big(t + t', y, z, t + t' + y, t + t' + z\Big) =$$
$$\Phi\Big(t, y, z, t+y, t+z\Big) + \Phi\Big(t', y, z, t'+y, t'+z\Big),$$

where $t = J(K), t' = J(K'), t, t' \in [0, +\infty]$.

Among all h of the Prop. n.1, only differentiable functions are considered. Here it is used the same procedure of [12].

The equation $[(e_4)]$ is

$$h^{-1}(h(t{+}t'){-}h(y){-}h(z){+}h(t{+}t'{+}y){+}h(t{+}t'{+}z))$$
$$= h^{-1}(h(t) - h(y) - h(z) + h(t + y) + h(t + z)) +$$
$$h^{-1}(h(t') - h(y) - h(z) + h(t' + y) + h(t' + z)).$$

Now, the function h will be characterized.
Putting $y = z$,

$$h^(h(t{+}t'){-}h(y){-}h(y){+}h(t{+}t'{+}y){+}h(t{+}t'{+}y))$$
$$= h^{-1}(h(t) - h(y) - h(y) + h(t + y) + h(t + y)) +$$
$$h^{-1}(h(t') - h(y) - h(y) + h(t' + y) + h(t' + y)),$$

i.e. it is

$$h^{-1}\Big(2\,h(t + t' + y) + h(t + t) - 2\,h(y)\Big) = \tag{6}$$

$$h^{-1}\Big(2\,h(t + y) + h(t) - 2\,h(y)\Big) +$$

$$h^{-1}\Big(2\,h(t' + y) + h(t') - 2\,h(y)\Big).$$

Setting

$$\varphi(t,y) = h^{-1}\Big(2\,h(t+y) + h(t) - 2\,h(y)\Big) \qquad (7)$$

the equation (6) becomes

$$\varphi(t+t',y) = \varphi(t,y) + \varphi(t',y). \qquad (8)$$

Fixed $y = y^*$, the (8) is the classical Cauchy equation [7], whose solution is the continuous function φ :

$$\varphi(t,y^*) = \lambda(y^*)t. \qquad (9)$$

So, from (7),

$$\lambda(y^*)t = h^{-1}\Big(2\,h(t+y^*) + h(t) - 2\,h(y^*)\Big) \quad i.e.$$

$$h\Big(\lambda(y^*)t\Big) = 2\,h(t+y^*) + h(t) - 2\,h(y^*). \qquad (10)$$

If $y^* = 0$, as $h(0) = 0$, from (10), one has

$$h\Big(\lambda(0)t\Big) = 2\,h(t) + h(t), \quad i.e.$$

$$h\Big(\lambda(0)t\Big) = 3\,h(t). \qquad (11)$$

Taking inspiration by [7, 8, 13, 14] one will prove that

$$h\Big(\lambda(0)t\Big) = 3\,h(t) \Longrightarrow \lambda(0) = 3. \qquad (12)$$

Set $\lambda(0) = c$, from (11), one will solve the equation

$$h(c\,t) = 3\,h(t); \qquad (13)$$

by differentiating $ch'(c\,t) = 3\,h'(t)$ from which

$$\frac{c\,h'(c\,t)}{h(c\,t)} = \frac{h'(t)}{h(t)}. \qquad (14)$$

Setting

$$v(t) = \frac{h'(t)}{h(t)}, \qquad (15)$$

the (14) is

$$v(c\,t) = \frac{v(t)}{c}, \quad \forall\, t. \qquad (16)$$

The function $v(t) = \frac{1}{t}$ is the unique solution admitting a Laurent expansion about 0. By substituing in (15), one obtain the equation

$$\frac{h'(t)}{h(t)} = \frac{1}{t} \qquad (17)$$

whose solution is

$$h(t) = k\,t, t \in [0,+\infty], k > 0. \qquad (18)$$

By substituing (18) in (13), it is $c = \lambda(0) = 3$. So, the function h satisfies the following condition:

$$h(3\,t) = 3\,h(t). \qquad (19)$$

From (10),

$$\varphi(x,t) = 3\,t = h^{-1}\Big(2\,h(t+y) + h(t) - 2\,h(y)\Big)$$

$$i.e. \quad h(3\,t) = 2\,h(t+y) + h(t) - 2\,h(y),$$

taking into account (19), it is

$$3\,h(t) = 2\,h(t+y) - 2\,h(y) + h(t)$$

$$i.e. \quad h(t) + h(y) = h(t+y),$$

which is the classical Cauchy equation [7], whose solution is

$$h(x) = c\,x, c > 0. \qquad (20)$$

Now, it is possible to give the following
Proposition n.2
The solution of the system $[(e_1) - (e_4)]$ is

$$\Phi(x,y,z,u,v) = x - y - z + u + v. \qquad (21)$$

Proof: It is easy to check that (21) holds, by applying (20) in the (4).

In the independent case, given H and S as in (1), from (21), information of any set $A \in \mathcal{A}$ conditioned by H with the aim S is

$$J_H(A \to S) = J(A) - J(H) - J(S) + \qquad (22)$$

$$J(A \cap H) + J(A \cap S).$$

V. CONCLUSION

First, by axiomatic way, it has been defined general conditional information with an aim, on fuzy setting. By using its properties, it has been possible to find a class of this measure (5).

Then, taking into account the $J-$independence property, it has been obtained a particular measure (22).

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