Selecting the most Appropriate Fuzzy Implication based on Statistical Data

P. Pagouropoulos, C. D. Tzimopoulos and B. K. Papadopoulos*

Abstract—Fuzzy implications are used in inference system applications involving fuzzy control, approximate reasoning, and artificial intelligence, among others. In applications where propositional logic is employed for reasoning, fuzzy implications play a fundamental role as logical connectives. In applications where multiple fuzzy implications are to be engaged, it is necessary that the most appropriate of these implications be selected, on the grounds that it best represents the notion of induction of an application, pertaining to it. This study introduces a method for the selection of the most appropriate fuzzy implication among others under consideration. The method's resulting most appropriate fuzzy implication is the one, whose corresponding fuzzy propositions best represent the inference making from the data of an application, regarding the expert's opinion on the data application.

Keywords—Fuzzy implications, Similarity measures, Fuzzy propositions, Fuzzy sets.

I. Introduction

PROPOSITIONAL logic is one of the basic concepts of reasoning systems and its logical techniques play an important role in the implementation of artificial intelligence and knowledge-based systems. It involves logic variables and logic functions (usually called logical connectives) between logic variables, which assess the truth value of logic propositions. Logic variables take place in propositions of the form 'p: s is *P*' where s is a subject and *P* is a predicate that describes a property. Then, proposition p takes a truth value to gauge its validity. The most common logical connectives used in propositional logic are the negation (not), conjunction (and), disjunction (or), which form other logical functions like the implications (if-then). An implication is an If-Then rule of the form 'If *A*, Then *B*' and it assigns a value to the two logic variables derived from the If-part *A*, and the Then-part *B*. The If-part of the implication is a logic variable called antecedent and the Then-part is a logic variable called consequent.

In classical logic the truth value of a logic variable or a proposition belongs to the set of two elements {0, 1} (i.e. true or false). However, in fuzzy logic the truth value of a proposition or a logical variable is a matter of degree, and it is a number of the interval [0, 1]. This result makes fuzzy propositions suitable for applications where the predicates are variables which cannot be expressed by a unique number, unlike those in classical logic. For example, the variable 'high altitude' is not binary for all the values of distance, but takes values in an interval instead. What is more, when these variables are combined with linguistic hedges, the result is the formation of other variables, called linguistic variables, which reflect levels of the initial variables [29]. For example, from the variable 'altitude' and the hedges 'high', 'medium', and 'low' the linguistic variables 'high altitude', 'medium altitude', and 'low altitude' are obtained. When logic variables of the form 'p: s is *F*' are considered as fuzzy sets, the subject s is an object of a universal crisp set *X* and the degree of truth of the logic variable is the value of the membership function *F(x)* of the fuzzy set *F* for each one of the

B. K. Papadopoulos is with the Department of Civil Engineering, Democritus Thrace University, Vas. Sofias 12, 67100, Xanthi, Greece, Corresponding Author (phone: +3025410-79747;email: papadob@civil.duth.gr).

Christos D. Tzimopoulosis with the Department of Rural and Surveying Engineering, Aristotle University of Thessaloniki 54124, Greece (email: ctzimop@gmail.com).

P. Pagouropoulos is with the Department of Civil Engineering, Democritus Thrace University, Vas. Sofias 12, 67100, Xanthi, Greece (email: ppagouro@civil.duth.gr).

objects in the crisp set *X*. Furthermore, in fuzzy propositions the logical connectives are not unique, since they are fuzzy operations. That said, combining different fuzzy negations, fuzzy conjunctions and fuzzy disjunctions, a variety of fuzzy implications is created. A fuzzy implication maps the values of two logic variables, as they are fuzzy sets, to a number in the close interval [0, 1]. That means that a fuzzy implication involved in a proposition, expresses the relationship between the premise and the consequent.

A general categorization of fuzzy propositions is based on whether they are unconditional or conditional propositions. In this study the conditional fuzzy propositions are used. The If-Then fuzzy rules, involving fuzzy implications, are conditional fuzzy propositions. The canonical form of such fuzzy propositions is 'p: If *x* is *A*, Then *y* is *B*', where *x*, *y* are values in the crisp sets *X* and *Y* and *A*, *B* are fuzzy sets on $X \times Y$. Fuzzy inference rules play a crucial role in fuzzy control and approximate reasoning. Many inference system applications in engineering and industry implement fuzzy propositional logic and a ruled-base system where fuzzy implications are involved [23-26]. The fact that fuzzy implications affect the degree of truth of conditional fuzzy propositions that are involved in makes the selection of suitable fuzzy implications a part of major importance of the system design procedure.

The theoretical suitability of fuzzy implications is based on tautologies called generalized inference rules, which are derived from inference rules of classical logic and implications. Those are the generalized modus ponens, generalized modus tollens and the generalized hypothetical syllogism. Given a fuzzy proposition, an implication is suitable, theoretically, when the generalized tautologies coincide with their classical counterparts [7], [27], [28]. Furthermore, a method for the selection of the most suitable fuzzy implication, which is close to the theoretical viewpoint, is described in [13], and it is based on the distance between fuzzy implications and the modus ponens.

Contrary to the theoretical concept, the method introduced in this study for choosing the most appropriate fuzzy implication among other fuzzy implications is beyond the notion of theoretical suitability of implications. More specifically, it introduces an implication selection method, which is based on the application approach, and so in a way that is closer to the dataset an the expert's opinion, rather than the theoretical scope.

The rest of the paper is structured as follows. In Section II some useful definitions and notations are presented, in Section III the methodology and procedure is described, and in Section IV an application of the method is demonstrated. Finally, Section V is devoted to the conclusions.

II. Preliminaries

All the definitions regarding fuzzy implications and fuzzy sets used in this paper can be found in [6, 7, 11, 14-16].

A. Implications

Definition 1 An implication in classical logic is a function:
$l:\{0,1\}\times\{0,1\}\rightarrow\{0,1\}$,which satisfies the boundary conditions:

i) $l(0,0)=1$

ii) $l(0,1)=1$ (Falsity implies anything)

iii) $l(1,0)=0$

iv) $l(1,1)=1$

Definition 2 A fuzzy implication is a relation:
$I:[0,1]\times[0,1]\rightarrow[0,1]$, which satisfies at least some of the following axioms, which are listed from the weakest to the strongest $\forall a,b,c\in[0,1]$:

1) $a\le b$ implies $I(a,x)\ge I(b,x)$

2) $a\le b$ implies $I(x,a)\le I(x,b)$

3) $I(0,a)=1$ (Falsity implies anything)

4) $I(1,b)=1$

5) $I(a,a)=1$

6) $I(a,I(b,x))=I(b,I(a,x))$

7) $I(a,b)=1$ iff $a\le b$

8) $I(a,b)=I(c(b),c(a))$, where c is a fuzzy complement.

9) I is a continuous function.

Letting $a,b,c\in\{0,1\}$, fuzzy implications satisfy the boundary conditions i) - iv) of those in classical logic. An important property is that, since fuzzy implications are formed by a combination of fuzzy complements (i.e. negation), t-norms (i.e. conjunction) and t-conorms (i.e. disjunction), this

triplet of fuzzy operations must satisfy the De Morgan laws.
Some common fuzzy implications are listed below. Furthermore, these implications are employed in Section 4:

1. The Larsen rule
$R_1(a,b) = ab$

2. The Mamdani rule
$R_2(a,b) = \min\{a,b\}$

3. The Kleene-Dienes implication
$R_3(a,b) = \max\{1-a,b\}$

4. The Lukasiewicz implication
$R_4(a,b) = \min\{1,1-a+b\}$

5. The Early Zadeh implication
$R_5(a,b) = \max\{1-a,\min\{a,b\}\}$

6. The Reichenbach implication
$R_6(a,b) = 1-a+ab$

7. The Willmott implication
$R_7(a,b) =$
$= \min\{\max\{1-a,b\},\max\{1-a,a\},\max\{1-b,b\}\}$

8. The Klir and Yuan 1 implication
$R_8(a,b) = 1-a+a^2b$

The first two operators, the Mamdani rule and the Larsen rule, are the fuzzy products, as discussed in [2], [8], and [10]. When considered as implications they are called engineering implications as mentioned in [12], and since they satisfy the boundary condition $I(0,a)=0$ (i.e. falsity implies nothing), in opposition to the above-mentioned axiom ii) of implications in classical logic and axiom 3) of those in fuzzy logic (i.e. falsity implies anything), they make no distinction between premise and conclusion. That being the case, they are suitable for applications where cause and effect are confused, as in [17]. Moreover, when these fuzzy relations are used as membership functions they construct fuzzy sets, as explained in [7, pp. 120-121]. That means that given the crisp sets X and Y, and $A=\{(x,A(x))\,|\,x\in X\}$, $B=\{(y,B(y))\,|\,y\in Y\}$ two fuzzy sets on X and Y, with membership functions A and B respectively, a fuzzy implication I can be treated as a fuzzy set on the set $X\times Y$, as the fuzzy set $I=\{((x,y),I(x,y))\,|\,(x,y)\in X\times Y\}$ $\forall x\in X$ and $\forall y\in Y$, where $I(x,y)=I(A(x),B(y))$ is the membership function of the fuzzy set *I*, which is the expression of the very fuzzy implication *I.*

B. Final Conditional fuzzy propositions

Propositions of this type are expressed by the canonical form:

$p:\ If\ x\ is\ A,\ \text{Then}\ y\ \text{is}\ B,$

where *x*, *y* are, in general, variables whose variables are in crisp sets *X* and *Y* respectively, *A*, *B* are fuzzy sets on *X*, *Y*. These propositions can be also expressed by a similar form:

$p:\ \langle x,\ \mathrm{y}\rangle\ \text{ is }\ R,$

where R is a fuzzy binary relation $R(x,y)=R(A(x),B(y))$, thus a fuzzy set (as explained in the introduction) on $X\times Y$, of the form $R=\{((x,y),R(x,y))\,|\,(x,y)\in X\times Y\}$. In inference systems applications, suitable fuzzy implications are used as fuzzy relations, involved in fuzzy propositions, thus constructing compositional rules.
The degree of truth of a fuzzy proposition is, in general, a function $T(p)$ of the membership function of the predicate, called truth qualifier, and it expresses the quality of truth of the proposition (i.e. True, Very true, Very-very true, False, Fairly false etc.), as discussed in [7, pp. 222 - 225], and [19]. In the case of a conditional fuzzy proposition, $T(p)$ is a function of the fuzzy implication, and it indicates how true is that the predicate implies the consequent. When the truth qualifier of a conditional fuzzy preposition is considered to be the identity function, the degree of truth of the proposition is the value of the corresponding fuzzy implication of the proposition.

C. Measures of similarity

In literature, a large variety of measures of similarity between fuzzy values has been studied and discussed as in [20], [21], and [22]. The measures of similarity are categorized as distance-based similarity measures, which are based on the distance of fuzzy sets, set-theoretic measures of similarity, and fuzzy implicators-based similarity measures. Some commonly used set-theoretic measures of similarity are stated as follows [13]:

1) The grade of similarity M of the fuzzy sets *A* and *B*, defined by

$$M(A,B) = \begin{cases} 1, & \text{if } A = B = \varnothing \\ \dfrac{\Sigma_{x\in X} \min(A(x), B(x))}{\Sigma_{x\in X} \max(A(x), B(x))}, & \text{otherwise} \end{cases}$$

2) The grade of similarity L of the fuzzy sets *A* and *B*, defined by

$$L(A,B) = 1 - \max_{x\in X} \left| A(x) - B(x) \right|$$

3) The grade of similarity S of the fuzzy sets *A* and *B*, defined by

$$S(A,B) = \begin{cases} 1, & \text{if } A = B = \varnothing \\ 1 - \dfrac{\Sigma_{x\in X} \left| A(x) - B(x) \right|}{\Sigma_{x\in X} (A(x) + B(x))}, & \text{otherwise} \end{cases}$$

In this study the M grade of similarity is employed to measure the degree of similarity of two fuzzy sets.

III. Method for the Selection of the Fuzzy Implication

Given an initial set of implications, in order to determine which implication is the most appropriate one, a special fuzzy set L is defined to represent the ideal fuzzy implication. The membership function of this fuzzy set takes the value one for each one of the values of the data of an application, since they are real observations, therefore the inference drawn that the premise implies the conclusion has to be the absolute truth (i.e. 'True').

Let *X*, *Y* be the crisp sets that contain the data, and *A*, *B* be fuzzy sets on *X*, *Y*. The fuzzy set *L* is:

$L = \{((x_i, y_i), L(x_i, y_i)) \mid (x_i, y_i) \in X \times Y\}$, where $L(x_i, y_i) = L(A(x_i), B(y_i)) = 1, \forall (x_i, y_i) \in X \times Y$.

As a consequence, when this fuzzy set is considered as implication, the degree of truth of the corresponding fuzzy conditional proposition is equal to 1, for every increasing truth qualifier *T(p)*.

The mechanism of the method introduced, comprises six steps:

Step 1: Creation of the fuzzy sets which are the linguistic variables for each variable of the set of paired data.

Step 2: Creation of a partition of the data, where each subset of the partition is randomly populated.

Step 3: Calculation of the mean value for each of the subset of the partition.

Step 4: Calculation of the values of the fuzzy implications for the mean values in Step 3 using the membership functions of the linguistic variables created in Step 1.

Step 5: Evaluation of the implications under consideration for the membership functions values of Step 4.

Step 6: Evaluation of the degree of similarity between each fuzzy implication and the fuzzy set *L*.

The implication with the highest or that with the lowest degree of similarity with the fuzzy set *L* for each of the linguistic fuzzy sets is considered as the most appropriate fuzzy implication according to the expert's opinion about the inference drawn from the degree of truth of the corresponding conditional fuzzy proposition for the application.

IV. Application of the Method

The method is applied on a data set, provided for Hellenic National Meteorological Service, that contains yearly measurements of rainfall and overflow, which took place in Vogatsiko village, located in Northern Greece in the region of Macedonia (see Table I). The truth qualifier of the conditional fuzzy propositions, is considered to be the identity function, thus the degree of truth of each proposition is the value of the corresponding fuzzy implication of the proposition.

Step 1: For the creation of the linguistic variables the linguistic hedges 'High', 'Medium', and 'Low' for the variables Rainfall and Overflow of the paired data. For the construction of the membership functions of the fuzzy sets 'Medium Rainfall' and 'Medium Overflow', the method of least-square curve fitting, has been applied, as demonstrated in [7, pp. 292-295]. More specifically, the class of skew-normal distributions, as discussed in [1], [4] has been employed to fit the data using a selection of the normalized frequencies of Rainfall and Overflow regarding their maximum frequency, so that their values belong to the close interval [0, 1].

The skew-normal distribution is defined by:

Table I
The Data Set

Rainfall (mm)	Overflow (mm)
359	3,58159609600000
406,300000000000	13,8432332100000
410,200000000000	14,8100169300000
410,400000000000	14,8600890800000
459,900000000000	28,7205915500000
461,800000000000	29,3102586100000
501,200000000000	42,4756771500000
501,900000000000	42,7255362900000
504	43,4784072200000
538,100000000000	56,3852183600000
542	57,9418692300000
542,600000000000	58,1827966000000
555,400000000000	63,4135143800000
556,800000000000	63,9960925800000
578,400000000000	73,2416595700000
583,600000000000	75,5386096400000
593,600000000000	80,0320145100000
599,100000000000	82,5456641700000
608,500000000000	86,8904356000000
635,700000000000	100,015965400000
640,300000000000	102,300745100000
656,700000000000	110,603044500000
669,100000000000	117,039559900000
681,900000000000	123,823898900000
683,200000000000	124,520771300000
687,100000000000	126,619963500000
691,100000000000	128,786266200000
703,700000000000	135,696926300000
714	141,442348200000
751,200000000000	162,886273300000
779,600000000000	179,952821600000
787,400000000000	184,740380400000
836,700000000000	215,939070500000
851	225,282868200000
854,600000000000	227,653805000000
882,500000000000	246,286734700000

$$f(x;a,b,c,d) = a\frac{2}{c}\varphi\left(\frac{x-b}{c}\right)\Phi\left(d\left(\frac{x-b}{c}\right)\right),$$

where b and c represent the usual location and scale parameters, d determines the skewness, a controls the height of the function, and φ, Φ denote the pdf and cdf of a standard Gaussian deviate. Hence, the linguistic variable 'Medium Rainfall' is: $R_M = \{(x_i, R_M(x_i) \mid x_i \in X\}$, where X is the set of Rainfall measurements and R_M is the membership function:

$$R_M(x) = \\ = 3.5297\varphi\left(\frac{x-723.7342}{220.6916}\right)\cdot\Phi\left(-2.1556\left(\frac{x-723.7342}{220.6916}\right)\right) \quad (1)$$

The linguistic variable 'Medium Overflow' is:

$O_M = \{(y_i, O_M(y_i) \mid y_i \in Y\}$, where Y is the set of Overflow measurements and O_M is the membership function:

$$O_M(y) = \\ = 3.2316\varphi\left(\frac{y-3.7951}{109.3525}\right)\cdot\Phi\left(2.3429\left(\frac{y-3.7951}{109.3525}\right)\right) \quad (2)$$

For the construction of the membership functions of the linguistic variables 'Low Rainfall' and 'Low Overflow', the family of the decreasing sigmoidal membership functions is used, as explained in [5], under the formula:

$$f(x;a,c) = \frac{1}{1+e^{-a(x-c)}},$$

where a controls the slope and c the center of the curve which is the mean value of Rainfall and that of Overflow values. These fuzzy sets are:

$R_L = \{(x_i, R_L(x_i) \mid x_i \in X\}$, with membership function

$$R_L(x) = \frac{1}{1+e^{-(-0.039)(x-617.1833)}} \quad (3)$$

and $O_L = \{(y_i, O_L(y_i) \mid y_i \in Y\}$, with membership function

$$O_L(y) = \frac{1}{1+e^{-(-0.11)(y-99.5990)}} \quad (4)$$

Similarly, the membership functions of the fuzzy sets 'High Rainfall' and 'High Overflow' are created based on the increasing family of sigmoidal functions:

$$f(x;a,c) = \frac{1}{1+e^{-a(-x+c)}}.$$

These fuzzy sets are:

$R_H = \{(x_i, R_H(x_i) \mid x_i \in X\}$, where

$$R_H(x) = \frac{1}{1+e^{-(-0.038)(-x+617.1833)}} \quad (5)$$

and $O_H = \{(y_i, O_H(y_i) \mid y_i \in Y\}$, where

$$O_H(y) = \frac{1}{1+e^{-(-0.07)(-y+99.5990)}} \qquad (6)$$

The graphical representation of the membership functions (1)-(6) of Rainfall and Overflow for the hedge 'Medium' are shown in Fig. 1 and Fig. 2 respectively, for the hedge 'Low' in Fig. 3 and Fig. 4, while for the hedge 'High' in Fig. 5 and Fig. 6.

Step 2: The paired data of rainfall and overflow measurements are divided into six sets of six entries. The six subsets of the partition are populated randomly. Two subsets of the partition are used as control subsets to test the results. The partition is given in Table II.

Step 3: The arithmetic mean of each subset of the partition of the variables Rainfall and Overflow is calculated. The values are contained in Table III.

Step 4: The membership functions of the six linguistic variables created in Step 1 are evaluated for the mean value of each subset of the partition. The values are presented in Table III.

Step 5: The eight implications described in Section 2 are evaluated using the membership functions values of the previous step of the mechanism. The implication values are included in Table IV.

Step 6: Evaluation of the degree of similarity *M* between each of the eight fuzzy implications and the fuzzy set *L*.

The membership function of the fuzzy set *L* for the fuzzified mean values of the subsets of the partition is: $L\left(\bar{x}_i, \bar{y}_i\right) = L\left(R\left(\bar{x}_i\right), O\left(\bar{y}_i\right)\right) = 1, \qquad \forall i = 1,...,4$, where $\bar{x}_i, \bar{y}_i$ are the mean values of the $i-th$ subset, $R(x)$ represents one of the membership functions described by (1), (3), and (5), and $O(y)$ one of the membership functions (2), (4), and (6) respectively. The resulting degree of similarity *M* between each one of the eight fuzzy implications and the implication *L*, for each of the three hedges 'Low', 'Medium' and 'High' are contained in Table V, showing that the highest and the lowest degrees of similarity with the implication *L* correspond to the Lukasiewicz and the Larsen implication respectively.

The degree of similarity between the Lukasiewicz and the implication L is:
'Low Rainfall implies Low Overflow' with $S(I_4, L) = 0,974790009267612$, 'Medium Rainfall implies Medium Overflow' with $S(I_4, L) = 1$, and 'High Rainfall implies High Overflow' with $S(I_4, L) = 0,970511917148573$. That means that since fuzzy implication L which maps the values of all the subsets to 1, and the Lukasiewicz implication is the most similar implication to the L implication among the other seven implications, it takes the highest values for all the subsets of the partition and all the three hedges. This conclusion is confirmed by the implication values in Table IV. Moreover, this signifies that the premise of the conditional fuzzy propositions involving the Lukasiewicz implication implies the consequent with the highest degree of truth. For example the proposition '595,583333333333 mm is Low Rainfall to the degree of 0,698970442851426 implies that 92,5026916966667 mm is Low Overflow to the degree of 0,685808578280278' is true, with degree of truth 1 (i.e. absolutely true).

The degree of similarity between the Larsen and the implication L is:
'Low Rainfall implies Low Overflow' with $S(I_1, L) = 0,405985443893697$, 'Medium Rainfall implies Medium Overflow' with $S(I_1, L) = 0,872957343116749$, and 'High Rainfall implies High Overflow' with $S(I_1, L) = 0,229674263532507$. The result that for every hedge 'Low', 'Medium', and 'High' the degree of similarity between the Larsen and the implication L is the lowest means that the Larsen implication takes the lowest values of all the rest seven implications and for every subset of the data as shown in Table IV. For instance, the proposition '595,583333333333 mm is Low Rainfall to the degree of 0,698970442851426 implies that 92,5026916966667 mm is Low Overflow to the degree of 0,685808578280278' is true, with degree of truth 0,479359925671873. The fact that the Larsen implication takes the lowest values means that the corresponding conditional fuzzy propositions take the lowest degree of truth. Furthermore, the method is tested on the control subsets. The Lucasiewicz implication gives the highest values for the three control subsets of all the rest implications for each of the three hedges, whereas the Larsen implication

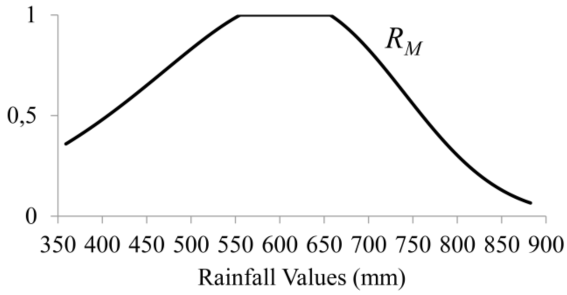

Fig. 1. Membership function of Medium Rainfall

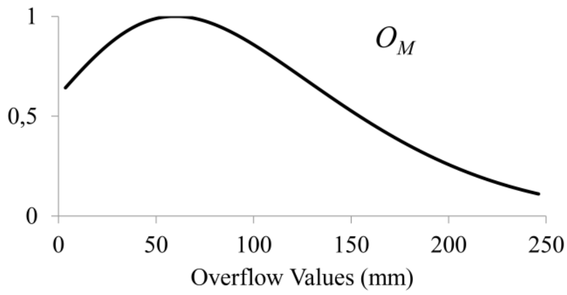

Fig. 2. Membership function of Medium Overflow

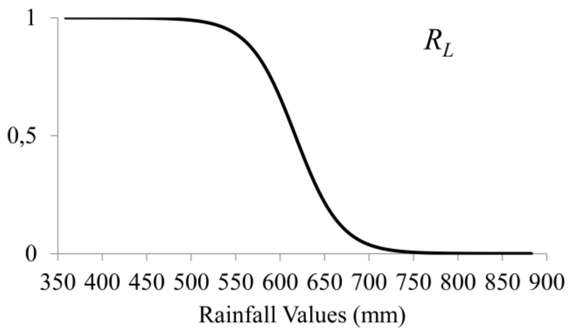

Fig. 3. Membership function of Low Rainfall

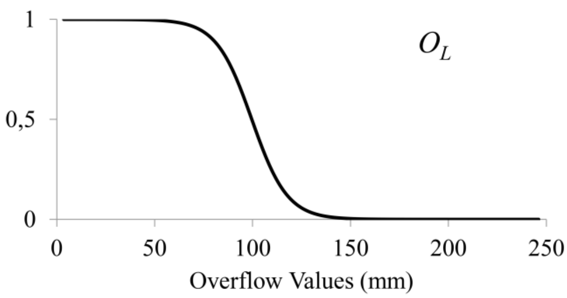

Fig. 4. Membership function of Low Overflow

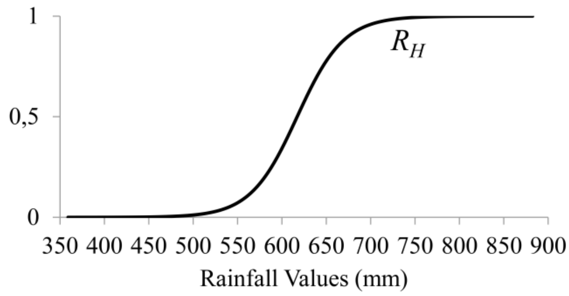

Fig. 5. Membership function of High Rainfall

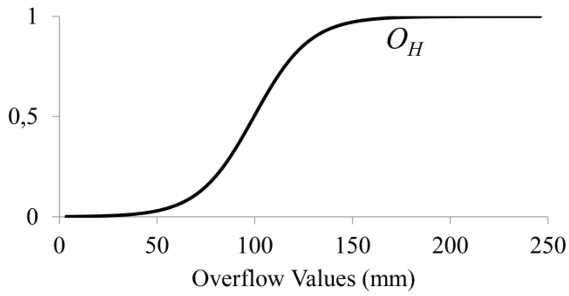

Fig. 6. Membership function of High Overflow

Table II

Partition of the nine subsets S1-S6 (last two for control) and six entries each

Subsets	**Rainfall**	**Overflow**
S1	640,30000000000	102,30074510000
	851	225,28286820000
	578,40000000000	73,241659570000
	410,20000000000	14,810016930000
	410,40000000000	14,860089080000
	683,20000000000	124,52077130000
S2	504	43,478407220000
	681,90000000000	123,82389890000
	656,70000000000	110,60304450000
	703,70000000000	135,69692630000
	406,30000000000	13,843233210000
	542	57,941869230000
S3	714	141,44234820000
	501,90000000000	42,725536290000
	787,40000000000	184,74038040000
	461,80000000000	29,310258610000
	555,40000000000	63,413514380000
	882,50000000000	246,28673470000
S4	542,60000000000	58,182796600000
	608,50000000000	86,890435600000
	836,70000000000	215,93907050000
	359	3,5815960960000
	669,10000000000	117,03955990000
	599,10000000000	82,545664170000

Table II

(*Continued*)

Control Subsets	**Rainfall**	**Overflow**
S5	593,600000000	80,03201451
	854,600000000	227,6538050
	687,100000000	126,6199635
	635,700000000	100,0159654
	779,600000000	179,9528216
	751,200000000	162,8862733
S6	459,900000000	28,72059155
	583,600000000	75,53860964
	501,200000000	42,47567715
	556,800000000	63,99609258
	691,100000000	128,7862662
	538,100000000	56,38521836

Table III

Values of membership functions on the arithmetic mean of Rainfall and Overflow

Subsets	$\overline{x}$	R_L	R_M	R_H	$\overline{y}$	O_L	O_M	O_H
S1	595,583333333333	0,698970442851426	1	0,305593868489408	92,5026916966667	0,685808578280278	0,901132175628007	0,378306383303007
S2	582,433333333333	0,794986613639948	1	0,210735118535510	80,8978965600000	0,886665441276491	0,955923664835160	0,212641060468765
S3	650,500000000000	0,214274431217568	1	0,780062963866185	117,986462096667	0,116848491438761	0,741972727212592	0,783659479658116
S4	602,500000000000	0,639374423691585	1	0,364018064225803	94,0298538110000	0,648535558984595	0,892800804791239	0,403755423092722
Control Subsets								
S5	716,966666666667	0,0200052985303838	0,740823007266648	0,977941819898813	146,193473885000	0,00590877810232455	0,551576430746269	0,963088130726514
S6	555,116666666667	0,918384728360827	0,999319309699934	0,0863898834131590	65,9837425800000	0,975818306499594	0,996510096756681	0,0868221070922182

Table IV

Values of the eight implications for the three hedges ‘Low’, ‘Medium’, ‘High’ of the subsets S1-S4 of the partition of the dataset

Subsets		I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8
S1	Low	0,479359925671873	0,685808578280278	0,698970442851426	1	0,685808578280278	0,793551347391594	0,685808578280278	0,642940570829289
	Med.	0,901132175628007	0,901132175628007	1	1	0,901132175628007	1	0,901132175628007	0,910907022324058
	High	0,115608111147803	0,305593868489408	0,621693616696993	0,927287485186401	0,621693616696993	0,737301727844796	0,621693616696993	0,665428903105810
S2	Low	0,704887156591968	0,794986613639948	0,794986613639948	0,908321172363458	0,794986613639948	0,818221715315477	0,794986613639948	0,738333640473257
	Med.	0,955923664835160	0,955923664835160	1	1	0,955923664835160	1	0,955923664835160	0,957866388156723
	High	0,0448109390834017	0,210735118535510	0,787358939531235	0,998094058066744	0,787358939531235	0,832169878614636	0,787358939531235	0,796887585138530
S3	Low	0,0250376440416714	0,116848491438761	0,883151508561239	1	0,883151508561239	0,908189152602910	0,785725568782432	0,886077119496689
	Med.	0,741972727212592	0,741972727212592	1	1	0,741972727212592	1	0,741972727212592	0,808550800714699
	High	0,611303736363943	0,780062963866185	0,780062963866185	0,996403484208069	0,780062963866185	0,827644256705826	0,780062963866185	0,695394488293913
S4	Low	0,414657049269275	0,639374423691585	0,639374423691585	0,990838864706991	0,639374423691585	0,766121490284681	0,639374423691585	0,620384282250157
	Med.	0,892800804791239	0,892800804791239	1	1	0,892800804791239	1	0,892800804791239	0,904292472244645
	High	0,146974267534883	0,364018064225803	0,596244576907278	0,960262641133080	0,596244576907278	0,743218844442160	0,596244576907278	0,655586234479567

Table V

The degree of similarity between each implication and the fuzzy set L for the three hedges ‘Low’, ‘Medium’, ‘High’

$S(I,L)$	Low	Medium	High
I_1	0,405985443893697	0,872957343116749	0,229674263532507
I_2	0,559254526762643	0,872957343116749	0,415102503779226
I_3	0,754120747186050	1	0,696340024250423
I_4	0,974790009267612	1	0,970511917148573
I_5	0,750830281043263	0,872957343116749	0,696340024250423
I_6	0,821520926398666	1	0,785083676901855
I_7	0,726473796098561	0,872957343116749	0,696340024250423
I_8	0,721933903262348	0,895404170860031	0,703324302754455

Table VI

Values of the eight implications for the control clusters (S5, S6)

Subsets		I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8
S5	Low	0,000118206869886797	0,00590877810232455	0,994091221897675	1	0,994091221897675	0,994209428767562	0,979994701469616	0,994091920355840
	Medium	0,408620510162855	0,551576430746269	0,740823007266648	1	0,551576430746269	0,857044079416586	0,551576430746269	0,673809011779078
	High	0,941844159285633	0,963088130726514	0,977941819898813	1	0,963088130726514	0,978756028559119	0,963088130726514	0,943990800075571
S6	Low	0,896176630344152	0,918384728360827	0,918384728360827	0,942566421861233	0,918384728360827	0,920358323844558	0,918384728360827	0,898687255247349
	Medium	0,995831781999900	0,996510096756681	0,999319309699934	1	0,996510096756681	0,999321685243219	0,996510096756681	0,995846328677418
	High	0,00750055170938154	0,0863898834131590	0,913177892907782	0,999567776320941	0,913177892907782	0,920678444617163	0,913177892907782	0,913829106611544

takes the lowest ones. The results can be found in Table VI.

Taking into account the nature of this application example and the dataset, since the fuzzy products do not efficiently interpret the cause-effect relationship, it is more reasonable that the Lukasiewicz implication be selected as the most appropriate fuzzy implication.

Lastly, the method was applied to all seven different partitions of the thirty six measurements of Rainfall and Overflow and the resulting two fuzzy implications were the same in every case; the Lukasiewicz implication, being the fuzzy set with the highest degree of similarity to the implication L and taking the highest values, and the Larsen implication which gives the lowest ones.

V. Conclusions

Selecting an appropriate fuzzy implication for reasoning under each particular application is, in general, a difficult problem. In the literature, there are methods addressing the suitability of fuzzy implications under the theoretical guidelines. The theoretical results of these methods, which involve the generalized inference rules, does not always hold for each situation. To overcome this effect, other methods, as mentioned in the introduction section of this article, assess the suitability of fuzzy implications according to their distance from the corresponding generalized inference rules, and in addition they make the selection of the most suitable one, which corresponds to the smallest distance. However, these approaches are based on the theoretical guidelines of the fuzzy propositions, and so they do not take under account the induction form the data, but the inference rules instead. Furthermore, these methods do not take under consideration the expert's opinion about the ability of the resulting suitable implications to interpret the data application.

This article introduces a method for the selection of the most appropriate fuzzy implication among others. The method introduced does not incorporate the theoretical aspect of suitability of fuzzy implications, but it is based on the appropriateness of fuzzy implications from the application view point instead. This is achieved by introducing a special fuzzy implication, which makes the propositions that it is involved in suitable for making valuable inference from the data of applications, since due to its construction it represents the ideal fuzzy implication. It employs the measures of similarity between fuzzy implications and the special aforementioned implication, as a criterion of appropriateness of the implications. What is more, the final selection of the most appropriate implication from the resulting implications relies on the expert's opinion about the essence and characteristics of an application and on which degree the resulting implications are consistent with the inference drawn from the evaluation of this special implication. Moreover, the method has been applied on a dataset of real observations. From the two resulting implications the one with the highest value of grade of similarity with the ideal implication for this application was selected as the most appropriate one, thus it best interprets the data. What is more, this selection is made considering which of the two resulting implications are best agreeable to the nature of the application, according to the expert.

Our aim for the future work is to generalize the conception of the special fuzzy implication L introduced, to a more expert opinion based fuzzy set, whose membership function takes positive values in a close interval $[a,1]$, instead of the value one for each value of the variable. In this way, the expert's opinion about the appropriateness of the given dataset will be included in the induction process.

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